A MINIMAL RK-DEGREE

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We construct a minimal rK-degree, continuum many, in fact. We also show that every minimal sequence, that is, a sequence with minimal rK-degree, must have very low descriptional complexity, that every minimal sequence is rK-reducible to a random sequence and that there is a random sequence with no minimal sequence rK-reducible to it.

1. Introduction

This article continues the study of relative randomness via rK-reducibility initiated in [3] and pursued in [9].

One of the most popular definitions of absolute algorithmic randomness states that an infinite binary sequence R is random if it is incompressible, that is, if

$$\exists d \, \forall n \, . \, K(R[n]) \geq n - d,$$

where $K(\sigma)$ is the prefix-free descriptional complexity of the string σ. Under this same paradigm of incompressibility, one can define relative algorithmic randomness as follows. An infinite binary sequence A is less random than an infinite binary sequence B if A is completely compressible given B, that is, if

$$\exists d \, \forall n \, . \, K(A[n] | B[n]) < d,$$

where $K(\sigma|\tau)$ is the conditional prefix-free descriptional complexity of σ given τ. In this case, we write $A \leq_{\mathrm{rK}} B$ for short and say "A is rK-reducible to B".[a]

The rK-reducibility, which is fairly easily seen to be reflexive and transitive, enjoys the following properties, all of which we will use throughout.

Theorem 1.1 (Downey, Hirschfeldt and LaForte [3]): *For infinite binary sequences A and B, $A \leq_{\mathrm{rK}} B$ is equivalent to both of*

- $\exists d\,\forall n\,.\,C(A[n] \mid B[n]) < d$,
- *There exists a partial computable function φ such that*

$$\exists d\,\forall n\,\exists i < d\,.\,\varphi(i, B[n]) = A[n],$$

and implies all three of

- $\exists d\,\forall n\,.\,K(A[n]) \leq K(B[n]) + d$,
- $\exists d\,\forall n\,.\,C(A[n]) \leq C(B[n]) + d$,
- $A \leq_{\mathrm{T}} B$.

Notice that the fifth bullet says that $\leq_{\mathrm{rK}}$ really is a reducibility in the sense of computability theory as there is a way to compute A from B. Moreover, a sequence is computable iff it is rK-reducible to any other sequence, that is, the computable sequences are those of least relative randomness, as they should be.

In what follows we answer a basic question: is there a sequence of minimal relative randomness, that is, a sequence with only the computable sequences strictly less random ($<_{\mathrm{rK}}$) than it? Indeed, as our title indicates, there is such a degree and this is our main result. In fact, there are even continuum many such minimal degrees. We prove these claims in Section 2 and follow them up in Section 3 with three notes on such minimal sequences.

Before beginning, let us set some notation and conventions. $\mathbb{N}$ will denote the set of natural numbers $\{0, 1, 2, \ldots\}$, $\{0,1\}^*$ the set of binary strings and $\{0,1\}^\infty$ the set of infinite binary sequences. 'String' and 'sequence' without further qualification will mean 'binary string' and 'infinite binary sequence', respectively. For strings σ and τ, $|\sigma|$ will denote the length of σ, and $\sigma\tau$ or, when that might cause confusion, $\sigma\frown\tau$ the concatenation of σ and τ. Also

[a]The 'rK' stands for 'relative Kolmogorov' complexity, where 'relative' refers to the fact that one uses 'conditional complexity' and 'Kolmogorov' is just used in place of 'plain Kolmogorov' or 'prefix-free Kolmogorov' since by Theorem 1.1 it does not matter which variant is considered.

$\sigma \subseteq \tau$ and $\sigma \subset \tau$ will mean σ is an initial segment of τ and σ is a proper initial segment of τ, respectively. For a sequence A and a positive natural n, $A[n]$ will denote the length n initial segment of A, that is, the string $\langle A(0), A(1), \ldots, A(n-1)\rangle$. Trees are subsets of $\{0,1\}^*$ closed under initial segments. A path of a tree T is a sequence such that every initial segment of it is a member of T. The set of all paths of T will be denoted by $[T]$. A Π^1_0-class is a set of paths through a Π^0_1-tree.

Lastly, our notation for other notions from the theory of computability follows the books of Odifreddi [8] and Soare [10].

2. The main result

Theorem 2.1: *There is a minimal rK-degree.*

Proof: We construct a special binary tree, suitable paths of which will have minimal rK-degree. Roughly speaking, we make the set of splitting nodes of our tree very sparse so that any incomputable path of hyperimmune-free Turing degree can be recovered in two guesses from its image under an rK-reduction. More precisely, we build a Π^0_1 tree T (a tree whose complement is recursively enumerable) such that

(1) T has no computable paths;
(2) for every computable function $\Phi : \mathbb{N} \to \mathbb{N}$ (thought of as a functional) and for every path X of T there is a string $\star \subset X$ such that
   (a) either Φ^X and Φ^Y are compatible for every path Y of T extending $\star$
   (b) or Φ^Y and Φ^Z are incompatible for all distinct paths Y, Z of T extending $\star$,
(3) the set S of splitting nodes of T is very sparse, to wit, for all computable functions $g : \mathbb{N} \to \mathbb{N}$ we have

$$\forall^\infty \sigma \in S\, \forall \tau \in S\, .\, \sigma \subset \tau \to g(|\sigma|) < |\tau|.$$

Constructing T. We build T in stages, beginning with the full binary tree and pruning it stage by stage. To describe this pruning we use moving markers in the style of [11]. For notational niceness stage subscripts are suppressed whenever possible.

Let $\{m_\sigma \mid \sigma \in \{0,1\}^*\} \subseteq \{0,1\}^*$ denote the set of markers of T. These are on the splitting nodes of T. At stage zero, $T = \{0,1\}^*$ and each $m_\sigma = \sigma$. At later stages when necessary T is pruned via the $\textsc{Cut}$ procedure. If $\sigma \subset \tau$ then $\textsc{Cut}(m_\sigma, m_\tau)$ cuts off all paths of T that extend m_σ but not m_τ and then updates the positions of all the markers, preserving their order, as follows: m_σ moves to m_τ, each $m_{\sigma\epsilon}$ moves to $m_{\tau\epsilon}$ and all other markers stay where they are. Since Cut is the only action ever taken, T will be a perfect tree without leaves at every stage.

At stage $s > 0$ the construction runs as follows, where each check is performed only when the markers involved have indices of length $\leq s$; also, the computations involved are only up to stage s.

- If there exist σ, $i < 2$ and $e \leq |\sigma|$ such that for all $x \leq |\sigma|$, $\Phi_e(x) = m_{\sigma i}(x)$, then $\textsc{Cut}(m_\sigma, m_{\sigma(1-i)})$.
- If there exist σ, δ, ϵ and $e \leq |\sigma|$ such that $\Phi_e^{m_{\sigma 0}}$ and $\Phi_e^{m_{\sigma 1}}$ are compatible for all arguments up to $|\sigma|$, but $\Phi_e^{m_{\sigma 0\delta}}$ and $\Phi_e^{m_{\sigma 1\epsilon}}$ are incompatible at some argument up to $|\sigma|$, then $\textsc{Cut}(m_{\sigma 0}, m_{\sigma 0\delta})$ and $\textsc{Cut}(m_{\sigma 1}, m_{\sigma 1\epsilon})$.
- If there exist σ, τ, υ and $e \leq |\sigma|$ such that $\sigma \subset \tau \subset \upsilon$ and $|m_\tau| \leq \Phi_e(|m_\sigma|) < |m_\upsilon|$, then $\textsc{Cut}(m_\tau, m_\upsilon)$.

It is not difficult to check that each marker eventually settles and that, in the limit, T satisfies properties (1)-(3).

A suitable path of T. Let A be a path of T of hyperimmune-free Turing degree.[b] Such a path exists by the Hyperimmune-Free Basis Theorem [5] since $[T]$ is a nonempty Π^0_1 class (since T is a Π^0_1 tree). We show that A has minimal rK-degree. By (1), A is incomputable. Let $B \leq_{\mathrm{rK}} A$ be an incomputable set. We need to show that $A \leq_{\mathrm{rK}} B$. To this end, observe that $B \leq_{\mathrm{T}} A$ and, in fact, $B \leq_{tt} A$ since A has hyperimmune-free Turing degree; see page 589 in Odifreddi's book [8]. Let Φ be a computable functional (total on all oracles) that witnesses the truth-table reduction.

We come now to the heart of the argument: building an rK-reduction from B to A. Let $\star$ be the magic string of (2) for Φ and A. Given $B[n]$ for n sufficiently large, run through the computable approximation (that thins) to T until a stage t is reached such that T_t (the stage t approximation of T) has at most two extensions of $\star$ of length n with extensions in T_t that map to $B[n]$ under Φ. The key here is that such a stage is guaranteed to exist by Lemma 2.2 below. To find these extensions effectively from $B[n]$ we use the fact that Φ is total on all oracles and has a computable use function. Output the (at most) two strings of length n found; one will be $A[n]$. Except for finitely many short lengths, this procedure describes an rK-reduction

[b]That is, for every total function $f \leq_{\mathrm{T}} A$, there exists a computable function g such that for all x, $g(x) \geq f(x)$. Put more concisely, every total function computable from A is majorized by a computable one.

from B to A. Extending it to all lengths gives the final reduction. $\square$

Lemma 2.2: *Let $\star$ be the magic string of (2) for Φ and A. For almost all lengths n and almost all stages t, T_t has at most two extensions of $\star$ of length n with extensions in T_t that map to $B[n]$ under Φ.*

Proof: Let φ be the computable use function for the tt-reduction Φ. Let f be the function defined for $m \geq |\star|$ by $f(m)$ equals the first stage s such that for all strings $\nu \supset A[m]\widehat{\ }(1-A(m))$ of length $\varphi(s)$ on T_s, there exists $x \leq s$ such that $\Phi^{\nu}(x)\downarrow \neq \Phi^{A}(x)$. (Notice that all ν extend $\star$.) For $m < |\star|$, define $f(m)$ to be 0, say. It is unimportant. Note that f is total, for if not, then for all s there exists a string $\nu_s \supset A[m]\widehat{\ }(1-A(m))$ of length $\varphi(s)$ on T_s such that for all $x \leq s$, $\Phi^{\nu_s}(x) = \Phi^{A}(x)$. (Remember that Φ is total on all oracles.) Then the sequence Y defined by $Y(n) = \liminf_s \nu_s(n)$ is a path of T different from A such that $\Phi^Y = \Phi^A$. But this is a contradiction, because (2b) holds for $X = A$ since B is incomputable. Also, f is A-computable by definition. Thus, since A has hyperimmune-free Turing degree, there is a computable function g majorizing f.

Now, fix n bigger than the length of $\star$, the length that (3) takes effect for g and the length of the first splitting node of A on T. Let τ be the last splitting node of T on $A[n]$ and let $\sigma \subset \tau$ be any other splitting node of T extending $\star$. Then by (3) we have for $s = f(|\sigma|)$ that

$$s \leq g(|\sigma|) < |\tau| \leq n.$$

So by stage s, every $\nu \in T_s$ extending $A[|\sigma|]\widehat{\ }(1-A(|\sigma|)) = \sigma\widehat{\ }(1-A(|\sigma|))$ will have some number $x \leq s$ with $\Phi^{\nu}(x)\downarrow \neq \Phi^{A}(x) = B(x)$, so that ν cannot map to $B[n]$ under Φ. Since σ was an arbitrary splitting node of T below the last splitting node of $A[n]$, we see that only the strings extending the last splitting node of $A[n]$ can map to $B[n]$ under Φ. So the result holds. $\square$

In fact, by a generalized Hyperimmune-Free Basis Theorem below, the tree of the proof of Theorem 2.1 has continuum many paths of hyperimmune-free Turing degree. Thus, since every rK-degree is countable, there are continuum many minimal rK-degrees.

Theorem 2.3: *Every nonempty Π^0_1 class with no computable members has $2^{\aleph_0}$ paths of hyperimmune-free Turing degree.*

Proof: By basic facts from the theory of Π^0_1 classes, we can assume without loss of generality that our Π^0_1 class is the set of paths through a binary tree

T_0 that is infinite, computable and has no computable paths. We modify slightly the proof of the Hyperimmune-Free Basis Theorem in [5] by way of an extra parameter sequence X. For each sequence X we construct (uniformly relative to $X \oplus \emptyset''$) computable subtrees $S_1 \supset T_1 \supseteq S_2 \supset T_2 \supseteq \ldots$ of T_0 such that their only common path Y has hyperimmune-free Turing degree. We then show that the map $X \mapsto Y$ is one-to-one.

To this end, fix X and, starting from T_0, let S_e and T_e be defined recursively as follows. Let $U_{e,x}$ be the computable tree $\{\tau \mid \Phi^{\tau}_{e,|\tau|}(x)\uparrow\}$.

(1) If for all x, $T_e \cap U_{e,x}$ is finite, then let $S_e = T_e$. Otherwise, choose x least such that $U_{e,x}$ is infinite and let $S_e = T_e \cap U_{e,x}$.
(2) Since S_e is an infinite tree with no computable paths, it has at least two paths. Let σ be the length-lexicographic least node of S_e such that $\sigma 0$ and $\sigma 1$ have paths in S_e through them.
(3) Let $T_{e+1} = \{\tau \in S_e \mid \tau \subseteq \sigma\,\hat{}\,X(e) \vee \tau \supset \sigma\,\hat{}\,X(e)\}$.

By induction each $[T_e]$ and $[S_e]$ is nonempty, so that $\bigcap_e [T_e] \cap [S_e]$ is nonempty, being the intersection of a decreasing sequence of closed nonempty sets in the compact space $\{0,1\}^\infty$. Choose (the unique) sequence $Y \in \bigcap_e [T_e] \cap [S_e]$. It will have hyperimmune-free Turing degree, for fix a natural e and consider the function Φ^Y_e. If for every x, $T_e \cap U_{e,x}$ is finite, then there is a total and computable function g majorizing Φ^Y_e where g is chosen as

$$g(x) = \max\{\Phi^{\tau}_{e,|\tau|}(x) \mid \tau \in T_e \wedge |\tau| = l_x\}$$

and l_x is the least value such that $\Phi^{\tau}_{e,|\tau|}(x)$ is defined for each $\tau \in T_e$ of length l_x. If there exists some x such that $T_e \cap U_{e,x}$ is infinite, then $\Phi^{\tau}_{e,|\tau|}(x)$ is undefined for infinitely many $\tau \in T_e$ and S_e is the set of all these τ. Since all prefixes of Y are in S_e, this means $\Phi^Y_e(x)$ is undefined, so that Φ^Y_e is not total.

Also, the map $X \mapsto Y$ is one-to-one, for if two sequences X_1 and X_2 differ and e is the first place at which this happens, then the corresponding trees $S_e(X_1)$ and $S_e(X_2)$ are the same, but the intersection of $T_{e+1}(X_1)$ and $T_{e+1}(X_2)$ is finite since one contains the nodes above $\sigma 0$ and the other the ones above $\sigma 1$. Thus $Y(X_1)(|\sigma|) \neq Y(X_2)(|\sigma|)$. □

3. Three notes

From now on let us call a sequence with minimal rK-degree a 'minimal sequence'. As one might expect, minimal sequences have low initial segment complexity. Indeed, so low that they are close to being computable in the sense of Chaitin's characterization (see [1]): a sequence X is computable iff $\exists d\,\forall n\,.\,C(X[n]) \leq C(n) + d$.

Proposition 3.1: *If A is a minimal sequence, then for any computable unbounded increasing function $g : \mathbb{N} \to \mathbb{N}$,*

$$\begin{aligned} &\exists d\,\forall n\,.\,C(A[n]) \leq C(n) + g(n) + d \quad \textit{and} \\ &\exists d\,\forall n\,.\,K(A[n]) \leq K(n) + g(n) + d. \end{aligned}$$

In particular, A cannot be random.

For the proof, we need the notion of dilutions.

Definition 3.2: For $X \in \{0,1\}^\infty$ and $f : \mathbb{N} \to \mathbb{N}$ strictly increasing, the *f-dilution of X* is the sequence defined by

$$X_f(n) = \begin{cases} X(m) & \text{if } n = f(m) \text{ for some (unique) } m \\ 0 & \text{else.} \end{cases}$$

Notice that for any sequence X and any strictly increasing computable function f, $X_f \leq_{\mathrm{rK}} X$ and $X_f \equiv_{\mathrm{T}} X$. Now we prove Proposition 3.1.

Proof: Fix A and g as in the hypothesis. The idea is that since A is a minimal sequence, it is rK-reducible to every one of its computable dilutions. Picking a dilution appropriate to g will give the desired complexity bound.

We prove the bound for K. The argument for C is identical. Define the function $f : \mathbb{N} \to \mathbb{N}$ recursively by

$$\begin{aligned} &f(0) = 0; \\ &f(x) = \text{the least } n \text{ such that } n > f(x-1) \text{ and } g(n) \geq 4x. \end{aligned}$$

Since g is unbounded and increasing, f is well-defined. Also, by construction f is computable, total and strictly increasing. Furthermore, for any given n, if x is the greatest number such that $f(x) \leq n$, then $g(n) \geq 4x$.

Since A is minimal, $A \leq_{\mathrm{rK}} A_f$ via some $[\varphi, e]$. Now fix n and choose x greatest such that $f(x) \leq n$. Observe that inserting zeros into $A[x]$ in the appropriate computable places produces $A_f[n]$. So to describe $A[n]$, besides a few computable partial functions given ahead of time, one only needs the correct $i < e$ such that $\varphi(i, A_f[n]) = A[n]$, the value n and $A[x]$. This information can be coded, up to a uniform constant, by a string of length

$K(n) + 2K(A[x])$. The factor of 2 comes from concatenating strings in a prefix-free way. So, up to a uniform additive constant, for all n,

$$K(A[n]) \leq K(n) + 2K(A[x]) \leq K(n) + 4x \leq K(n) + g(n),$$

as desired. Now fixing g as, say, $g(n) = \lfloor \lg(n+1) \rfloor$, we see that A cannot be random. □

Using dilutions again, we also get the following.

Proposition 3.3: *Every minimal sequence is rK-reducible to a random sequence.*

Proof: Fix a minimal sequence A and choose a random sequence $R \geq_{\mathrm{wtt}} A$ with the use being majorized by $f(n) = 2n$. This is possible since every sequence has such a random; see [4, 6] and also [7] for a more recent proof using martingales. Then $R \geq_{\mathrm{rK}} A_f \geq_{\mathrm{rK}} A$, by the minimality of A, as desired. □

Do all sequences have randoms rK-above them? That question is still open and seemingly difficult. We end with one last note, a contrast to Proposition 3.3.

Proposition 3.4: *There is a random sequence with no minimal sequence rK-reducible to it.*

Proof: Let R be a random sequence of hyperimmune-free Turing degree. Such a sequence exists by the Hyperimmune-Free Basis Theorem applied to the complement of any member of a universal Martin-Löf test. Then R has no minimal sequence reducible to it.

To see this, assume (toward a contradiction) there is some minimal sequence A such that $A \leq_{\mathrm{rK}} R$. Since R has hyperimmune-free Turing degree, so does A and $A \leq_{\mathrm{tt}} R$. Since A is incomputable and truth-table reducible to a random, A is Turing equivalent to some random S (see [2]). Since A has hyperimmune-free Turing degree, $S \leq_{\mathrm{tt}} A$ via some computable function with computable use function f. Thus, disregarding floor functions and uniform constants for ease of reading, we have that for all n,

$$\begin{array}{rlll} n & \leq & K(S[n]) & \text{(since } S \text{ is random)} \\ & \leq & K(A[f(n)]) + K(n) & \text{(using the tt-reduction)} \\ & \leq & K(f(n)) + \lg n + K(n) & \text{(by Proposition 3.1)} \\ & \leq & 2K(n) + \lg n & \text{(since } f \text{ is computable)} \\ & \leq & 4 \lg n, & \end{array}$$

a contradiction. □

Remark 3.5: Do maximal rK-degrees exist? That basic question is still open.

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