Partial functions and domination*

C. T. Chong[1], Gordon Hoi[1], Frank Stephan[1,2] and Dan Turetsky[3]

[1] Department of Mathematics, National University of Singapore,
Singapore 119076, Republic of Singapore.
chongct@math.nus.edu.sg, hoickg@gmail.com
[2] Department of Computer Science, National University of Singapore,
Singapore 117417, Republic of Singapore.
fstephan@comp.nus.edu.sg
[3] Kurt Gödel Research Center, Währinger Straße 25,
1090 Wien, Austria.
dturets@gmail.com

Abstract. The current work introduces the notion of pdominant sets and studies their recursion-theoretic properties. Here a set A is called pdominant iff there is a partial A-recursive function ψ such that for every partial recursive function ϕ and almost every x in the domain of ϕ there is a y in the domain of ψ with $y \leq x$ and $\psi(y) > \phi(x)$. While there is a full Π^0_1-class of nonrecursive sets where no set is pdominant, there is no Π^0_1-class containing only pdominant sets. No weakly 2-generic set is pdominant while there are pdominant 1-generic sets below K. The halves of Chaitin's Ω are pdominant. No set which is low for Martin-Löf random is pdominant. There is a low r.e. set which is pdominant and a high r.e. set which is not pdominant.

1 Introduction

It is known since the beginning of recursion theory that there is a close relation between the computational power of a set A and the classes of recursive and partial recursive functions which can be dominated by an A-recursive function [13, 14, 21]: The halting problem K is Turing reducible to A iff every partial recursive function is majorised by some A-recursive total function iff some fixed A-recursive function dominates all partial recursive functions. Furthermore, Martin showed that a set A has high Turing degree iff there is an A-recursive function f dominating every total recursive function. This generalises to r.e. Turing degrees by the fact that an r.e. set A is low$_2$ iff some K-recursive function dominates all total A-recursive functions.

For r.e. sets, one has also the further result that $A \leq_T B$ iff the convergence module of A is majorised by some B-recursive function. It is also known that for sets $A, B \leq_T K$, $A \leq_T B$ iff for every A-recursive function f there is a B-recursive function g majorising f. This does, however, not generalise to the larger class of the sets below K': There is a nonrecurisve set $A \leq_T K'$ for

* C. T. Chong's research was partially supported by NUS grant WBS C146-000-025-00. F. Stephan's research was partially supported by NUS grants WBS R252-000-420-112, WBS R146-000-181-112 and WBS R146-000-184-112 (MOE2013-T2-1-062).

which every A-recursive function is majorised by a recursive function; such a set A is said to have a hyperimmune-free degree [16].

There have also been various attempts to generalise domination to relations between the partial recursive functions of two relativised worlds. For example, Stephan and Yu [17] considered the notion where A is strongly hyperimmune-free relative to B; here A is strongly hyperimmune-free relative to B iff for every partial A-recursive function f there is a partial B-recursive function g such that $dom(f) \subseteq dom(g)$ and $f(x) \leq g(x)$ for all $x \in dom(f)$. Their results include the result that, when $B \not\geq_T K$ then A is strongly hyperimmune-fre relative to B iff $A \leq_T B$. So real power in terms of the degrees of being strongly hyperimmune-free is only achieved by being above K. One of the reasons for this phenomenon was that the domains have to be compatible in the sense that whenever f is defined so is g and furthermore g is above f.

The notion investigated in the current work tries to overcome this shortcoming by permitting g to be above the value $f(x)$ not at x itself but at places below x. In this initial work, the emphasis is more on domination than majorisation; that is, on the question of when there is a partial B-recursive function dominating all partial A-recursive functions in this sense. Then B must be properly above A and already the question when this can be done with the lower oracle being recursive turned out to be an interesting and challenging notion.

The notion itself stems from a principle in reverse mathematics, more precisely, it is motivated by the bounded monotone enumeration principle BME. Chong, Slaman and Yang [4] introduced BME in order to analyse the rate of growth of a finite collection of partial recursive functions in a nonstandard model of a fragment of Peano arithmetic. The ability to control the growth rate in that model was key to separating stable Ramsey's theorem for pairs from Ramsey's theorem for pairs over the system RCA_0.

The formal definition of when a function pdominates another function or a set of functions is the following.

Definition 1. If ψ and ϕ are partial functions, ψ pdominates ϕ iff for almost all $x \in dom(\phi)$ there is a $y \in dom(\psi)$ with $y \leq x$ and $\psi(y) > \phi(x)$; ψ pdominates a set of partial functions iff it pdominates every partial function in it. A set A is pdominant iff some partial A-recursive function pdominates the set of all partial recursive functions.

This property can indeed be formulated in a stricter way and therefore the following equivalent property turned out to be fruitful for various proofs.

Proposition 2. *A set A is pdominant iff there is a partial A-recursive function ψ such that for all constants k and all partial recursive functions ϕ and almost all $x \in dom(\phi)$ there is a $y \in dom(\psi)$ with $y \leq x/k$ and $\psi(y) > \phi(x)$.*

Proof. Given ψ and a partial recursive function ϕ and a constant k, consider the functions ϕ_ℓ given by $\phi_\ell(x) = \phi(kx + \ell)$. For each ℓ and almost all x in the domain of ϕ_ℓ there is a $y \leq x$ in the domain of ψ with $\psi(y) > \phi_\ell(x)$. It follows that for almost all x in the domain of ϕ there is a $y \leq x/k$ in the domain of ψ with $\psi(y) > \phi(x)$. $\square$

Clearly the sets above the halting problem K are pdominant. One might ask whether there are natural examples of sets which are pdominant but not above the halting problem. Indeed, taking the half of Chaitin's Ω [2, 3, 24] provides such an example. Martin-Löf random sets [15] are those sets A for which there is not a uniformly r.e. sequence of open classes $C_0, C_1, \ldots$ such that $\mu(C_n) < 2^{-n} \wedge A \in C_n$ for all n.

Example 3. *Let Ω be a left-r.e. Martin-Löf random set and A be given by $A(n) = \Omega(2n)$. Then A is low and A is pdominant.*

Proof. The lowness of A is well-known [18, Corollary 3.4.11]. Now it is shown that A is pdominant. Let Ω_s be an approximation of Ω from the left. On input $x = 2^n + \sum_{m<n} 2^m \cdot a_m$, let $\psi^A(x)$ be the first stage s such that $\Omega_s(0)\Omega_s(1)\ldots\Omega_s(2n) = A(0)a_0A(1)a_1\ldots A(n-1)a_{n-1}A(n)$. The function ψ^A is partial recursive and in the case that $x = 2^n + \sum_{m<n} 2^m \cdot \Omega(2m+1)$, $\psi^A(x)$ is the time to compute $\Omega(0)\Omega(1)\ldots\Omega(2n)$ from the left. Assume now by way of contradiction that there would be a partial recursive function ϕ such that for infinitely many x in the domain of ϕ there is no $y \leq x$ with $\psi^A(y) > \phi(x)$. Taking now n such that $2^{n+1} \leq x < 2^{n+2}$ for such an x, it would follow that $s = \phi(x)$ satisfies $\Omega_s(m) = \Omega(m)$ for $m = 0, 1, \ldots, 2n$. So one could compute Ω up to $2n$ for infinitely many n from an input to ϕ which has $n+1$ binary bits. Hence, for infinitely many n it would hold that $C(\Omega(0)\Omega(1)\ldots\Omega(2n)) \leq n + c$ for some constant c in contradiction to the fact that Ω is Martin-Löf random. $\square$

Note that in a similar spirit, one can show the following property: If A is a set which is Martin-Löf random relative to Ω, then Ω is also Martin-Löf random relative to A and therefore the convergence module c_Ω of Ω is a total function which pdominates the set of all partial A-recursive functions. Thus, if one relativises the notion, one can have two oracles, namely A and Ω, such that a total Ω-recursive function dominates all partial A-recursive functions though $A \not\leq_T \Omega$.

For further properties of Ω, see the usual textbooks on Kolmogorov complexity and recursion theory [5, 12, 18, 19, 22, 23]. In the following sections, it will be investigated which types of sets can be pdominant and which not. The first results will deal with Π^0_1 classes which are a convenient tool to construct sets of various types; for the construction of pdominant sets, they turn out to be of limited value as no Π^0_1 class consists entirely of pdominant sets and some do not contain any pdominant set.

2 Π^0_1-classes and pdominance

For the purposes of this paper, Π^0_1-classes are classes of sets represented by their $\{0, 1\}$-valued characteristic functions and they are not empty. Furthermore, a Π^0_1-class is called nonrecursive iff it contains only nonrecursive sets. Π^0_1-classes correspond to the sets of infinite branches of recursive trees.

Theorem 4. *Every Π^0_1-class contains a set which is not pdominant.*

Proof. Let a Π_1^0-class be given as the infinite branches of a recursive tree T_0. Now one defines inductively a sequence of trees $T_1, T_2, \ldots$ such that the following holds for all trees:

- $T_0 \supseteq T_1 \supseteq T_2 \supseteq \ldots$;
- Each e is linked to two numbers a_e, b_e such that T_e is computed by the function φ_{a_e} and b_e is the minimum possible size of the set $\{0, 1, \ldots, 2^{\langle a_e, e \rangle - 2}\} \cap dom(\varphi_e^B)$ when B ranges over the infinite branches of T_e;
- T_{e+1} is a recursive tree having exactly those infinite branches of T_e for which the intersection $\{0, 1, \ldots, 2^{\langle a_e, e \rangle - 2}\} \cap dom(\varphi_e^B)$ contains at most b_e elements.

Note that the intersection of a descending sequence of binary trees with infinite branches is not empty, as each tree defines a compact subset of the set of possible infinite branches and the intersection of a descending intersection of nonempty compact sets is not empty in the Cantor Space. Let A be an infinite branch which is on all trees T_e.

Now consider the following function ϕ: The algorithm for $\phi(x)$ searches first for numbers a, e, b with $x = 2^{\langle a, e \rangle} + b$ and $b < 2^{\langle a, e \rangle}$. If these numbers can be found, then the algorithm goes on with determining for each set B the first number t found such that either $B(0)B(1)\ldots B(t)$ is not on the tree given by φ_a or there are b distinct numbers $u_1, u_2, \ldots, u_b \leq 2^{\langle a, e \rangle - 2}$ satisfying for each $y \in \{u_1, u_2, \ldots, u_b\}$ that $\varphi_{e,t}^{B(0)B(1)\ldots B(t)}(y)$ is defined and below t. In the case that these searches terminate for all oracles B, for compactness reasons, the algorithm can find a uniform proper upper bound t' on all these t and it then outputs this t'.

Assuming now by a way of contradiction that ψ is the partial function witnessing that A is pdominant. Then there are infinitely many e such that $\varphi_e^A = \psi$. For each such e and the corresponding a_e, b_e it holds that (a) φ_e^A is defined on b_e places below $2^{\langle a_e, e \rangle - 2}$ for all infinite branches of T_e and (b) $\psi(2^{\langle a_e, e \rangle} + b_e)$ returns a value t' which is larger than $\psi(y)$ for all y in the domain of ψ satisfying $y \leq 2^{\langle a_e, e \rangle - 2}$; these y contain every $y \in dom(\psi)$ with $y \leq (2^{\langle a_e, e \rangle} + b_e)/8$. Hence the function ψ does not witness in sense of Proposition 2 that A is pdominant. As the choice of ψ was arbitrary, it follows that A is not pdominant. $\square$

Theorem 5. *There is a nonrecursive Π_1^0-class not containing any pdominant set.*

Proof. The idea is to construct a partial recursive $\{0, 1\}$-valued function ϑ and let the Π_1^0-class be the class of all total extensions. At each stage s, let $a_{m,s}$ be the m-th non-element of the domain of ϑ_s. Now one updates ϑ according to that of the following actions which has highest priority, that is, the lowest parameter m being involved.

- If $\varphi_{m,s}$ and ϑ_s are consistent, that is, not different in the common domain, and if $\varphi_{m,s}(a_{m,s})$ is defined, then make $\vartheta_{s+1}(a_{m,s})$ be defined and different from $\varphi_{m,s}(a_{m,s})$.
- If there is a string $\tau \in \{0, 1\}^s$ consistent with $\vartheta_{m,s}$ and the value

$$u = \langle m, a_{0,s}, \tau(a_{0,s}), a_{1,s}, \tau(a_{1,s}), \ldots, a_{m,s}, \tau(a_{m,s}) \rangle$$

does not satisfy that $\varphi_{e,|\tau|}^{\tau}(y) \downarrow \Rightarrow \varphi_{e,a_{m+1,s}}^{\tau}(y) \downarrow$ for all $e, y \leq 2^{u-2}$ then let $\vartheta_{s+1}(z) = \tau(z)$ for $z = a_{m,s} + 1, a_{m,s} + 2, \ldots, s - 1$.

Here it is assumed that at any pairing of finitely many coordinates the formed pair is larger than every single coordinate. Note that the requirements are set up in a way that it happens only at finitely many stages that $\vartheta_{s+1}(a_{m,s})$ becomes defined, hence for each m the limit a_m of the $a_{m,s}$ exists and is outside the domain of ϑ. By the first of the two requirement for m, every member of the resulting Π^0_1-class will be different from φ_m whenever the latter is total. So the resulting Π^0_1-class is nonrecursive.

Now assume by way of contradiction that A is in the resulting Π^0_1-class and that $\psi = \varphi_e^A$ witnesses that A is pdominant. Then there is, for each $m > e$, a value u of the form

$$u = \langle m, a_0, A(a_0), a_1, A(a_1), \ldots, a_m, A(a_m)\rangle.$$

Let b be the number of places y such that $\varphi_e^A(y)$ is defined below 2^{u-2}. Now one can define a partial recursive function ϕ such that whenever $x = 2^u + b$ for u, b as above then $\psi(x)$ decodes $m, a_0, a_1, \ldots, a_m, A(a_0), A(a_1), \ldots, A(a_m)$ from u and waits until an s is found such that $a_{k,s} = a_k$ for $k \leq m$ and there are b values $y \leq 2^{u-2}$ for which $\varphi^A_{e,a_{m+1,s}}(y)$ is defined and is below s. Then $\phi(x)$ takes the value $s+1$.

One can easily verify that whenever the parameters e, b are taken as above, all the involved computations halt and the corresponding s is found and the value $s+1$ returned is strictly larger than $\psi(y)$ for all $y \leq x/8$ which are in the domain of ψ. Hence, by Proposition 2, A cannot be pdominant in contradiction to the assumption. In other words, no member A of the constructed Π^0_1-class is pdominant. $\Box$

Remark 6. One can construct a Π^0_1-class with one nonrecursive branch of the same Turing degree as K and some recursive branches. Hence there is a nontrivial Π^0_1-class where all nonrecursive members are pdominant.

3 Pdominant sets and genericity

An extension function F is a partial function which on input σ is either undefined or maps σ to a string extending σ. A set A meets an extension function F iff there is an n such that either $F(A(0)A(1)\ldots A(n)\sigma)$ is undefined for all σ or $F(A(0)A(1)\ldots A(n)) = A(0)A(1)\ldots A(m)$ for some $m \geq n$.

A set A is called 1-*generic* iff it meets every partial recursive extension function; a set A is called *weakly* 2-*generic* iff it meets every total K-recursive extension function.

Note that every weakly 2-generic set meets each total K-recursive extension function infinitely often in the sense that for every such F there are infinitely many n with $F(A(0)A(1)\ldots A(n))$ being a prefix of A.

1-generic sets are used implicitly in Friedberg's Jump Inversion theorem [6]. Jockusch [9, 10] formally introduced 1-generic sets and its variants.

Theorem 7. *No weakly* 2*-generic set is pdominant.*

Proof. Let A be a weakly 2-generic set. Assume by way of contradiction that A is pdominant and that the partial A-recursive function ψ^A with oracle A witnesses this fact.

Define a partial recursive extension function γ such that $\gamma(\sigma, u)$ is the length-lexicographically first extension τ of σ such that there is an $x \leq u$ for which $\psi^\tau_{|\tau|}(x)$ is defined while $\psi^\sigma_{|\sigma|}(x)$ is undefined. $\Gamma(\sigma, u)$ is obtained by iterating γ as long as possible: $\Gamma(\sigma, u) = \sigma_n$ for the largest n where σ_n is defined where inductively $\sigma_0 = \sigma$ and $\sigma_{m+1} = \gamma(\sigma_m, e, u)$ whenever that is defined. Note that γ is partial recursive and that σ_{u+2} cannot exist as for each m where σ_{m+1} is defined there is at least one $x_m \leq u$ with $\psi^{\sigma_k}_{|\tau|}(x_m)$ being defined iff σ_k exists and $k \geq m+1$. Hence Γ is a total K-recursive function.

Now one considers the extension function which maps σ to $\Gamma(\sigma, 2^{no(\sigma)})$ where no is a recursive bijection from strings to natural numbers.

There are infinitely many prefixes σ_0 of A such that $\Gamma(\sigma_0, 2^{no(\sigma_0)-2})$ is a prefix of A as well. For any such prefix σ_0, let $c = no(\sigma_0) - 2$ and let n be the maximal number such that σ_n is defined where $\sigma_{m+1} = \gamma(\sigma_m, 2^{c-2})$ whenever σ_m is defined and $\gamma(\sigma_m, 2^{c-1})$ is defined.

Let $\phi(2^c+n)$ be the partial recursive function which computes c, n, σ_0 from 2^c+n, computes σ_n from σ_0 through n times iterating γ with the corresponding inputs and then outputs

$$1 + \max\{\psi^{\sigma_n}_{|\sigma_n|}(x) : x \in \mathrm{dom}(\psi^{\sigma_n}_{|\sigma_n|}) \wedge x \leq 2^{c-2}\}.$$

One can now easily see that $\phi(2^c+n) > \psi(y)$ for all $y \leq 2^{c-2}$ in the domain of ψ in the case that $\sigma_0, \Gamma(\sigma_0, 2^{no(\sigma_0)-2})$ are both prefixes of A and n is the number of extensions such that $\sigma_n = \Gamma(\sigma_0)$. So, for the constant $k = 8$ and the infinitely many x of the form $2^c + n$ in the domain of ϕ with n, c derived from the corresponding σ_0 there is no $y \leq x/8$ in the domain of ψ with $\psi(y) > \phi(x)$. Hence A is not pdominant. □

Theorem 8. *There is a pdominant* 1*-generic set.*

Proof. The idea is to construct the 1-generic set A as a sequence $\sigma_0\tau_0\sigma_1\tau_1\ldots$ such that the σ_n is chosen such that the n-th 1-genericity requirement is satisfied and τ_n contains information about the halting problem K which is sufficient to permit partial domination. More precisely, σ_n is the first string found such that $\sigma_0\tau_0\sigma_1\tau_1\ldots\sigma_{n-1}\tau_{n-1}\sigma_n$ is enumerated into the n-th r.e. set of strings; if no such string is found then one takes σ_n to be the empty string. These enumeration procedures are fixed from now on, so that the coding and decoding always refers to the same way of enumerating the n-th r.e. set. The string τ_n encodes whether $\varphi_e(x)$ halts for all $e, x < 2^{n+3}$: So τ_n has length 4^{n+3} and for all $e, x < 2^{n+3}$, if $\varphi_e(x)$ halts then $\tau_n(2^{n+3}e+x) = 1$ else $\tau_n(2^{n+3}e+x) = 0$.

Now one can construct the following function ψ^A using the algorithm below which is undefined whenever some of the computations on the way are not terminating:

- On input y, determine $n, a_0, \ldots, a_n$ such that $y = 2^{n+1} + \sum_{m \leq n} 2^m a_m$.
- For $m = 0, 1, \ldots, n$: if $a_m = 1$ then search for the first σ_m found such that the string $\sigma_0\tau_0\sigma_1\tau_1\ldots\sigma_{m-1}\tau_{m-1}\sigma_m$ is enumerated into the n-th r.e. set else let σ_m be the empty string; let τ_m be that string of length 4^{m+5} which makes $\sigma_0\tau_0\sigma_1\tau_1\ldots\sigma_m\tau_m$ to be a prefix of A. If either the search does not terminate or the prefix-condition cannot be satisfied then $\psi^A(y)$ is undefined.

– Having τ_n, one simulates all computations $\varphi_e(x)$ with $e, x < 2^{n+3}$ and $\tau_n(2^{n+3}e+x) = 1$ until they terminate with a value z. If all these simulations terminate then $\psi^A(y)$ is $z+1$ for the largest z returned in these simulations else $\psi^A(y)$ is undefined.

If now $\varphi_e(x)$ is defined, $n > e$ and $2^{n+2} \leq x < 2^{n+3}$ then one can choose $a_0, a_1, \ldots, a_n$ such that $a_m = 1$ whenever the search for σ_m finds this string and $a_m = 0$ otherwise. Now considering $y = 2^{n+1} + \sum_{m \leq n} 2^m \cdot a_m$, $\psi^A(y)$ is defined, $\psi^A(y) > \varphi_e(x)$ and $y < x$. Hence A is pdominant. $\square$

Note that the 1-generic set constructed is actually below K. Theorem 9 below shows that there are also 1-generic sets below K which are not pdominant; this holds as there are 1-generic sets which are low for Martin-Löf random.

4 Pdominance and lowness

As mentioned in the introduction, a pdominant set A can be low, that is, satisfy $A' \leq_T K$.

Furthermore, it can even be superlow, as one can construct the sequence of the σ_n and τ_n relative to Theorem 8 such that the σ_n is not taken to satisfy a 1-genericity requirement but to fix the n-th bit in the jump of A. Indeed, one can for each n ask how often approximations to the sequence of the $\sigma_0\tau_0\sigma_1\tau_1\ldots\sigma_n\tau_n$ changes where each change is either caused by some bit in some τ_m going from 0 to 1 or by finding a σ_m; the latter can be undone later if some σ_k is found with $k < m$ or some bit is enumerated into a τ_k with $k < m$. Nevertheless, one can compute from n an upper bound on the number of changes and a truth-table query to K permits then to get the amount of actual changes. Hence A' is truth-table reducible to K.

So the next step in making pdominant sets to be even lower than superlow would be to make them low for Martin-Löf random. But that goal can no longer be achieved as the following result shows.

Theorem 9. *Let A be a pdominant set. Then A is not low for Martin-Löf randomness.*

Proof. Recall that H denotes the prefix-free Kolmogorov complexity [1]. First, it is well-known that one can choose the universal machine U such that there is a recursive approximation H_t of H from above satisfying for infinitely many x that $H(x) = H_0(x)$. In other words, the first approximation H_0 is a Solovay function for H [7, Proof of Proposition 2.5].

Second, for each x let $s(x, c)$ denote the number of strings y which are of length x such that $H(y) < x + H_0(x) - c$. Fix the value c such that there are infinitely many x with $H(x) = H_0(x)$ and $s(x, c) < 2^x$: This value exists, since there is a constant c' with

$$\sum_{y \in \{0,1\}^x} 2^{-H(y)} > 2^{-c'-H(x)} \text{ and } \forall y \in \{0,1\}^x\, [H(y) < H(x) + c'].$$

The assumption $H(y) < x + H_0(x) - c'$ for all y of length x implies that

$$\sum_{y \in \{0,1\}^x} 2^{-H(y)} \leq \sum_{y \in \{0,1\}^x} 2^{-x-H_0(x)-c'} \leq 2^{-c'-H_0(x)}$$

and thus $H(x) < H_0(x)$. It follows that any $c > c'$ is a reasonable choice to satisfy the condition, so let $c = c' + 2$. For this c, there is furthermore a minimal constant d such that $s(x,c) \geq 2^{x-d}$ for infinitely many x satisfying $H(x) = H_0(x)$ and $s(x,c) < 2^x$: As mentioned, all strings y of length x satisfy $H(y) \leq H(x) + c'$. Thus there is a constant c'' such that for all d', $2^{x-2d'}$ of these strings y satisfy $H(y) \leq H_0(x) + x + c'' - d'$. From this fact, one can conclude the existence of some d with the desired properties and then take the minimal one among the possible values. Let

$$B = \{x : H(x) = H_0(x) \wedge 2^{x-d} \leq s(x,c) < 2^{x-(d-1)}\}$$

be the infinite set of these x.

Third, by Proposition 2, ψ^A is a partial A-recursive function such that for all φ_e, all k and almost all $a \in dom(\varphi_e)$ there is a $b \in dom(\psi^A)$ with $b \leq a \cdot 2^{-2k-1} \wedge \psi^A(b) > \varphi_e(a)$.

Fourth, now let φ_e be the function which on input a computes first the integer x with $2^{x-d} \leq a < 2^{x-(d-1)}$ and then searches for the first t such that there are a distinct strings $y \in \{0,1\}^x$ with $H_t(y) < x + H_0(x) - c$. Note that for all $x \in B$ the value $\varphi_e(s(x,c))$ is defined and returns the time t which is needed until all strings $y \in \{0,1\}^x$ with $H(y) < x + H_0(x) - c$ indeed satisfy $H_t(y) < x + H_0(x) - c$. For each number k and almost all $x \in B$ there is a $b \leq a \cdot 2^{-2k-1}$ with $\psi^A(b) > \varphi_e(s(x,c))$; note that $b \leq a \cdot 2^{-2k-1}$ implies $b < 2^{x-2k-1}$.

Fifth, consider any x and let p be any program for the universal machine on which H is based with output x. Now one makes a prefix-free machine M^A which takes inputs of the form $p \cdot 0^k 1 \cdot z$ where p is in the domain of the prefix-free universal machine and the length of z is $x - 2k - 1$ provided that the prefix-free universal machine outputs x on input p. On inputs of these form, M^A finds p and determines x, then M^A determines the binary value b of z and then M^A runs the computation $t = \psi^A(b)$. If all these computations halt then M^A outputs the lexicographic first string $y \in \{0,1\}^x$ with $H_t(y) \geq x + |p| - c$ (provided that this exists).

Sixth, in the case that $x \in B$ and p is a shortest program for x in the universal machine and $\psi^A(b) > \varphi_e(s(x,c))$ for some $b < 2^{x-2k-1}$ and z coding b in $x - 2k - 1$ binary bits, $M^A(p \cdot 0^k 1 \cdot z)$ returns a string y with $H(y) \geq x + H(x) - c$. Furthermore, the input for M^A to generate y is of length $|p| + x - k$ where $|p| = H(x)$. It follows that there is for each k a string y such that the partial A-recursive machine M^A generates y from a string which is $k - c$ bits shorter than its unrelativised Kolmogorov complexity. Hence A is not low for prefix-free Kolmogorov complexity and thus A is also not low for Martin-Löf randomness. □

The last result showed that every set which is low for Martin-Löf random cannot be pdominant. As there are sets of nonrecursive r.e. degree, sets of minimal Turing degree and 1-generic sets which are all low for Martin-Löf random, one gets by this result alternative proofs to the preceding ones that sets of those types might not be pdominant.

One generalisation of sets which are low for Martin-Löf random are those which are low for Ω. Therefore the following question is natural.

Open Problem 10. *Is there a pdominant set which is low for Ω?*

Furthermore, one might also like to know whether a further lowness notion is compatible with pdominance.

Open Problem 11. *Is there a pdominant set of hyperimmune-free Turing degree?*

5 R.e. sets and pdominance

In this section it is shown that pdominant sets are orthogonal to the low-high-hierarchy. That is, there is a low r.e. set which is pdominant and a high r.e. set which is not pdominant.

Theorem 12. *There is a low r.e. set which is pdominant.*

Proof. The idea would be to construct in the limit a set A such that its characteristic function consists of a sequence $\sigma_0\tau_0\sigma_1\tau_1\ldots$ where each σ_n can become longer over time and each τ_n has the length 2^{n+3}. As in Theorem 8, $\tau_n(2^{n+3}e + x) = 1$ if $\varphi_e(x)$ halts and $\tau_n(2^{n+3}e + x) = 0$ if $\varphi_e(x)$ does not halt for all $e, x < 2^{n+3}$. Furthermore, the σ_n can be made longer and takes then the current values of A_s at the corresponding positions, that is, increasing the length of σ_n does never change the values of A_s. Whenever an interval σ_n becomes longer than it is made to extend up to s and the strings $\tau_n, \sigma_{n+1}, \ldots$ keep their length but have all bits reset to 0, as A_s does not contain elements beyond s. The algorithm is now the following:

- Identify the first string σ_n or τ_n in the characteristic function which needs attention.
- Here σ_n occupying an interval from i to j with $j < s$ needs attention when $\varphi_n^{A_s}(n)$ converges but needs more than j and less than s steps. If σ_n receives attention then σ_n is adjusted to cover the interval from i to s and $A_{s+1} = A_s$, that is, σ_n takes the value $A_s(i)A_s(i{+}1)\ldots A_s(s)$. Each of the strings $\tau_n, \sigma_{n+1}, \tau_{n+1}, \sigma_{n+2}, \tau_{n+2}, \ldots$ keeps it length but has its values set to 0, so if τ_{n+1} was $b_1b_2\ldots b_k$ for some k then its new value is 0^k.
- Here τ_n occupying an interval from i to j with $j < s$ needs attention when $\tau_n(2^{n+3}e+x) = 0$ and $\varphi_e(x)$ converges within s computational steps. If τ_n receives attention then $i + 2^{n+3}e + x$ is put into A_{s+1} and thus $\tau_n(2^{n+3}e + x)$ becomes 1. Furthermore, if σ_{n+1} does not cover s then it is enlarged and takes the value $A_s(j + 1)A_s(j + 2)\ldots A_s(s)$ and each of the strings $\tau_{n+1}, \sigma_{n+2}, \tau_{n+2}, \sigma_{n+3}, \tau_{n+3}, \ldots$ keeps it length but has its values reset to 0, so if τ_{n+1} was $b_1b_2\ldots b_k$ for some k then its new value is 0^k.

Note that when $\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_{n-1}, \tau_{n-1}$ have found their final values then σ_n will receive attention at most once in order to change $A'(n)$ from 0 to 1; afterwards τ_n might receive attention up to 4^{n+3} times as each of its bits can go from 0 to 1.

Note that $A'(n)$ can only change from 1 to 0 when some string of priority higher than σ_n receives attention or when the right end of σ_n is beyond s. This happens only finitely often and so A is low.

Now define $\psi^A(y)$ as follows. Determine $n, a_0, a_1, \ldots, a_n$ such that $y = 2^{n+1} + \sum_{m \le n} 2^m \cdot a_m$ (if these values exist). Now find the first s such that, inductively, the following holds for $m = 0, 1, \ldots, n$: σ_m has received exactly a_m times attention after the last time that some interval before σ_m has received attention; the string $\sigma_0\tau_0\sigma_1\tau_1\ldots\sigma_n\tau_n$ is a prefix of A. If such a stage s is found, then simulate all computations $\varphi_e(x)$ with $e, x < 2^{n+3}$ until they deliver an output z and return the least proper upper bound of these values z.

Now consider any φ_e and any n, x with $e < n$ and $2^{n+2} \leq x < 2^{n+3}$. Furthermore, choose $a_0, a_1, \ldots, a_n$ such that $a_m = 1$ iff in the construction σ_m has received attention after all the higher priority strings $\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_{n-1}, \tau_{n-1}$ have received attention for the last time. Then, in the above definition, $\psi^A(y)$ with $y = 2^{n+1} + \sum_{m \leq n} 2^m \cdot a_m$ is defined and is a proper upper bound of $\varphi_e(x)$ whenever the latter is defined. This witnesses that A is pdominant. $\square$

Remark 13. Actually one can show in Theorem 12 that A is even superlow. This is done by observing that a string σ_n receives attention at most once in any interval of time where no higher priority string receives attention. Furthermore, τ_n receives attention at most 4^{n+3} times during any interval of time where no higher priority string receives attention. This permits to conclude that the strings $\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_n, \tau_n$ altogether receive at most $2^{n+1} \cdot 5^3 \cdot 5^4 \cdot 5^5 \cdot \ldots \cdot 5^{n+3}$ times attention. One can therefore find out with a tt-reduction to K how often these intervals receive attention and then simulate the construction until that attention was granted for the last time and then one knows whether $A'(n)$ holds or not. So $A' \leq_{wtt} K$ and indeed $A' \leq_{tt} K$ as wtt-reductions to K can be turned into tt-reductions. It follows that A is superlow.

Theorem 14. *There is a high r.e. set which is not pdominant.*

Proof. Let $\langle \phi_e \rangle_{e \in \omega}$ be an effective listing of all partial recursive functions, and let $\langle \Psi_e \rangle_{e \in \omega}$ be an effective listing of all Turing functionals. We let ψ_e denote the use functional of Ψ_e. We make the assumption that for any oracle X, if $\Psi_e^X(y)$ converges at stage s, then both y and $\psi_e^X(y)$ are less than s.

We build an r.e. set A, an A-recursive function Γ^A, and partial recursive functions $\langle f_e^0 \rangle_{e \in \omega}$ and $\langle f_e^1 \rangle_{e \in \omega}$ meeting the following requirements:

G**:** Γ^A is total.
P_e**:** If ϕ_e is total, Γ^A dominates ϕ_e.
R_e**:** Ψ_e^A fails to pdominate at least one of f_e^0 and f_e^1.

G with the P_e's guarantees that A is high, while the R_e's ensure that A is not pdominant. The R_e-requirement will not be met directly, but will instead be met through subrequirements:

$R_{e,n}$**:** There is an $i < 2$ and an $x > n$ with $x \in \mathrm{dom} f_e^i$ such that for every $y \leq x$ with $y \in \mathrm{dom} \Psi_e^A$, $\Psi_e^A(y) < f_e^i(x)$.

Clearly if infinitely many of these subrequirements are met, the R_e-requirement will be met.

Our strategy for meeting an $R_{e,n}$-requirement will involve guessing which of the $\phi_{e'}$ are total, which we accomplish via a tree of strategies. We define the tree by devoting all nodes at level $2e$ to the P_e-requirement, and all nodes at level $2\langle e, n \rangle + 1$ to the $R_{e,n}$-requirement. The global G-requirement does not appear on the tree.

Nodes devoted to a P_e-requirement have two possible outcomes, `inf` and `fin` (denoting whether or not ϕ_e is total). All other nodes have only one possible outcome, which we will simply call `outcome`. Thus we can define the tree of strategies recursively: the empty string $\langle\rangle$ is the root of the tree; if σ is on the tree at level $2e$, then σ`inf` and σ`fin` are its two children; if σ is on the tree at level $2\langle e, n \rangle + 1$, then σ`outcome` is its unique child.

If τ_0 and τ_1 are nodes on the tree, we say that τ_0 is to the left of τ_1 if there is a node σ on the tree with $\sigma\texttt{inf} \subseteq \tau_0$ and $\sigma\texttt{fin} \subseteq \tau_1$. It is simple to verify that this is a transitive relation.

For a node σ, accessible at stage s and of length $\leq s$, we describe what actions σ takes at stage s, and if $|\sigma| < s$, which outcome of σ is next accessible. We also describe the action of the G-strategy.

Strategy for meeting the G-requirement. We partition ω in some effective fashion as $\omega = \{\langle e, x\rangle \mid e < x\}$ with $\langle e, x\rangle < \langle e, x'\rangle$ whenever $x < x'$. At stage s, for every $x < s$, if $\Gamma^A(x)\,[s]$ is not defined, we define

$$\Gamma^A(x)\,[s] = \max\{\phi_{e,s}(x) \mid e < x \ \wedge\ \phi_{e,s}(x)\downarrow\}$$

with use $\gamma^A(x)\,[s] = \max\{\langle e, x\rangle \mid e < x\} + 1$.

Strategy for meeting the P_e-requirement. Whenever this strategy is initialised, we choose a threshold m. It suffices to let m be the stage at which the strategy was initialised. Let $\ell_s(e)$ be largest such that $\phi_{e,s}\restriction_{\ell_s(e)}$ converges.

At stage s, consider all $x < \ell_s(e)$. If $\langle e, x\rangle > m$, we enumerate $\langle e, x\rangle$ into A. After considering all such x, we pass to the next accessible node.

Let $t < s$ be the last stage at which $\sigma\texttt{inf}$ was accessible ($t = 0$ if there is no such stage). If $\ell_t(e) < \ell_s(e)$, we let $\sigma\texttt{inf}$ be accessible and initialise all strategies $\tau \supseteq \sigma\texttt{fin}$. Otherwise, we let $\sigma\texttt{fin}$ be accessible.

Strategy for meeting the $R_{e,n}$-requirement. When σ is initialised, it will claim an interval of the form $[k, 3k]$ on which it will define f^0_e and f^1_e. We must have $k > n$, and the interval must be disjoint from any interval previously claimed by any $R_{e',n'}$-requirement. We can take this as our definition of k: k is the least number greater than n with $[k, 3k]$ disjoint from every interval previously claimed by any $R_{e',n'}$-requirement. Since only finitely many intervals can have been claimed by any stage, there will always be such a k. At a stage s when σ is accessible, let

$$\begin{aligned}
b &= \min\{\langle e', \ell_s(e')\rangle \mid \tau\texttt{inf} \subseteq \sigma \text{ for some } P_{e'}\text{-strategy } \tau\};\\
p &= \max\{\Psi^A_e(y)\,[s] \mid y \in \texttt{dom}\Psi^A_e\,[s] \ \wedge\ y \leq 3k \ \wedge\ \psi^A_e(y)\,[s] < b\};\\
q &= \max\{f^i_{e,s}(x) \mid i < 2 \ \wedge\ k \leq x \leq 3k \ \wedge\ x \in \texttt{dom}f^i_{e,s}\}.
\end{aligned}$$

If $p < q$, we pass to the next accessible node.

Otherwise, let $i < 2$ and $x \in [k, 3k]$ be such that $f^i_{e,s}(x)\uparrow$ (we will later argue that such i and x exist). We define $f^i_{e,s}(x) = p + 1$, initialise all $\tau \supset \sigma$ and pass to the next accessible node.

Since σ has only one possible outcome, this unique child is the next accessible node.

Construction. We begin stage s by letting $\langle\rangle$ be accessible and acting according to its described strategy. This strategy determines one of its children to be the next accessible node, which then acts according to its strategy. We continue in this fashion, letting each accessible node choose one of its children to be the next accessible node until we reach a node of length s. Once we have acted according to the strategy for the node of length s, we let the G-strategy act and then proceed to stage $s + 1$.

Verification. We define the true path of the construction recursively. The empty string $\langle\rangle$ is on the true path. If σ is on the true path and σ is devoted to some $R_{e,n}$-requirement, then $\sigma\texttt{outcome}$ is on the true path. If σ is on the true path and σ is devoted to some P_e-requirement, and $\sigma\,\text{inf}$ is accessible at infinitely many stages, then $\sigma\,\text{inf}$ is on the true path; otherwise, $\sigma\texttt{fin}$ is on the true path. Note that an equivalent definition would be that the true path is formed by choosing from every level of the tree the leftmost node which is accessible at infinitely many stages.

We perform the verification as a sequence of claims.

Claim 15. *If τ_0 and τ_1 are nodes on the tree of strategies with τ_0 to the left of τ_1, and τ_0 is accessible at stage s, then τ_1 is initialised at stage s*

Proof. Fix σ with $\sigma\texttt{inf} \subseteq \tau_0$ and $\sigma\texttt{fin} \subseteq \tau_1$. Then σ is a P_e-strategy that was accessible at stage s and chose $\sigma\texttt{inf}$ to be the next accessible node. By construction, all nodes $\tau \supseteq \sigma\texttt{fin}$ are initialised at this stage, including τ_1.

Claim 16. *Γ^A is total.*

Proof. At every stage s, we define $\Gamma^A(x)\,[s]$ for all $x < s$. Since A is r.e., and the use $\gamma^A(x)\,[s]$ never increases, $\Gamma^A(x)$ is defined for all x.

Claim 17. *If σ is an $R_{e,n}$-requirement, σ has outcome $\texttt{inf}$ at stage s, σ is not initialised between stages $s < t$, and $y \leq 3k$ is such that $\Psi_e^A(y)\,[s]$ was included in the maximum from which p was defined at stage s, then $y \in dom\ \Psi_e^A\,[t]$ and $\Psi_e^A(y)\,[t] = \Psi_e^A(y)\,[s]$.*

Proof. The only way the conclusion might fail is if some $P_{e'}$-strategy were to enumerate an element into A below $\psi_e^A(y)\,[s]$ between stages s and t. We consider the possibilities.

Since σ was not initialised, no $P_{e'}$-strategy to the left of σ was accessible to do this. Similarly, no $P_{e'}$-strategy τ with $\tau\texttt{fin} \subseteq \sigma$ could enumerate a new element into A without $\ell_s(e')$ increasing, causing $\tau\texttt{inf}$ to be accessible and initialising σ.

Any $P_{e'}$-strategy to the right of σ or extending σ would have been initialised at stage s and so its next threshold would be no less than s, which by our earlier assumption is greater than $\psi_e^A(y)\,[s]$. So such a strategy cannot enumerate an element below $\psi_e^A(y)\,[s]$.

Any $P_{e'}$-strategy τ with $\tau\texttt{inf} \subseteq \sigma$ has $\langle e', \ell_s(e')\rangle > \psi_e^A(y)\,[s]$, and thus has already enumerated by stage s all the elements below $\psi_e^A(y)\,[s]$ that it ever will.

Claim 18. *Let σ be an $R_{e,n}$-requirement. Suppose σ was initialised at stage s_0, choosing an interval $[k, 3k]$, and σ was not initialised between stages $s_0 < t$. Then there is an $i < 2$ and an $x \in [k, 3k]$ such that $f^i_{e,t}(x)\uparrow$.*

Proof. Since $[k, 3k]$ was chosen disjoint from every previously claimed interval, and the functions f^i_e are only defined within a claimed interval, if s is a stage with some $f^i_e(x)$ defined with $x \in [k, 3k]$, then $s_0 \leq s$. By Claim 17, at every such stage $s \leq t$, the domain of $\Psi_e^A\,[s]$ grows by at least one element which is no more than $3k$ and which is still in dom $\Psi_e^A\,[t]$. Thus this can happen at most $3k+1$ many times. Since at most one $f^i_e(x)$ definition is made by σ at each such stage s, and there are $4k+2$ many pairs (i, x) with $i < 2$ and $x \in [k, 3k]$, there is always a pair with $f^i_{e,t}(x)\uparrow$.

Claim 19. *If σ is a strategy along the true path, σ is only initialised finitely many times.*

Proof. By induction on $|\sigma|$. The case when $\sigma = \langle\rangle$ is trivial.

A strategy σ can only be initialised by the actions of strategies $\tau \subset \sigma$.

If τ is a P_e-requirement, and $\tau\mathtt{inf} \subseteq \sigma$, τ cannot initialise σ.

If τ is a P_e-requirement, and $\tau\mathtt{fin} \subseteq \sigma$, then $\tau\mathtt{fin}$ is on the true path. By definition of the true path, there is some stage after which τ never again has outcome $\mathtt{inf}$. After this stage, τ can never again initialise σ.

If τ is an $R_{e,n}$-requirement, let s_0 be the last stage at which τ is ever initialised. By Claim 17, at every stage $s > s_0$ at which τ initialises σ, the domain of $\Psi_e^A[s]$ grows by at least one element which is no more than $3k$ and which never subsequently leaves. Thus this can happen at most $3k+1$ many times.

Claim 20. *If ϕ_e is total, Γ^A dominates ϕ_e.*

Proof. Let σ be the P_e-strategy along the true path. Let s_0 be the final stage at which σ is initialised, and let m be the threshold chosen at this stage. For an x with $\langle e, x\rangle > m$, let s be the least stage such that $\ell_s(e) > x$. By construction, $\langle e, x\rangle$ will not enter A before stage s.

At the first stage $t \geq s$ when σ is accessible, σ will enumerate $\langle e, x\rangle$ into A if no other P_e-strategy has already done so. When this happens, the G-strategy will redefine $\Gamma^A(x) \geq \phi_e(x)$. By construction, all future redefinitions of $\Gamma^A(x)$ will be at least $\phi_e(x)$. Thus $\Gamma^A(x) \geq \phi_e(x)$ for all x with $\langle e, x\rangle > m$.

Claim 21. *For every $n \in \omega$, there is an $i < 2$ and an $x > n$ with $x \in \mathtt{dom} f^i$ such that for every $y \leq x$ with $y \in dom\ \Psi_e^A$, $\Psi_e^A(y) < f_e^i(x)$.*

Proof. Let σ be the $R_{e,n}$-strategy along the true path. Let s_0 be the final stage at which s is initialised, and let $[k, 3k]$ be the interval chosen at stage s_0. Recall that we chose $k > n$. Let $i < 2$ and $x \in [k, 3k]$ be the final pair for which σ defines $f_e^i(x)$. We claim these are the desired values.

Suppose there were a $y \leq x$ with $y \in \mathrm{dom}\ \Psi_e^A$ and $\Psi_e^A(y) \geq f_e^i(x)$. Let $s_1 > s_0$ be a stage such that $y \in \mathrm{dom}\ \Psi_{e,s_1}^A$ and $A_{s_1}\restriction_{\phi_e^A(y)} = A\restriction_{\phi_e^A(y)}$. Let $s_2 > s_1$ be a stage such that $\langle e', \ell_s(e')\rangle > \phi_e^A(y)$ for every $\tau\,\mathrm{inf} \subseteq \sigma$ a $P_{e'}$-strategy. Let $s_3 > s_2$ be a stage at which σ is accessible.

At stage s_3, $p \geq \Psi_e^A(y) \geq f_e^i(x) = q$. By construction, a new pair $i' < 2$ and $x' \in [k, 3k]$ will be chosen and $f_e^{i'}(x')$ defined, contradicting our choice of i and x.

This completes the proof. $\square$

6 The opposite of dominance

Hyperimmune-free degrees are in a certain way the counterpart of high degrees: Every high set A computes a function f which grows faster than every recursive function [13, 14]; every hyperimmune-free set A only computes functions f for which there is a recursive function growing faster than them (this recursive function depends on f). One can say that besides being low also having hyperimmune-free Turing degree is something like the opposite of being high. Then, every total A-recursive function is majorised by a recursive one. One could ask whether the same could be possible for the type of sets investigated in the present work.

Definition 22. Call a set A to be phif (hyperimmune-free with respect to partial functions) iff for every partial A-recursive function ψ there is a partial recursive function ϕ and a constant k such that for every $x \in dom(\psi)$ there is a $y \in dom(\phi)$ with $y \leq kx$ and $\phi(y) > \psi(x)$.

Theorem 23. *Every phif set A has hyperimmune-free Turing degree.*

Proof. It is shown that a set A of hyperimmune Turing degree cannot be phif.

To show this, the following fact is used: There is an A-recursive function F such that for every infinite r.e. set W_e and every n there are infinitely many $m \in W_e$ with $F(m) = n$.

One can define such a function F inductively by starting a monotonically increasing function $G \leq_T A$ not majorised by any recursive function and then letting $F(m) = n$ for the least triple $\langle e, n, k\rangle$ such that either $\langle e, n, k\rangle = m$ or $m \in W_{e,G(m)}$ and there are less than k numbers $o < m$ with $F(o) = n \wedge o \in W_{e,G(m)}$. It is clear that F is a total A-recursive function. Furthermore, a requirement given by a triple $\langle e, n, k\rangle$ is satisfied when there are at least k numbers $o \in W_e$ with $F(o) = n$. One can see that whenever W_e is infinite then each requirement $\langle e, n, k\rangle$ becomes eventually satisfied as there are infinitely many m with $m \in W_{e,G(m)}$ where $\langle e, n, k\rangle$ requires attention unless the requirement is satisfied; here the m-th requirement can only be denied attention when a higher order requirement, that is the 0-th, 1-st, ... or $m-1$-th requirement is done and will be satisfied by defining F at one place accordingly. This happens of course only finitely often and then the m-th requirement will be satisfied eventually. So F has the requested properties.

Given F, define a partial A-recursive function ψ as follows. On input x with $2^m \leq x < 2^{m+1}$ and $F(m) = \langle i, j, k\rangle$, $\psi(x)$ simulates the computations of $\varphi_i(y)$ for inputs $y \leq 2^{m+1} \cdot k$ until at least $h = j \cdot 2^m + (x - 2^m)$ of these computations have converged with outputs $z_0, z_1, \ldots, z_{h-1}$. Then $\psi^A(x)$ takes the value $1 + z_0 + z_1 + \ldots + z_{h-1}$.

Now assume by way of contradiction that there is a partial recursive function φ_i and a k such that for almost all $x \in dom(\psi)$ there is a $y \leq kx$ in the domain of φ_i with $\varphi_i(y) > \psi(x)$. Let j be the lim sup for $m \to \infty$ of the numbers j_m where each j_m is the rounded-down value of $2^{-m} \cdot |dom(\varphi_i) \cap \{0, 1, \ldots, 2^{m+1} \cdot k\}|$. As $j_m \in \{0, 1, 2, \ldots, 2k+1\}$ for each m, the lim sup of the j_m must exist and is a number j. The set $E = \{m : j_m = j\}$ is r.e. and infinite. Hence there are infinitely many $m \in E$ with $F(m) = \langle i, j, k\rangle$. For each of these m there is an x such that $2^m \leq x < 2^{m+1}$ and $j \cdot 2^m + (x - 2^m) = |dom(\varphi_i) \cap \{0, 1, \ldots, 2^{m+1} \cdot k\}|$. Now $\psi(x)$ is defined and $\psi(x) > \varphi_i(y)$ for every $y \in dom(\varphi_i)$ with $y \leq 2^{m+1} \cdot k$. This produces the desired contradiction and therefore i, k cannot destroy the witness ψ against A being phif. Hence, every phif set is hif, that is, has hyperimmune-free Turing degree. □

The next result shows that phif sets are even recursive. Kummer's Cardinality Theorem [8, 11] implies the following result, which confirmed a conjecture by Owings [20].

Theorem 24. *If a set A of natural numbers, a natural number k and a function g with k inputs satisfy*

$$\forall x_1, x_2, \ldots, x_k \; [x_1 < x_2 < \ldots < x_k \Rightarrow g(x_1, \ldots, x_k) \in \{0, 1, \ldots, k\} - \{|A \cap \{x_1, \ldots, x_k\}|\}]$$

then $A \leq_T g$.

The next result uses this theorem now to prove that A is recursive; the proof first shows that A is K-recursive and then invokes the above result that A has hyperimmune-free Turing degree in order to show that A is recursive.

Theorem 25. *If A is phif, then A must be recursive.*

Proof. Suppose that A is phif. Let $\psi^A(2^e + 2^{e+1}x) = \phi_e^A(x)$. Let p and ℓ be such that for every $x \in \mathtt{dom}(\psi)$, there is a $y \in \mathtt{dom}(p)$ with $y \leq \ell x$ and $p(y) > \psi(x)$.

We will construct a partial A-recursive function $h = \phi_e^A$, and by the usual recursion theorem argument we may assume we already know e. Let $k = 2^{e+2}\ell$ and let f be an A-recursive function with $f(m) = |A \cap \{x_1, \ldots, x_k\}|$ for the m-th tuple $(x_1, \ldots, x_k)$ with respect to a recursive enumeration of all k-tuples. For all sufficiently large $x \in \mathtt{dom}(h)$, there is a $y \in \mathtt{dom}(p)$ with $y \leq kx$ and $p(y) > h(x)$.

Construct a pair of sequences $\langle b_n \rangle_{n\in\omega}$ and $\langle c_n \rangle_{n\in\omega}$ as follows:

- $b_0 = 1$;
- $b_{n+1} = (k+1)b_n + 1$;
- $c_0 = k$;
- $c_{n+1} = (k+1)(b_{n+1} - b_n) - 1$.

The important properties of these sequences are the following:

$$c_{n+1} < (b_{n+1} - b_n)(k+1) \text{ and } c_{n+1} = kb_{n+1}.$$

Let

$$g(n) = |\{m : m \in \mathtt{dom}(p) \wedge m < c_{n+1}\}| \% (k+1).$$

That is, $g(n)$ is the residue when the number of elements in the domain of p less than c_{n+1} is reduced mod $(k+1)$. Note that g is K-recursive.

We construct h as follows: for $(b_n + i) \in [b_n, b_{n+1})$, wait until $(k+1)i + f(n)$ many elements less than c_{n+1} have entered the domain of p. Then define $h(b_n + i)$ greater than the largest value of p seen so far.

For any n with $f(n) = g(n)$, let i be such that

$$(k+1)i + f(n) = |\{m : m \in \mathtt{dom}(p) \wedge m < c_{n+1}\}|.$$

Note that $i \leq \frac{c_{n+1}}{k+1} < b_{n+1} - b_n$, so $b_n \leq b_n + i < b_{n+1}$. Then by construction, $h(b_n + i)$ is larger than $p(y)$ for all $y < c_{n+1}$. So $h(b_n + i)$ is larger than $p(y)$ for all $y < (b_n + i)k$. By our choice of p and k, this can only happen finitely many times. So $f(n) = g(n)$ for at most finitely many n. By Theorem 24, $A \leq_T g$ and so $A \leq_T K$. As A has also hyperimmune-free Turing degree, it follows that A is recursive. $\square$

Remark 26. One might also look at the other opposite of highness, that is, the analogue of lowness. The corresponding question would be: When is K pdominant relative to A? More precisely, for which A does it hold that there is a partial K-recursive function ψ such that for all partial A-recursive functions ϕ and almost all $x \in dom(\phi)$ there is a $y \in dom(\psi)$ with $y \leq x \wedge \psi(y) > \phi(x)$. Indeed, there are uncountably many such A as one could take all sets which are low for Ω, that is, all sets A relative to which Ω is Martin-Löf random. Such sets then satisfy that there is no partial A-recursive function ϕ satisfying that $\phi(x) > c_\Omega(x)$ for infinitely many $x \in dom(\phi)$ where c_Ω is the convergence module of Ω; otherwise Ω would not be Martin-Löf random relative to A. Hence the K-recursive function $\psi(x) = c_\Omega(2x)$ witnesses the corresponding lowness property.

7 Weak truth-table reducibility

Recall that a partial A-recursive function ψ is wtt-reducible to A iff there is an index e and a recursive function f such that $\psi(x) = \varphi_e^{\{y \in A: y < f(x)\}}(x)$ for all x. Now a function is *wttpdominant* iff there is a partial function $\psi \leq_{wtt} A$ such that for every partial recursive ϕ and almost all $x \in dom(\phi)$ there is an $y \in dom(\psi)$ with $y \leq x \wedge \psi(y) > \phi(x)$. Arslanov showed that an r.e. set is wtt-complete iff it wtt-computes a fixed-point free function. The next result shows that wttpdominance is a further criterion for wtt-completeness of r.e. sets, which stands in contrast to the situation at Turing reducibility.

Theorem 27. *An r.e. set is wttpdominant iff it is wtt-complete.*

Proof. Only the direction "wttpdominant ⇒ wtt-complete" needs to be shown. Let $\psi \leq_{wtt} A$ be given and let f be the use of the wtt-reduction. Now define that $\phi(x)$ with $2^m \leq x < 2^{m+1}$ takes the value $\psi^{A_s}(x - 2^m)$ iff s is the time which m needs to be enumerated into K; $\phi(x)$ is undefined when $m \notin K$. By Proposition 2 there is for almost all m and all $x \in dom(\phi)$ with $2^m \leq x < 2^{m+1}$ a y in the domain of ψ such that $y < 2^m$ and $\psi(y) > \phi(x)$. Given now any m, let $y < 2^m$ be chosen such that $\psi(y)$ is the largest value among these y and $x = 2^m + y$. It follows from above property that either $\phi(x)$ is undefined and $m \notin K$ or $\phi(x) < \psi(y)$ and therefore A_s below $f(2^m)$ different from A below $f(2^m)$. Now define the A-recursive function g with $g(m)$ being the first time s where A_s equals A below $f(2^m)$, one has for almost all m that $m \in K_{g(m)}$ iff $m \in K$. This shows that A is wtt-complete. □

Furthermore, one can show that no hyperimmune set and thus also no 1-generic set is wttp-dominant. On the other hand, Example 3 actually shows that the half A of Ω is wttpdominant. Hence there is a wtt-incomplete wttpdominant set $A \leq_{wtt} K$. Note that the negative results on pdominance transfer to wttpdominance.

Acknowledgments. The authors want to thank Noam Greenberg for discussions and correspondence on this subject; furthermore they are greatful to the anonymous referees for detailed and comprehensive comments.

References

1. Cristian S. Calude. *Information and Randomness: An Algorithmic Perspective*, Second Edition, Springer, Heidelberg, 2002.
2. Gregory J. Chaitin. A theory of program size formally identical to information theory. *Journal of the ACM*, 22:329–340, 1975.
3. Gregory J. Chaitin. Incompleteness theorems for random reals. *Advances in Applied Mathematics*, 8(2):119–146, 1987.
4. C. T. Chong, Theodore A. Slaman and Yue Yang. The metamathematics of stable Ramsey's theorem for pairs. *Journal of the American Mathematical Society*, 27(3):863-892, 2014.
5. Rod Downey and Denis Hirschfeldt. *Algorithmic randomness and complexity.* Springer, New York, 2010.
6. Richard Friedberg. A criterion for completeness of degrees of unsolvability. *The Journal of Symbolic Logic*, 22:159–160, 1957.
7. Santiago Figueira, Frank Stephan and Guohua Wu. Randomness and universal machines. *Journal of Complexity*, 22:738–751, 2006.
8. Valentina S. Harizanov, Martin Kummer and James C. Owings. Frequency Computations and the Cardinality Theorem. *The Journal of Symbolic Logic* 57(2):682–687, 1992.
9. Carl G. Jockusch. Simple proofs of some theorems on high degrees of unsolvability. *Canadian Journal of Mathematics*, 29:1072–1080, 1977.
10. Carl G. Jockusch. Degrees of generic sets. *London Mathematical Society Lecture Notes*, 45:110–139, 1981.
11. Martin Kummer. A Proof of Beigel's Cardinality Conjecture. *The Journal of Symbolic Logic* 57(2):677–681, 1992.
12. Ming Li and Paul Vitányi. *An introduction to Kolmogorov complexity and its applications.* Graduate Texts in Computer Science. Third edition. Springer, New York, 2008.
13. Donald Martin. Completeness, the recursion theorem and effectively simple sets. *Proceedings or the American Mathematical Society*, 17:838–842, 1966.
14. Donald Martin. Classes of recursively enumerable sets and degrees of unsolvability. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 12:295–310, 1966.
15. Per Martin-Löf, The definition of random sequences. *Information and Control*, 9:602–619, 1966.
16. Webb Miller and Donald A. Martin. The degrees of hyperimmune sets. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 14:159–166, 1968.
17. Frank Stephan and Liang Yu. A reducibility related to being hyperimmune-free. *Annals of Pure and Applied Logic*, 165(7–8):1291–1300, 2014.
18. André Nies. *Computability and randomness.* Oxford University Press, New York, 2009.
19. Piergiorgio Odifreddi. *Classical recursion theory. Studies in Logic and the Foundations of Mathematics, vol. 125.* North-Holland, Amsterdam, 1989.
20. James C. Owings. A Cardinality Version of Biegel's Nonspeedup Theorem. *The Journal of Symbolic Logic* 54(3):761–767, 1989.

21. Emil Post. Recursively enumerable sets of positive integers and their decision problems. *Bulletin of the American Mathematical Society*, 50, 284–316, 1944.
22. Hartley Rogers, Jr. *Theory of recursive functions and effective computability.* MIT Press, Cambridge, 1987.
23. Robert I. Soare. *Recursively enumerable sets and degrees.* Perspectives in Mathematical Logic. Springer, Berlin, 1987.
24. Alexander K. Zvonkin and Leonid A. Levin. The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms. *Russian Mathematical Surveys*, 25(6):83–124, 1970.