Learnability and Positive Equivalence Relations*

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Abstract. Prior work of Gavryushkin, Khoussainov, Jain and Stephan investigated what algebraic structures can be realised in worlds given by a positive (= recursively enumerable) equivalence relation which partitions the natural numbers into infinitely many equivalence classes. The present work investigates the infinite one-one numbered recursively enumerable (r.e.) families realised by such relations and asks how the choice of the equivalence relation impacts the learnability properties of these classes when studying learnability in the limit from positive examples, also known as learning from text. For all choices of such positive equivalence relations, for each of the following entries, there are one-one numbered r.e. families which satisfy it: (a) they are behaviourally correctly learnable but not vacillatorily learnable; (b) they are explanatorily learnable but not confidently learnable; (c) they are not behaviourally correctly learnable. Furthermore, there is a positive equivalence relation which enforces that (d) every vacillatorily learnable one-one numbered family of languages closed under this equivalence relation is already explanatorily learnable and cannot be confidently learnable. In addition, if there is for a positive equivalence relation a confidently learnable class, then there is also a class which is confidently but not finitely learnable.

1 Introduction

Consider a learning scenario where all positive examples of a given target concept L belonging to a concept class $\mathcal{L}$ are shown sequentially to a learner M. After processing each example, M

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makes a conjecture as to the identity of the target concept, based on some fixed representation system of all concepts in $\mathcal{L}$. M is said to successfully identify L if its sequence of conjectures converges to a correct hypothesis describing L. This learning paradigm, due to Gold [21], is well-studied and has inspired the development of a large number of other learning models in inductive inference [22,37].

In this work, we study how the interrelations between the elements of a domain X influences the learnability of classes of languages defined over X. The domain of interest throughout this work is $\mathbb{N}$. We will be concerned with *recursively enumerable (r.e.)* equivalence relations defined on $\mathbb{N}$ that induce *infinitely* many equivalence classes. The main motivation for focussing on such relations comes from the study of r.e. structures. Here, r.e. structures are given by a domain, recursive functions representing basic operators in the structure, and some recursively enumerable predicates, among which there is a recursively enumerable equivalence relation η with infinitely many equivalence classes which plays the role of equality in the given structure. Such structures have been studied for a long time; for example, Novikov [34] constructed a finitely generated group with undecidable word-problem; in other words, there is a group which can be represented using an r.e. but nonrecursive equivalence relation (as equality of the group) but one cannot represent it using a recursive equivalence relation E. On the other hand, for Noetherian rings [33], Baur [5] showed that every r.e. Noetherian ring is a recursive ring, implying that the underlying equality η is always a recursive relation and that its equivalence classes are uniformly recursive. Another example of an r.e. equivalence relation is the relation of *provable equivalence* with respect to any formal system, say Peano Arithmetic (PA), where

$x \sim_{\mathrm{PA}} y$ holds iff $\alpha \leftrightarrow \beta$ is provable in PA.

Here x and y are the Gödel numbers of α and β, respectively, according to some fixed Gödel numbering.

Fokina, Gavryushkin, Jain, Khoussainov, Semukhin, Stephan and Turetsky [16,19,20] focussed in a sequence of papers on the question of which type of structures could be realised by an r.e. equivalence relation on $\mathbb{N}$ with infinitely many equivalence classes and how different relations compare with respect to their ability to realise structures of certain type. Ershov [13,14], and following him Odifreddi [35], call r.e. equivalence relations *positive* equivalence relations. Fokina, Kötzing and San Mauro [17] studied Gold-style learnability of equivalence structures (with no computability restrictions on the learner), and gave a structural characterisation of families of equivalence structures with no infinite equivalence classes that are learnable in the limit from informant. In follow-up work by Bazhenov, Fokina and San Mauro [6], a model-theoretic characterisation of families of general equivalence structures that are learnable in the limit from informant was obtained.

Now a structure is realised by a positive equivalence relation η iff there is a bijection between the elements in the domain of the structure and the equivalence classes of η and all relations involved are r.e. and all functions are realised by recursive functions which respect η. In the simplest case of functions from the domain to the domain, they respect η if they map η-equivalent numbers to η-equivalent numbers. This work focusses on the study of learnability within the framework of families realised by positive equivalence relations. In particular the topic of the investigations is to what extent separations between learning criteria known from inductive inference can be witnessed by η-closed sets, that is, which of the positive equivalence relations witness a separation of two learning criteria or collapse them. Furthermore one asks, whether certain learning criteria can be void (non-existent) for certain equivalence relations η. It could be shown that one single equivalence relation ϑ witnesses the two above mentioned collapses. The study of learning in a world given by some η is similar to that in complexity relative to an oracle; one wants to know how robust the results from the non-relativised world are and how much they generalise.

As mentioned earlier, a major topic investigated in this work is the relationship between various learning criteria for a fixed but arbitrary positive equivalence relation η on $\mathbb{N}$. In particular, we study the question of whether the strict hierarchy of learnability notions

$$\mathrm{Fin} \subset \mathrm{Conf} \subset \mathrm{Ex} \subset \mathrm{Vac} \subset \mathrm{BC}$$

(see Definition 2), which holds with respect to the general class of uniformly r.e. families (see [22]), carries over to the class of η-families for any given positive equivalence relation η. Here for any learning criteria A and B, we write $A \subset B$ to mean that the class of uniformly r.e. families that are learnable under criterion A is a strict subset of the class of uniformly r.e. families that are learnable under criterion B. The above inclusions follow directly from the definitions of the respective learning criteria, so the main interesting question is whether the inclusions are strict. It will be shown that many learnability relations that hold when η is just the usual equality relation on $\mathbb{N}$ carry over for general η. In particular, the fact that the family of explanatorily learnable classes of languages is a strict subset of the family of behaviourally correctly learnable classes of languages, which is well-known when η is the equality relation [12, Theorem 3.1], holds for *any* positive equivalence relation η (see Theorem 11). On the other hand, we show that for a special positive equivalence relation ϑ, every vacillatorily learnable ϑ-family is already explanatorily learnable (see Theorem 20); this is quite an interesting contrast to the case when η is the equality relation on $\mathbb{N}$, for which vacillatory learning is known to be strictly more powerful than explanatory learning [8, Theorem 3.3]. Furthermore, we extend the "non-union property" of explanatory and behaviourally correct learning (that is, the family of explanatorily (resp. behaviourally correctly) learnable classes is not closed under union) [21] by showing that for any positive equivalence relation η, explanatory learning is not closed under union with respect to η-families; we also obtain a characterisation of all positive equivalence relations η for which behaviourally correct learning is closed under union with respect to η-families (see Theorem 16). The current work thus serves a dual purpose: first, casting inductive inference results in a more general framework, where the elements of the domain over which concepts are defined can be interrelated via any positive equivalence relation; second, modelling the learning of r.e. structures endowed with any r.e. equality relation.

2 Preliminaries

Any unexplained recursion-theoretic notation may be found in [35,38,40]. The set of natural numbers is denoted as $\mathbb{N} = \{0, 1, 2, \ldots\}$; the set of all *partial recursive functions* and of all *recursive functions* of one, and two arguments over $\mathbb{N}$ is denoted by $\mathcal{P}$, $\mathcal{P}^2$, $\mathcal{R}$ and $\mathcal{R}^2$ respectively. Any function $\psi \in \mathcal{P}^2$ is called a *numbering of partial-recursive functions* — this numbering may or may not include all partial recursive functions. Moreover, let $\psi \in \mathcal{P}^2$; the notion ψ_e stands for $\lambda x.\psi(e,x)$ and $\mathcal{P}_\psi = \{\psi_e \mid e \in \mathbb{N}\}$. A numbering $\varphi \in \mathcal{P}^2$ is said to be an *acceptable numbering* or *Gödel numbering* of all partial recursive functions if $\mathcal{P}_\varphi = \mathcal{P}$ and for every numbering $\psi \in \mathcal{P}^2$, there is a $c \in \mathcal{R}$ such that $\psi_e = \varphi_{c(e)}$ for all $e \in \mathbb{N}$ (see [38]). Throughout this paper, $\varphi_0, \varphi_1, \varphi_2, \ldots$ is a fixed acceptable numbering of all partial recursive functions and $W_0, W_1, W_2, \ldots$ is a fixed *numbering of all recursively enumerable sets* (abbr. r.e. sets) of natural numbers, where W_e is the domain of φ_e for all $e \in \mathbb{N}$.

Let $e, x \in \mathbb{N}$; if $\varphi_e(x)$ is defined then we say that $\varphi_e(x)$ *converges*. Otherwise, $\varphi_e(x)$ is said to *diverge*. Furthermore, if the computation of $\varphi_e(x)$ halts within s steps of computation then we write $\varphi_{e,s}(x)\downarrow = \varphi_e(x)$; otherwise $\varphi_{e,s}(x)$ diverges. For all $e, s \in \mathbb{N}$ the set $W_{e,s}$ is defined as the domain of $\varphi_{e,s}$. Given any set S, S^* denotes the set of all finite sequences of elements from S. By $D_0, D_1, D_2, \ldots$ we denote any fixed canonical indexing of all finite sets of natural numbers. *Turing reducibility* is denoted by $\leqslant_T$; $A \leqslant_T B$, where $A, B \subseteq \mathbb{N}$, holds if A can be computed via a Turing machine which uses B as an oracle; that is, it can give information on whether or not x belongs to B. $A \equiv_T B$ means that $A \leqslant_T B$ and $B \leqslant_T A$ both hold. The symbol K denotes the *diagonal halting problem*, i.e., $K = \{e : e \in \mathbb{N},\ \varphi_e(e)\downarrow\}$. Moreover, the *jump* of K, denoted by K', is the relativised halting problem $\{e : e \in \mathbb{N},\ \varphi_e^K(e)\downarrow\}$, where $\varphi_0^K, \varphi_1^K, \varphi_2^K, \ldots$ is an acceptable numbering of all partial-K-recursive functions, which are partial functions depending on the input as well as on answers to queries of the form "is x in K?".

For $\sigma \in (\mathbb{N} \cup \{\#\})^*$ and $n \in \mathbb{N}$ we write $\sigma(n)$ to denote the element in the nth position of σ. For any finite sequence σ we use $|\sigma|$ to denote the length of σ. Further, whenever $n \leqslant |\sigma|$, $\sigma[n]$ denotes the sequence $\sigma(0), \sigma(1), \ldots, \sigma(n-1)$. The concatenation of two sequences σ and τ is denoted by $\sigma \circ \tau$; for convenience, and whenever there is no possibility of confusion, this is occasionally denoted by $\sigma\tau$.

A class $\mathcal{L}$ is said to be *uniformly r.e.* (or just *r.e.*) if there is an r.e. set $S \subseteq \mathbb{N}$ such that $\mathcal{L} = \{W_i : i \in S\}$. A class is said to be *one-one r.e.*, if the r.e. set S as above additionally satisfies the condition that for $i, j \in S$, $W_i = W_j$ iff $i = j$. An r.e. class $\mathcal{L} = \{B_0, B_1, \ldots\}$ is said to be *uniformly recursive* or an *indexed family* if there exists a recursive function $f \in \mathcal{R}^2$ such that for all $i, x \in \mathbb{N}$, if $x \in B_i$ then $f(i,x) = 1$ else $f(i,x) = 0$.

3 Learnability

Recursion-theoretic notation mainly follows the textbooks of Odifreddi [35], Rogers [38] and Soare [40]. Background on inductive inference may be found in [22]. Let $\mathcal{L}$ be a class of r.e. languages. Throughout this paper, the mode of data presentation is that of a *text*. A text is any infinite sequence of natural numbers and the # symbol, where the symbol # indicates a pause in the data presentation. More formally, a *text* T_L for a language $L \in \mathcal{L}$ is any total mapping $T_L : \mathbb{N} \to \mathbb{N} \cup \{\#\}$ such that $L = \mathrm{range}(T_L) - \{\#\}$. We use $\mathrm{content}(T)$ to denote the set $\mathrm{range}(T) - \{\#\}$, i.e., the content of a text T contains only the natural numbers appearing in T. Furthermore, for every $n \in \mathbb{N}$ we use $T[n]$ to denote the finite sequence $T(0), \ldots, T(n-1)$, i.e., the *initial segment* of length n of T. Analogously, for a finite sequence $\sigma \in (\mathbb{N} \cup \{\#\})^*$ we use content(σ) to denote the set of all numbers in the range of σ.

Description 1. Further basic ingredients of the notions considered in this paper are as follows.

(1) For each positive equivalence relation η, one can define an infinite sequence $a_0, a_1, \ldots$ of least representatives of the equivalence classes where each a_n is the ascending limit of approximations $a_{n,t}$ where $a_{n,t}$ is the least natural number which is not η_t-equivalent to any $a_{m,t}$ with $m < n$ (where η_t denotes the approximation to η after t steps of enumeration and is closed under reflexivity and transitivity). Alternatively, one can obtain η from a construction of such a sequence where the $a_{n,t}$ approximate the a_n from below and whenever an $a_{n,t+1} \neq a_{n,t}$ then $a_{n,t+1} = a_{m,t}$ for some $m > n$ and whenever $a_{n,t}$ is not in the list at $t+1$ then it is put into the equivalence class of some $a_{m,t}$ with $m < n$. Some algorithms to construct the equivalence relation η explain on how to update these approximations to $a_0, a_1, \ldots$ and one should note that (the construction implies) the limit satisfies $a_0 < a_1 < \ldots$ and that for each n there are only finitely many t with $a_{n,t} < a_{n,t+1}$.
(2) The classes whose learnability are considered are given by a uniformly r.e. one-one numbering of sets $B_0, B_1, \ldots$ where each set B_k is a union of η-equivalence classes; however, the indices k of B_k are usual natural numbers and not equivalence-classes of η. Such a family is called an *η-family* below and note that η-families are always infinite.
(3) Infinite indexed families as considered by Angluin [1] are too restrictive, as they might not exist for some η; however, every infinite indexed family has a one-one numbering and thus using the notion of infinite uniformly r.e. one-one numberings is the adequate choice for the present work.
(4) The learner sees an infinite sequence $x_0, x_1, \ldots$ of members of one set B_k (such sequences are called texts and can have pauses represented by a special pause symbol #) and the learner has to find in the limit an r.e. index for B_k, which may or may not be equal to k.
(5) The hypotheses issued by the learners are always indices from a hypothesis space given as a numbering of η-closed r.e. sets; the most general hypothesis space is that of the η-closures of the members of a given acceptable numbering of all r.e. sets.
(6) The present work focusses on the following learning criteria [3,8,11,15,21,22,36,37]: *Explanatory learning*, where the hypotheses of the learner converge on every text for a set B_k to a single index of B_k; *Confident learning*, which is explanatory learning with the additional requirement that the learner also on texts not belonging to any language in the class has to converge to some index; *Behaviourally correct learning*, which is more general than explanatory learning and where the learner is only required to output on any text for B_k almost always an index for B_k but these indices can all be different; *Vacillatory learning*, where a learner is vacillatory iff it is a behaviourally correct learner for the class, with the additional constraint that on every text for a language B_k in the class, the set of all hypotheses issued in response to this text is finite (thus, some of these hypotheses are output infinitely often).

We now provide formal definitions of these criteria as well as the criterion of *finite* learning (sometimes known as *one-shot learning* in the literature; see [21,42]), which is a more restrictive version of explanatory learning. In the following definitions, a learner M is a recursive function mapping $(\mathbb{N} \cup \{\#\})^*$ into $\mathbb{N} \cup \{?\}$; the ? symbol allows M to abstain from conjecturing at a stage. If M is presented with a text T for any η-closed set L, it is enough to assume that content(T) contains *at least one* element of each η-equivalence class contained in L; since η is r.e., M on T could simulate a complete text for L by enumerating at each stage s the s-th approximation of the current input's η-closure.

The main learning criteria studied in this paper are *explanatory learning* (also called *learning in the limit*) introduced by Gold [21] and *behaviourally correct learning*, which goes back to Feldman [15], who called it *matching in the limit*; later, it was also studied by Bārzdiņš, Case, Lynes and Smith [3,4,11,12]. A fairly natural learning constraint on these criteria is *conservativeness* [1], which requires that any syntactic (resp. semantic) mind change by the explanatory (resp. behaviourally correct) learner occur only if the learner's original conjecture does not account for all the data revealed in the text so far.

Furthermore, we will also consider *finite learning* (see [21,42]), which is sometimes known as *one-shot learning* in the literature. Other learning criteria studied in this paper include *confident learning* (see [22,37]), which requires the learner to converge syntactically on every text for any language (even if it is outside the class to be learnt), and *vacillatory learning* (see [8]), according to which a learner is permitted to vacillate between finitely many correct indices in the limit.

Definition 2 (Angluin [1], Bārzdiņš [3], Case and Smith [12], Feldman [15], Gold [21], Osherson, Stob and Weinstein [37], Trakhtenbrot and Bārzdiņš [42]). Let $\mathcal{L}$ be any class of r.e. languages.

(1) *M explanatorily (Ex) learns* $\mathcal{L}$ if, for every L in $\mathcal{L}$ and each text T_L for L, there is a number n for which $L = W_{M(T_L[n])}$ and, for every $j \geqslant n$, $M(T_L[j]) = M(T_L[n])$.
(2) *M behaviourally correctly (BC) learns* $\mathcal{L}$ if, for every L in $\mathcal{L}$ and each text T_L for L, there is a number n for which $L = W_{M(T_L[j])}$ whenever $j \geqslant n$.
(3) *M finitely (Fin) learns* $\mathcal{L}$ if, for every L in $\mathcal{L}$ and each text T_L for L, there is a number n for which $L = W_{M(T_L[n])}$ and for every $m < n$, $M(T_L[m]) = ?$ and for every $j \geqslant n$, $M(T_L[j]) = M(T_L[n])$.
(4) *M confidently (Conf) learns* $\mathcal{L}$ if M Ex learns $\mathcal{L}$ and M converges to some number or to '?' on every text for any language, that is, for every $L \subseteq \mathbb{N}$ and text T_L for L, there is some $n \in \mathbb{N}$ such that for every $j \geqslant n$, $M(T_L[j]) = M(T_L[n])$.
(5) *M vacillatorily (Vac) learns* $\mathcal{L}$ if M BC learns $\mathcal{L}$ and for every L in $\mathcal{L}$ and each text T_L for L, $\{M(T_L[n]) : n \geqslant 1\}$ is finite.
(6) *M conservatively explanatorily learns* $\mathcal{L}$ if M Ex learns $\mathcal{L}$ and for all $n \in \mathbb{N}$, $k \geqslant 1$, $M(T[n]) \neq M(T[n+k])$ only if content$(T[n+k]) \not\subseteq W_{M(T[n])}$.
(7) *M conservatively behaviourally correctly learns* $\mathcal{L}$ if M BC learns $\mathcal{L}$ and for all $n \in \mathbb{N}$, $k \geqslant 1$, $W_{M(T[n])} \neq W_{M(T[n+k])}$ only if content$(T\ [n+k]) \not\subseteq W_{M(T[n])}$.

We will also consider the learning constraint of *monotonicity* [25,30,43]. Furthermore, we give some results concerning the dependence of learnability on the choice of hypothesis space. When considering learnability based on the choice of hypothesis space $\{H_0, H_1, H_2, \ldots\}$, one replaces the hypotheses $W_0, W_1, \ldots$ in Definition 2 by $H_0, H_1, \ldots$. This work studies three types of requirements on the learner's hypothesis space: the hypothesis space can be *class-comprising*, *class-preserving* or *exact* [29].

A learner M is said to be *monotonic* [43] if, for all texts T for some language in the target class, whenever M on input $T[n]$ outputs j_n and then at some subsequent step outputs j_{n+k} on input $T[n+k]$, the condition $W_{j_n} \cap \text{content}(T) \subseteq W_{j_{n+k}} \cap \text{content}(T)$ holds.

A learner M is said to be *strongly monotonic* [25] if, for all texts T, whenever M on input $T[n]$ outputs j_n and then at some subsequent step outputs j_{n+k} on input $T[n+k]$, the condition $W_{j_n} \subseteq W_{j_{n+k}}$ holds.

A learner M is said to be *weakly monotonic* [30] if, for all texts T, whenever M on input $T[n]$ outputs j_n and then at some subsequent step outputs j_{n+k} on input $T[n+k]$, the condition $\text{content}(T[n+k]) \subseteq W_{j_n} \Rightarrow W_{j_n} \subseteq W_{j_{n+k}}$ holds.

A learner M is said to be *class-comprising* [29,31] if it learns the target class $\mathcal{L}$ with respect to at least one hypothesis space $\{H_0, H_1, \ldots\}$; note that learnability automatically implies $\mathcal{L} \subseteq \{H_0, H_1, \ldots\}$.

Unless otherwise stated, the learner's hypothesis space will be taken as the η-closure of $W_0, W_1, \ldots$; in other words, the learner is class-comprising. For the notions of class-preserving and exact learning, one considers some other recursively enumerable hypothesis space $H_0, H_1, \ldots$ for interpreting the hypothesis of the learner. Only uniformly recursively enumerable hypothesis spaces are considered.

A learner M is said to be *class-preserving* [29,31] if it learns the target class $\mathcal{L}$ with respect to at least one hypothesis space $\{H_0, H_1, \ldots\}$ such that $\{H_0, H_1, \ldots\} = \mathcal{L}$.

A learner M is said to be *exact* [29,31] if it learns the target class $\mathcal{L} := \{L_0, L_1, \ldots\}$ using $\{L_0, L_1, \ldots\}$ as its hypothesis space (in other words, for all $i \in \mathbb{N}$, the learner outputs i to signal that its conjecture is L_i).

Lange and Zeugmann [31] showed that for learning uniformly recursive families, learnability is to some extent independent of the choice of hypothesis space in that exact, class-preserving and class-comprising explanatory learnability all coincide. Jain, Stephan and Ye [24] observed more generally that for any finite (resp. explanatorily) learnable uniformly r.e. family $\mathcal{L}$ and hypothesis space $\mathcal{H} := \{H_0, H_1, \ldots\}$ such that $\mathcal{H}$ and $\mathcal{L}$ consist of the same set of members (possibly ordered differently), one can uniformly construct a finite (resp. explanatory) learner for $\mathcal{L}$ with respect to $\mathcal{H}$ from an r.e. index of $\{\langle d, x \rangle : x \in H_d\}$. Thus, as the present paper deals exclusively with uniformly r.e. families, it will be assumed throughout that finite (resp. explanatory) learning, class-preserving finite (resp. explanatory) learning and exact finite (resp. explanatory) learning are all equivalent (this equivalence may, however, fail if other learning constraints are imposed at the same time).

In some cases we consider learners using oracles. In this case the learning criterion I when the learners are allowed use of oracle A is denoted by $I[A]$.

For every learning criterion I considered in the present paper, there exists a recursive enumeration $M_0, M_1, \ldots$ of the learning machines such that if a class is I-learnable, then some M_i I-learns the class. We fix one such enumeration of learning machines. We will also consider combinations of various learning criteria; for example, one could require a learner to be both confident and behaviourally correct.

Throughout this work, we only consider positive equivalence relations that induce *infinitely* many equivalence classes. For any positive equivalence relation η and $x \in \mathbb{N}$, let $[x]$ be $\{y : y\,\eta\,x\}$. Furthermore, for any $D \subseteq \mathbb{N}$, $[D]$ denotes $\bigcup_{x \in D}[x]$. For any finite $\{i_0, \ldots, i_n\} \subseteq \mathbb{N}$, the set $[\{a_{i_0}, a_{i_1}, \ldots, a_{i_n}\}]$ will simply be denoted by $[a_{i_0}, a_{i_1}, \ldots, a_{i_n}]$. An *$\eta$-family* $\mathcal{L}$ is a uniformly r.e. one-one infinite family, each of whose members is a union of η-equivalence classes. Note that uniformly recursive infinite families might not exist for some η and therefore an η-family is the nearest notion to a uniformly recursive family which exists for each positive equivalence relation η. A set is *η-infinite* (resp. η-finite) if it is equal to a union of infinitely (resp. finitely) many η-equivalence classes; note that an η-infinite set may not necessarily be recursively enumerable. A set is *η-closed* if it is either η-finite or η-infinite. In this paper, all families are assumed to consist of only η-closed sets (for some given η). For brevity's sake, we do not use any notation to indicate the dependence of a_n on η; the choice of η will always be clear from the context. A family $\mathcal{A}$ of sets is called a *superfamily* of another family $\mathcal{B}$ of sets iff $\mathcal{A} \supseteq \mathcal{B}$.

A useful notion that captures the idea of the learner converging on a given text is that of a *locking sequence*, or more generally that of a *stabilising sequence*. A sequence $\sigma \in (\mathbb{N} \cup \{\#\})^*$ is called a *stabilising sequence* [18] for a learner M on some language L if $\mathrm{content}(\sigma) \subseteq L$ and for all $\tau \in (L \cup \{\#\})^*$, $M(\sigma) = M(\sigma \circ \tau)$. A sequence $\sigma \in (\mathbb{N} \cup \{\#\})^*$ is called a *locking sequence* [7] for a learner M on some language L if σ is a stabilising sequence for M on L and $W_{M(\sigma)} = L$. The following proposition due to Blum and Blum [7] will be occasionally useful.

Proposition 3 (Blum and Blum [7]). *If a learner M explanatorily learns some language L, then there exists a locking sequence for M on L. Furthermore, all stabilising sequences for M on L are also locking sequences for M on L.*

The following theorem due to Kummer [27] will be useful for showing that a given family of r.e. sets has a one-one numbering.

Theorem 4 (Kummer [27]). *Suppose $L_0, L_1, L_2, \ldots$ and H_0, $H_1, H_2, \ldots$ are two numberings such that the following conditions hold:*

(1) for all $i, j \in \mathbb{N}$, $L_i \neq H_j$;
(2) $H_0, H_1, H_2, \ldots$ is a one-one numbering;
(3) for all $i \in \mathbb{N}$ and all finite $D \subseteq L_i$, there are infinitely many j such that $D \subseteq H_j$.

Then $\{L_i : i \in \mathbb{N}\} \cup \{H_j : j \in \mathbb{N}\}$ has a one-one numbering.

4 Results for all Positive Equivalence Relations: Fin, Conf, Ex, Vac and BC Learning

In the present section, we investigate the relationship between the main learning criteria – namely, finite, confident, explanatory, vacillatory and behaviourally correct learning – with respect to families closed under any given positive equivalence relation. The first part of this section will study, for any general positive equivalence relation η, the learnability of a particular η-family known as the *ascending family for* η. As will be seen later, the ascending family provides a useful basis for constructing η-families that witness the separation of various learnability notions.

Definition 5. For all $n \in \mathbb{N}$, A_n denotes the set $[a_0, a_1, ..., a_{n-1}]$. The family $\{A_n : n \in \mathbb{N}\}$ will be denoted by $\mathcal{A}_\eta$, and is called the *ascending family for* η.

Note that each member of $\mathcal{A}_\eta$ is η-finite; furthermore, $\mathcal{A}_\eta$ is an η-family because η induces infinitely many equivalence classes and for all n, a_n can be approximated from below (c.f. Description 1, item (1)). $\mathcal{A}_\eta$ is analogous to the family INIT $:= \{\{y : y < x\} : x \in \mathbb{N}\}$ defined in [23, Section 4.3]. For brevity's sake, we do not use any notation to indicate the dependence of A_n on η; the choice of η will always be clear from the context.

In the second part of this section, we study the question of whether the learning hierarchy

$$\text{Fin} \subseteq \text{Conf} \subseteq \text{Ex} \subseteq \text{Vac} \subseteq \text{BC},$$

collapses at any level for the class of η-families (for any given positive equivalence relation η). It turns out that while the strict inclusion Fin $\subset$ Conf holds whenever there is at least one finitely learnable η-family, Conf $\subset$ Ex holds whenever there is at least one confidently learnable η-family and Vac $\subset$ BC holds for *all* positive equivalence relations, there is a positive equivalence relation ϑ for which every vacillatorily learnable ϑ-family is also explanatorily learnable. The construction of ϑ will be given in the next section. Moreover, there is some η for which no η-family is confidently learnable, and in this case $\emptyset = \text{Fin} = \text{Conf} \subset \text{Ex}$. We begin with a few basic examples of η-families to illustrate some of the notions introduced so far.

Example 6 (Ershov [13]). If A is a recursive and coinfinite set, then $x \, \eta_A \, y \Leftrightarrow (x = y \vee (x \in A \wedge y \in A))$ is a positive equivalence relation. $\mathcal{F} := \{A\} \cup \{\{x\} : x \notin A\}$ is an η_A-family since (1) every equivalence class of η_A is either A or a singleton $\{x\}$ with $x \notin A$, which implies that $\mathcal{F}$ is infinite and each member of $\mathcal{F}$ is η_A-closed, and (2) there is a uniformly recursive one-one numbering $\{F_i\}_{i \in \mathbb{N}}$ of $\mathcal{F}$; for example, one could set $F_0 = A$ and $F_{i+1} = \{x_i\}$ for all i, where $x_1, x_2, x_3, \ldots$ is a one-one recursive enumeration of $\mathbb{N} - A$. $\mathcal{F}$ is also finitely learnable via a learner that outputs ? until it sees the first number x in the input; if $x \in A$ then A is conjectured, and if $x \notin A$ then $\{x\}$ is conjectured.

Example 7 (Ershov [13]). If R is an r.e. set and $D_0, D_1, D_2, \ldots$ is a one-one numbering of all finite sets, then $x \, \eta_R \, y \Leftrightarrow D_x \triangle D_y \subseteq R$ is a positive equivalence relation ($\triangle$ denotes the symmetric difference). If $S \cap R = \emptyset$, then $L_S := \{x : D_x \cap S \neq \emptyset\}$ is η_R-closed. Suppose $\mathbb{N} - R$ contains an infinite r.e. set C. Let $\mathcal{F}$ consist of all sets $L_{C'}$ such that $C' = C - F$ for some finite set F. Then $\mathcal{F}$ is an η_R-family that is not behaviourally correctly learnable.

The next theorem shows that for any positive equivalence relation η, the ascending family witnesses that explanatory learning is strictly more powerful than confident learning.

Theorem 8. *For every positive equivalence relation η, the ascending family $\mathcal{A}_\eta$ is explanatorily learnable but not confidently learnable. One can add the set $\mathbb{N}$ to $\mathcal{A}_\eta$ and obtain an η-family which is not behaviourally correctly learnable.*

Proof. Note that all members of the ascending family are η-finite and for each member A_n, the learner can find in the limit n as follows: At each time t, the learner takes n_t – the t-th approximation to n – to be the least number such that $a_{n_t,t}$ has not yet appeared in the text and this n_t converges to n: From the time t onwards where all $a_{m,t}$ with $m \leqslant n$ have converged to their final value a_m and furthermore all a_m with $m < n$ have appeared in the text, it holds that $n_t = n$ and that therefore the conjecture is A_n which can be given by a default index for this set.

Now assume, by way of contradiction, that $\mathcal{A}_\eta$ has a confident learner N. By [22, Exercise 5–14], there is some $\sigma \in \mathbb{N}^*$ such that for all $\tau \in \mathbb{N}^*$, $N(\sigma\tau) = N(\sigma)$. Since $A_0 \subset A_1 \subset A_2 \subset \ldots$ and almost all of the A_i's contain content(σ), one can pick a least n such that $W_{N(\sigma)} \neq A_n$ and content$(\sigma) \subseteq A_n$. Then σ could be extended to a text T for A_n such that N on T converges to $N(\sigma)$ – an incorrect index for A_n, contradicting the fact that N confidently learns $\mathcal{A}_\eta$.

Furthermore, if one adds $\mathbb{N}$ to $\mathcal{A}_\eta$, the resulting family contains $\mathbb{N}$ and a strictly ascending sequence of sets whose union is $\mathbb{N}$, thus by Gold [21] the family is not learnable from positive data. This applies to all learning criteria considered in this paper. $\square$

The following proposition provides a method for establishing that a given uniformly r.e. family is an η-family.

Proposition 9. *Every uniformly r.e. superfamily of $\mathcal{A}_\eta$ that consists of η-closed sets is an η-family; in particular, the families of all η-finite sets and all η-closed r.e. sets are η-families.*

Proof. According to Kummer's theorem (cf. Theorem 4), a union of two disjoint uniformly r.e. families has a one-one numbering whenever every finite subset of any member of one family has infinitely many supersets in the other family and the latter family has a one-one numbering. So let $E_0, E_1, \ldots$ be the given r.e. family and let $A_0, A_1, \ldots$ be the default one-one enumeration of the ascending η-family. Now one chooses for the second family $\mathcal{F}_2$ the family of all sets A_{2k} and for the first family $\mathcal{F}_1$ all those sets E_k which are not in $\mathcal{F}_2$. Note that all sets A_{2k+1} are therefore in $\mathcal{F}_1$. For completing the proof, one has to show that $\mathcal{F}_1$ is a uniformly r.e. family.

Let $E_{k,t}$ denote the t-th approximation to E_k. For this one defines sets $U_{k,n,t}$ as follows:

If (i) $a_{n,t} \notin E_{k,t}$ and (ii) either n is odd and all $a_{m,t} \in E_{k,t}$ with $m < n$ or $a_{n+1,t} \in E_{k,t}$, then one enumerates into $U_{k,n,t}$ all elements of $E_{k,t}$ until either $a_{n,t}$ is enumerated into $E_{k,s}$ for some $s \geqslant t$ or there is an $m \leqslant n+1$ and $s \geqslant t$ with $a_{m,s} \neq a_{m,t}$.
If this happens, one lets h be the largest element enumerated into $U_{k,n,t}$ so far and one enumerates into $U_{k,n,t}$ all elements of A_{2h+1} which contains all elements enumerated into $U_{k,n,t}$ so far.

Note that A_{2h+1} mentioned at the end of this algorithm is not a member of the second numbering $\mathcal{F}_2$. It is easy to see that the numbering of all $U_{k,n,t}$ is a family of uniformly r.e. sets and that each of these sets is either equal to E_k in the case that the condition associated to (n,k) witnesses this (either n is odd and a_n is the least non-element of E_k or $a_n \notin E_k$ while $a_{n+1} \in E_k$) or the set is explicitly made to be an A_{2h+1} which is not contained in the second numbering $\mathcal{F}_2$ but in the numbering of the E_k. There is only one set which is not captured by this numbering but might be in the numbering of the E_k: This is the set $\mathbb{N}$. If this set is also in the numbering, then one has to add it afterwards explicitly to $\mathcal{F}_1$. In the case that one has a family which does not respect η, one can also for this achieve the result — although it is not the intention of the current work to consider such families — by adding the members of a further numbering of sets $\tilde{U}_{k,x,y,t}$ which are equal to E_k in the case that $x\,\eta_t\,y$ and $x \in E_{k,t}$ and $y \notin E_k$; in the case that this is not satisfied, the set $\tilde{U}_{k,x,y,t}$ is made equal to some set of the form A_{2h+1} when the violation of the condition is discovered.

As the class of all η-finite sets and the class of all η-closed r.e. sets are uniformly r.e. superfamilies of the ascending η-family, these classes are also η-families. $\square$

A minor modification of the proof of Proposition 9 reveals a slightly more general result: for any positive equivalence relation η and any strictly increasing recursive enumeration $e_0, e_1, e_2, \ldots$, every uniformly r.e. superfamily of $\{A_{e_i} : i \in \mathbb{N}\}$ is an η-family. This variant of Proposition 9 will be occasionally useful for showing that a given uniformly r.e. class is an η-family.

Proposition 10. *Let f be any strictly increasing recursive function. Then, for any given positive equivalence relation η, every uniformly r.e. superfamily of $\{A_{f(i)} : i \in \mathbb{N}\}$ consisting of η-closed sets is an η-family.*

The next result shows that for any given positive equivalence relation η, behaviourally correct learning is more powerful than explanatory learning with respect to the class of η-families.

Theorem 11. *For every positive equivalence relation η, there is an η-family which is behaviourally correctly learnable but not explanatorily learnable.*

Proof. The η-family of all η-finite sets is behaviourally correctly learnable: When D is the set of all data items the learner has seen so far, then it conjectures the set $\{x : \exists y \in D\,[x\,\eta\,y]\}$. It is straightforward to compute indices of these sets from lists of elements in the set D. Next, we distinguish two cases: first, when the family of all η-finite sets is explanatorily learnable; second, when this family is not explanatorily learnable. In the case that this family is not explanatorily learnable, the proof is already completed; so assume that it has an explanatory learner M. We construct another η-family that is behaviourally correctly learnable but not explanatorily learnable.

One considers the class which contains for each n the following sets:

1. $A_n = \{x : \exists k < n\,[x\,\eta\,a_k]\}$;
2. $F_n = \{x : \exists k \neq n\,[x\,\eta\,a_k]\}$;
3. If W_n is finite then for all $m \leqslant |W_n|$ and $m > n$ the set $B_{n,m} = \{x : \exists k < m\ [k \neq n \wedge x\,\eta\,a_k]\}$.

This class is a superfamily of the ascending η-family $A_0, A_1, \ldots$ and so by Proposition 9, it is sufficient to show that it is uniformly r.e. for proving that it is an η-family. The enumeration-procedure checks for each combination of certain indexing parameters (described in detail below) whether these witness that the corresponding set is in the class; when an error is discovered with respect to the parameters, the enumerated set is overwritten by enumerating some member of the ascending family which contains all the numbers enumerated so far (this is possible because any finite set D is contained in the ascending family $\{x : \exists k < \max(D) + 1\, [x\, \eta\, a_k]\}$, where $\max(\emptyset)$ is defined to be 1).

We now describe in detail the enumeration procedures for (i) A_n, (ii) F_n and (iii) $B_{n,m}$.

i. A_n can be enumerated from the parameter n and no check is needed.
ii. One can, in the limit, compute for every n the value a_n and a stabilising sequence σ_n of M for the set $[a_n]$. F_n can be enumerated based on a_n and a stabilising sequence σ_n of the explanatory learner M for $[a_n]$; specifically, one has that $F_n = \{x : \exists \tau \in ([a_n] \cup [x])^*\, [M(\sigma_n \circ x \circ \tau) \neq M(\sigma_n)]\}$. Thus, at time t, one can make a guess $e_{n,t}$ for the enumeration procedure of F_n based on the value of $a_{n,t}$ and a guess τ_s for the actual string σ_n, where, given a default enumeration $\tau_0, \tau_1, \tau_2, \ldots$ of all finite sequences, s is the least index s' such that $\tau_{s'} \in ([a_{n,t}]_t \cup \{\#\})^*$, and $\tau_{s'}$ "appears" to be a stabilising sequence of M for $[a_{n,t}]$, in the sense that for all $\tau' \in ([a_{n,t}]_t \cup \{\#\})^*$ of length at most t, $M(\tau_{s'} \circ \tau') = M(\tau_{s'})$ (if no such $\tau_{s'}$ exists, then set $\tau_s = \lambda$); here $[a_{n,t}]_t$ refers to the set of elements enumerated into $[a_{n,t}]$ up to and including stage t, based on some fixed enumeration of $[a_{n,t}]$. One may then choose $e_{n,t}$ such that $W_{e_{n,t}} = \{x : \exists \tau \in ([a_{n,t}] \cup [x])^*\, [M(\tau_s \circ x \circ \tau) \neq M(\tau_s)]\}$. The set F_n can be enumerated from a pair (n, t) which uses the guess of the index $e_{n,t}$ for the enumeration procedure of F_n at time t and which enumerates $W_{e_{n,t}}$ until an $s > t$ is found with $e_{n,s} \neq e_{n,t}$; if that latter happens, then the enumeration is stopped and some member of the ascending family is enumerated which contains all the numbers so far enumerated. For sufficiently large t, the procedure will enumerate from (n, t) the set F_n.
iii. For the set $B_{n,m}$, one considers all triples (n, m, t) where $W_{n,t}$ has at least m elements and then one will enumerate for this triple all elements equivalent to some $a_{k,t}$ with $k < m \wedge k \neq n$ until an $s > t$ is found such that $a_{k,t} \neq a_{k,s}$ for some $k \leqslant m$ or $W_{n,s} \neq W_{n,t}$. If that happens, the parameters are invalid and the set is made equal to some member of the ascending family which contains all the data enumerated so far.

The so defined family is a superset of all η-cosingletons F_n. If the family was explanatorily learnable, then every F_n would have a locking sequence τ_n which can be found in the limit. If W_n is finite, then τ_n must contain an element $h \geqslant |W_n|$, due to A_n being in the class, and $B_{n,|W_n|}$ being in the class and containing all elements of F_n strictly below $|W_n|$ and perhaps some others more. Therefore one can check using the halting problem oracle K whether W_n enumerates at least $\max(range(\tau_n)) + 1$ elements. If the answer is "yes" then W_n must be infinite; if the answer is "no" then W_n must be finite. Thus one could decide relative to K whether n is in the set $\mathrm{FIN} = \{n : W_n \text{ is finite}\}$. But this is impossible, as FIN is Σ_2-complete, while K is only Σ_1-complete—see [35, Proposition X.9.6]. Thus the family cannot be explanatorily learnable.

It remains to show that the so constructed η-family is behaviourally correctly learnable. So a behaviourally correct learner N would first find in the limit the following pieces of information, the indices issued on the way to find these pieces of information enumerate something, but might not enumerate any set in the constructed η-family:

1. The least number n such that a_n does not appear in the text;
2. An index e_n such that $W_{e_n} = F_n$.

Furthermore, let t be the number of input items (including repetitions and pause symbols) from the text processed so far and $m = \min\{k > n : a_{k,t}$ has not yet appeared in the text$\}$.

In the case that $m = n + 1$, the learner conjectures A_n. In the case that $m > n + 1$, the learner checks whether $W_{n,t}$ has more than m elements.

If the answer is "yes", then the learner conjectures an index for an r.e. set which first starts enumerating all elements of $B_{n,m}$ until it happens that there is an $s > t$ with $W_{n,s} \neq W_{n,t}$ or $\exists k \leqslant m\, [a_{k,s} \neq a_{k,t}]$; if that happens then the learner enumerates W_{e_n} which is equal to F_n.

If the answer is "no", then the learner directly enumerates all members of W_{e_n}.

There is always one n for which a_n is not contained in the set; it is clear that the learner finds the least such n in the limit; furthermore, in the case that it exists, the learner also finds the second least m such that a_m is not in the set in the limit. Now one of the following cases holds:

1. $m = n + 1$: In this case, the learner will conjecture A_n almost always, thus it behaviourally correctly learns that set and this is the only set for which a_n and a_m with the above condition are the two least non-elements.
2. $n + 1 < m$ and $m \leqslant |W_n|$ and W_n is finite: In this case, for all sufficiently large t, $W_{n,t} = W_n$ and $a_{k,t} = a_k$ for all $k \leqslant m$ and all a_k with $k < m \wedge k \neq n$ have appeared in the text; in these circumstances, the learner will conjecture $B_{n,m}$ and this set is correct.
3. $\max(n+1, |W_n|) < m$: This case does not occur, there is no set with the two least non-elements a_n and a_m in the η-family considered.
4. m does not exist: Then the set to be learnt is F_n. When t is sufficiently large then it holds that either W_n is infinite and all $k \leqslant n+1$ satisfy the below statement $(*)$ or W_n is finite and all $k \leqslant |W_n|$ satisfy $(*)$. Here $(*)$ is the statement that $a_{k,t} = a_k$ and whenever $k \neq n$ then $a_{k,t}$ has appeared among the first t input data items of the text. One can see that in both the "either-case" and the "or-case", the set conjectured by the learner is F_n.

This case-distinction completes the proof by verifying that N is a correct behaviourally correct learner for the family. □

Vacillatory learning, according to which a learner is allowed to switch between any finite number of correct indices in the limit, is known to be strictly weaker than behaviourally correct learning for general families of r.e. sets [8]. The next main result – Theorem 13 – asserts that for any given positive equivalence relation η, this relation between the two criteria holds even for certain η-families. We begin with the following proposition, from which the separation result may be deduced.

Proposition 12. *If the class of η-finite sets is vacillatorily learnable then one can relative to the halting problem K compute a sequence $e_0, e_1, \ldots$ of characteristic indices of η-finite sets $E_0, E_1, \ldots$ which form a partition of $\mathbb{N}$.*

Proof. Let M be the vacillatory learner for the class of all η-finite sets. Assume that all E_m and e_m with $m < n$ are computed. Now one finds with help of K the first k such that $a_k \notin \bigcup_{m<n} E_m$. Furthermore, there is a finite sequence σ of elements η-equivalent to a_k such that any further finite sequence τ of elements η-equivalent to a_k satisfies that $M(\sigma\tau)$ equals to $M(\rho)$ for some prefix ρ of σ; if such a sequence would not exist, one could construct a text for the equivalence class of a_k on which M outputs infinitely many different hypotheses. M has issued at most $|\sigma|+1$ many different indices on prefixes of σ. Now let F_n be the union of the sets $\bigcup_{m<n} E_m$ and the set of all x for which there is a τ consisting of elements which are η-equivalent to a_k or to x such that $M(\sigma x\tau)$ contains an index not in the finite set $\{M(\rho) : \rho \preceq \sigma\}$. Note that F_n does not contain any element η-equivalent to a_k by the choice of σ; furthermore, whenever for an x there is no prefix $\rho \preceq \sigma$ with $W_{M(\rho)}$ enumerating $\{y : y\,\eta\,x \vee y\,\eta\,a_k\}$ then $x \in F_n$. Furthermore, it follows from the definition of F_n that F_n is η-closed. Now let $R = \{\rho \preceq \sigma : W_{M(\rho)} \cap F_n = \emptyset\}$ and note that R can be determined using the halting problem K as an oracle. Note that R is finite. The set $E_n = \bigcup_{\rho\in R} W_{M(\rho)}$ is η-finite and the complement of F_n; thus one can from the enumeration-procedures of F_n and E_n compute a characteristic index e_n of E_n. Furthermore, the cardinality c_n of R is an upper bound on the number of a_k in E_n and E_n is η-finite. As every a_k appears in exactly one E_n and as all E_n are η-closed, the E_n form a partition of $\mathbb{N}$ into η-finite sets. □

Theorem 13. *For every positive equivalence relation η, there is an η-family which is behaviourally correctly learnable but not vacillatorily learnable.*

Proof. One can use Proposition 12 to construct, in the case that the class of η-finite sets is vacillatorily learnable, the following class which is behaviourally correctly learnable but not vacillatorily learnable: The family contains (i) all members A_n of the ascending family; (ii) all sets F_n as constructed in the proof of Proposition 12; (iii) in the case that W_n is finite, all sets $A_m \cap F_n$ with $m \leqslant |W_n|$. For this class, one can show as in Theorem 11 that it is an η-family and that it is behaviourally correctly learnable and that it is not explanatorily learnable; the latter proof can be adjusted to a proof that it is not vacillatorily learnable. To see this, consider a vacillatory learner; for this learner one can compute using K a sequence σ of elements in F_n such that every extension $\sigma\tau$ of it using for τ only elements from F_n satisfies that $M(\sigma\tau) = M(\rho)$ for a prefix ρ of σ. Now if W_n is finite then there are at least $|W_n| - c_n$ many sets of the form $A_m \cap F_n$, $m \in \mathbb{N}$, in the class (where c_n is computed using K as in Proposition 12) and for each of them, either some $M(\rho)$ with $\rho \preceq \sigma$ must be an index for it or σ contains an element outside the set $A_m \cap F_n$. Note that the smallest number outside A_m is at least m. Thus if W_n is finite and if one takes $d_n = \max(range(\sigma)) + |\sigma| + c_n$, then $d_n \geqslant |W_n|$. Therefore one can compute from n the number d_n using K and then check using K whether $|W_n| \leqslant d_n$. If this is true then W_n is finite else W_n is infinite. Thus the existence of a vacillatory learner would lead to an incorrect result about the arithmetic hierarchy [35, Proposition X.9.6] and therefore the η-family constructed is not vacillatorily learnable. □

Moving down the learning hierarchy given at the start of the present section, the following theorem shows that for any positive equivalence relation η, finite learning can be more restrictive than confident learning with respect to η-families.

Theorem 14. *Let η be any given positive equivalence relation such that there is at least one finitely learnable η-family. Then there is an η-family that is confidently learnable but not finitely learnable.*

Proof. Let $B_0, B_1, \ldots$ be any finitely learnable η-family. Now, for every set B_n there is a finite subset C_n such that B_n is the only superset in this family. These sets C_n must exist, as they are the data-items observed so far when the finite learner conjectures B_n from the default text of B_n and as the conjecture cannot be revised, there cannot be an $m \neq n$ with $C_n \subseteq B_m$. Now one considers the family in which one replaces B_2 by $\mathbb{N}$. This family has a confident learner which first issues the same conjectures as the finite learner up to the first conjecture of the finite learner which is not ?; once the finite learner issues the first such conjecture, say e, then the confident learner will output e until it has seen all data in $C_0 \cup C_1$. If that happens, $\mathbb{N}$ is the only consistent hypothesis and the learner makes a mind change to that set. As this new class is not inclusion-free, it is not finitely learnable. $\square$

5 The Non-Union Theorem and Finitely Learnable Classes

As Gold [21] observed, the class consisting of $\mathbb{N}$ and all finite sets is not learnable in any sense considered in the present paper. Thus adding only one set to the learnable class of all finite set makes it unlearnable. This is not true for all classes. For example, in the case that one considers languages made of strings, Angluin [1, Example 1] provided the class of all non-erasing pattern languages which can be learnt from positive data explanatorily. If one adds a single set, then the class is still explanatorily learnable, the reason being that the set of shortest strings in a non-erasing pattern language defines this pattern language uniquely; if now the set of shortest strings of the new language coincides with that of one of the members of the class, then one can add in one more string on which the two languages differ and use these finitely many strings to detect the difference.

Similarly the union of two disjoint infinite explanatorily learnable classes of r.e. languages may not even be behaviourally correctly learnable. Blum and Blum [7] noted that the family of explanatorily (resp. behaviourally correctly) learnable classes of recursive functions is also not closed under union. This section deals with the question of whether the non-union property of explanatory (resp. vacillatory, behaviourally correct) learning holds for the class of η-families, where η is any given positive equivalence relation. For any learning criterion I and any positive equivalence relation η, say that I is *closed under union with respect to* η iff for any η-families $\mathcal{L}$ and $\mathcal{H}$ such that $\mathcal{L} \cup \mathcal{H}$ is an η-family, if $\mathcal{L}$ and $\mathcal{H}$ are I-learnable, then $\mathcal{L} \cup \mathcal{H}$ is I-learnable. Somewhat interestingly, while explanatory and vacillatory learnability are not closed under union with respect to any η, the answer for behaviourally correct learning depends on whether or not there are at least two η-infinite r.e. sets.

Proposition 15. *Let η be any given positive equivalence relation. If $\mathbb{N}$ is the only η-infinite r.e. set, then $\mathbb{N}$ is not contained in any behaviourally correctly learnable η-family.*

Proof. Assume that $\{B_0, B_1, ...\}$ is a behaviourally correctly learnable η-family containing $\mathbb{N}$. By [2, Corollary 3], there is a tell-tale set H for $\mathbb{N}$. Here a set S is said to be a *tell-tale* with respect to $\mathcal{L}$ for a language $L \in \mathcal{L}$ if $S \subseteq L$, S is finite and for all $L' \in \mathcal{L}$, $S \subseteq L' \subseteq L$ implies $L = L'$ (cf. [1]). Now there is a maximal subset H' of H such that H' is contained in infinitely many B_i. If now $H' = H$ then there is a $B_i \neq \mathbb{N}$ which contains H and thus H is not a tell-tale for $\mathbb{N}$. If there is an $x \in H - H'$ then there is an n such that no B_m with $m > n$ contains $H' \cup \{x\}$ as a subset. Now one can one-one enumerate the union of all B_m with $m > n$ such that $H' \subseteq B_m$; this union is an η-infinite r.e. set which does not contain x, but such a set does not exist. So this case does not hold either. ◻

Theorem 16. *Let η be any given positive equivalence relation. Then the following hold.*

(a) There are disjoint, explanatorily (resp. vacillatorily) learnable η-families $\mathcal{L}_1$ and $\mathcal{L}_2$ for which $\mathcal{L}_1 \cup \mathcal{L}_2$ is an η-family that is not explanatorily (resp. vacillatorily) learnable.

(b) Behaviourally correct learning is closed under union with respect to η iff $\mathbb{N}$ is the only η-infinite r.e. set.

Proof. We split the proof into two main cases.

Case 1: $\mathbb{N}$ is the only η-infinite r.e. set. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be any two behaviourally correctly learnable η-families such that $\mathcal{L}_1 \cup \mathcal{L}_2$ is an η-family. By Proposition 15, both $\mathcal{L}_1$ and $\mathcal{L}_2$ contain only η-finite sets. Thus $\mathcal{L}_1 \cup \mathcal{L}_2$ contains only η-finite sets. Note that any class of η-finite sets is behaviourally correctly learnable: on input σ, a BC learner conjectures $\bigcup_{x \in \mathrm{content}(\sigma)} [x]$. Therefore $\mathcal{L}_1 \cup \mathcal{L}_2$ is also behaviourally correctly learnable, and so behaviourally correct learning is closed under union with respect to η. This proves the "if" direction of statement (b).

Next, it is shown that explanatory (resp. vacillatory) learning is not closed under union with respect to η. Set

$$\mathcal{L}_1 = \{A_{2n+1} : n \in \mathbb{N}\} \cup \{[a_k] : k \in \mathbb{N}\},$$
$$\mathcal{L}_2 = \{A_{2n+2} : n \in \mathbb{N}\} \cup \{[a_k, a_l] : k, l \in \mathbb{N} \wedge k < l\}.$$

Note that $\mathcal{L}_1$ has the following uniformly r.e. numbering: for all $n, s \in \mathbb{N}$,

$$L_{2n} = A_{2n+1},$$
$$L_{2\langle n,s\rangle+1} = [a_{n,s}].$$

Similarly, $\mathcal{L}_2$ has the following uniformly r.e. numbering $\{H_0, H_1, H_2, \ldots\}$. For all $n \in \mathbb{N}$, set $H_{2n} = A_{2n+2}$. For all $m, n, s \in \mathbb{N}$, if $a_{m,s} = a_m$ and $a_{m+n+1,s} = a_{m+n+1}$, set $H_{2\langle m,n,s\rangle+1} = [a_{m,s}, a_{m+n+1,s}]$, and if there is a least $t > s$ such that $a_{m,t} \neq a_{m,s}$ or $a_{m+n+1,t} \neq a_{m+n+1,s}$, set $H_{2\langle m,n,s\rangle+1} = A_{2n'+2}$, where n' is the least k such that A_{2k+2} contains all the elements enumerated into $H_{2\langle m,n,s\rangle+1}$ up until stage t. A similar proof shows that $\mathcal{L}_1 \cup \mathcal{L}_2$ has a uniformly r.e. numbering. Thus by Proposition 10, $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_1 \cup \mathcal{L}_2$ are η-families.

The following learner M explanatorily learns $\mathcal{L}_1$. On input σ, M first checks whether or not $a_0 \in \mathrm{content}(\sigma)$. If $a_0 \in \mathrm{content}(\sigma)$, then M conjectures A_{2n+1} for the least n such that $a_{2n,|\sigma|} \in \mathrm{content}(\sigma)$ and $a_{2n+1,|\sigma|} \notin \mathrm{content}(\sigma)$. If $a_0 \notin \mathrm{content}(\sigma)$ and $\mathrm{content}(\sigma) \neq \emptyset$, then M conjectures $[\min(\mathrm{content}(\sigma))]$. If $\mathrm{content}(\sigma) = \emptyset$, then M outputs a default index, say A_1. An explanatory learner for $\mathcal{L}_2$ can be constructed similarly.

Now assume, by way of contradiction, that $\mathcal{L}_1 \cup \mathcal{L}_2$ were vacillatorily learnable by some learner N. As observed earlier in the proof of Proposition 12, a slight variant of Proposition 3 applies to vacillatory learners: if N vacillatorily learns a language L, then there is a sequence σ of elements in L such that every extension $\sigma\tau$ of it using for τ only elements from L satisfies that $M(\sigma\tau) = M(\rho)$ for a prefix ρ of σ. In particular, there is such a sequence $\sigma_0 \in ([a_0] \cup \{\#\})^*$ and there is a finite subset $F_0 = \{M(\rho) : \rho \text{ is a prefix of } \sigma \text{ and } W_{M(\rho)} = [a_0]\}$ for N on $L = [a_0]$. By the choice of σ_0 and F_0 and the fact that N vacillatorily learns all sets of the form $[a_0, a_k]$, where $k > 0$, it follows that the r.e. set

$$\{x : \exists \tau \in [a_0, x]^* \, [N(\sigma_0 \circ \tau) \notin F_0]\}$$

is an η-infinite r.e. subset of $\mathbb{N} - [a_0]$. But this contradicts the assumption that there is no η-infinite r.e. set different from $\mathbb{N}$. Thus neither explanatory nor vacillatory learning is closed under union with respect to η.

Case 2: There is an η-infinite r.e. set that is different from $\mathbb{N}$. Let B be any η-infinite r.e. set that is different from $\mathbb{N}$, and fix some k such that $a_k \notin B$. Let η_s be the equivalence relation obtained by enumerating η-equivalent pairs of elements of B for s steps and keeping the relation closed transitively and reflexively. Furthermore, $b_{n,s}$ is the n-th element in the order of the enumeration satisfying that all elements before it in the enumeration are not η_s equivalent to $b_{n,s}$, and b_n is defined to be the limit of the sequence of $b_{n,s}$'s, that is, $b_n = b_{n,l}$ for some l such that $b_{n,l'} = b_{n,l}$ for all $l' \geqslant l$. Now set

$$\begin{aligned} \mathcal{L}_1 &= \{A_{n+k+1} : n \in \mathbb{N}\}, \\ \mathcal{L}_2 &= \{\{x : \exists m < n \, \exists s \, [x \, \eta_s \, b_{m,s}]\} : n \in \mathbb{N}\} \cup \{\mathbb{N}\}. \end{aligned}$$

By Proposition 10, $\mathcal{L}_1$ is an η-family. Furthermore, a proof similar to that of Proposition 9 shows that every uniformly r.e. superfamily of $\{\{x : \exists m < n \, \exists s \, [x \, \eta_s \, b_{m,s}]\} : n \in \mathbb{N}\}$ is an η-family; in particular, $\mathcal{L}_2$ is an η-family.

By Theorem 8, $\mathcal{L}_1$ is explanatorily learnable. An explanatory learner M for $\mathcal{L}_2$ works as follows: on input σ, if $a_k \in \mathrm{content}(\sigma)$, then M outputs a canonical index for $\mathbb{N}$; if $a_k \notin \mathrm{content}(\sigma)$, then M conjectures $\{x : \exists m < n \, \exists s \, [x \, \eta_s \, b_{m,s}]\}$ for the least n found such that $b_{n,|\sigma|} \notin \mathrm{content}(\sigma)$, where $b_{n,|\sigma|}$ is defined as earlier. If M is presented with a text for $\mathbb{N}$, then M will always output a canonical index for $\mathbb{N}$ after a_k appears in the text. If M is presented with a text T for some set $\{x : \exists m < n \, \exists s \, [x \, \eta_s \, b_{m,s}]\}$, then there is an s_0 large enough so that $b_m \in \mathrm{content}(T[s_0])$ for all $m < n$ and $b_{m,s_0} = b_m$ for all $m \leqslant n$; thus $M(T[s'])$ equals some canonical index for $\{x : \exists m < n \, \exists s \, [x \, \eta_s \, b_{m,s}]\}$ whenever $s' \geqslant s_0$.

Since $\mathcal{L}_1 \cup \mathcal{L}_2$ contains $\mathbb{N}$ and the infinite ascending chain $\{A_{k+1} \subset A_{k+2} \subset \ldots \subset A_{k+i} \subset \ldots\}$ whose union is $\mathbb{N}$, $\mathcal{L}_1 \cup \mathcal{L}_2$ is not behaviourally correctly learnable. Hence, under the present case assumption, explanatory (resp. vacillatory, behaviourally correct) learning is not closed under union with respect to η. (In particular, this proves the "only if" direction of statement (b).) $\square$

The next result establishes the non-union theorem for finite learning of η-families, where η is any positive equivalence relation such that at least one finitely learnable η-family exists. It may be worth noting that, in contrast to explanatory learnability, there is a positive equivalence relation ϑ for which no ϑ-family is finitely (or even confidently) learnable, as will be seen in the subsequent section.

Theorem 17. *Let η be any given positive equivalence relation such that at least one η-family is finitely learnable. Then there are finitely learnable η-families $\mathcal{L}_1$ and $\mathcal{L}_2$ for which $\mathcal{L}_1 \cup \mathcal{L}_2$ is an η-family that is not finitely learnable. Furthermore, $\mathcal{L}_2 - \mathcal{L}_1$ contains a single language.*

Proof. Let $\{B_0, B_1, B_2, \ldots\}$ be a finitely learnable η-family, as witnessed by some learner M. As observed earlier in the proof of Theorem 14, for every set B_n there is a finite subset F_n such that B_n is the only superset in this family. The family $\{F_0, F_1, F_2, \ldots\}$ is *uniformly recursively generable* [30] in that there is a total effective procedure that, on every input i, generates all elements of F_i and stops. Furthermore, if there were some $j \neq i$ such that $F_i \subseteq B_j$, then one could extend σ_i to a text T for B_j and so M would not finitely learn B_j on T because it outputs an index for B_i on σ_i. Thus F_i is a *characteristic sample* [28,32] for B_i in the sense that $F_i \subseteq B_i$ and for all $j \neq i$, $F_i \not\subseteq B_j$. The family $\mathcal{L}_1 := \{[F_0], [F_1], [F_2], \ldots\}$ is therefore an η-family. Note that $\mathcal{L}_1$ is finitely learnable: on any given text, a finite learner outputs ? until it identifies the least i (if there is any such i) such that — (1) F_i is contained in the range of the current input and (2) one has seen at least i data items, including pause symbols, in the text — then the learner conjectures $[F_i]$. Condition (2) is only delaying the learning and is needed to make the learner recursive. Furthermore, it is possible to check for any i whether or not F_i belongs to the range of the current input because $\{F_0, F_1, F_2, \ldots\}$ is uniformly recursively generable.

Let $\mathcal{L}_2$ be the family $\{[F_0 \cup F_1]\} \cup (\mathcal{L}_1 - \{[F_j] : [F_j] \subset [F_0 \cup F_1]\})$. Since $[F_0 \cup F_1]$ is η-finite, there are only finitely many indices j such that $[F_j] \subset [F_0 \cup F_1]$ and therefore $\mathcal{L}_2$ is a finite variant of $\mathcal{L}_1$. It follows that $\mathcal{L}_2$ is also an η-family. Note that $\mathcal{L}_2$ is finitely learnable: on input σ, a finite learner outputs ? until it identifies the least i (if there is any such i) such that F_i is contained in the range of the current input and either $i \in \{0, 1\}$ or $[F_i] \not\subseteq [F_0 \cup F_1]$. If $i \in \{0, 1\}$, then the learner conjectures $[F_0 \cup F_1]$, and if $i > 1$, then it conjectures $[F_i]$. Suppose the learner is presented with a text for $[F_0 \cup F_1]$. Then the least j such that F_j is contained in the range of the text and either $j \in \{0, 1\}$ or $F_j \not\subseteq [F_0 \cup F_1]$ must be either 0 or 1; thus the learner will settle on the conjecture $[F_0 \cup F_1]$. If the learner is presented with a text for some $[F_j]$ with $[F_j] \not\subseteq [F_0 \cup F_1]$, then, since $j \notin \{0, 1\}$ and j is the least index j' such that $F_{j'}$ is contained in the range of the text and $[F_{j'}] \not\subseteq [F_0 \cup F_1]$, the learner will settle on the conjecture $[F_j]$.

Now consider $\mathcal{L}_1 \cup \mathcal{L}_2$; this is an η-family because it is the union of the η-family $\mathcal{L}_1$ and $\{[F_0 \cup F_1]\}$. Since any finite subset of $[F_0]$ is contained in $[F_0 \cup F_1]$, $[F_0]$ does not have a finite characteristic sample – which, as was pointed out earlier, must exist for every language in a finitely learnable family. Thus $\mathcal{L}_1 \cup \mathcal{L}_2$ is not finitely learnable. □

Note that confidently learnable classes (where learners are possibly non-partial-recursive) are closed under union [37, Exercise 4.6.2C] and that similarly adding single languages does not destroy confident learnability. The remainder of this section considers a less restrictive notion of closure under union for learning criteria; this is motivated by the fact that the union of two η-families is not necessarily an η-family: Letting η to be equality-relation and A be a nonrecursive r.e. set, the η-families $\{2x : x \notin A\} \cup \{2x, 2x+1 : x \in A\}$ and $\{2x+1 : x \notin A\} \cup \{2x, 2x+1 : x \in A\}$ satisfy the property that their union is not an η-family.

Definition 18. (1) Given uniformly r.e. families (with corresponding one-one indexing) $\mathcal{L} := \{L_0, L_1, \ldots\}$ and $\mathcal{H} := \{H_0, H_1, \ldots\}$, say that $\mathcal{L}$ is *one-one reducible to* $\mathcal{H}$ iff there is a one-one recursive function f such that for all $e \in \mathbb{N}$, $L_e = H_{f(e)}$.

(2) An η-family $\mathcal{K}$ is the strong union of $\mathcal{L}$ and $\mathcal{H}$ iff $\mathcal{K} = \mathcal{L} \cup \mathcal{H}$ and there are one-one reductions from $\mathcal{L}$ to $\mathcal{K}$ and from $\mathcal{H}$ to $\mathcal{K}$.

(3) For any learning criterion I and any positive equivalence relation η, say that I is *strongly closed under union with respect to* η iff for any exactly I-learnable η-families $\mathcal{L}$ and $\mathcal{H}$, also every strong union of $\mathcal{L}$ and $\mathcal{H}$ is exactly I-learnable.

Theorem 19. *For any positive equivalence relation η, confident learning is strongly closed under union with respect to η but neither finite nor explanatory learning is strongly closed under union with respect to η.*

Proof. Suppose $\mathcal{L} := \{L_0, L_1, L_2, \ldots\}$ and $\mathcal{H} := \{H_0, H_1, H_2, \ldots\}$ are exactly confidently learnable η-families such that $\mathcal{L} \cup \mathcal{H} = \{B_0, B_1, B_2, \ldots\}$ is an η-family, $\mathcal{L}$ is one-one reducible to $\mathcal{L} \cup \mathcal{H}$ and $\mathcal{H}$ is one-one reducible to $\mathcal{L} \cup \mathcal{H}$. Let f and g be recursive functions such that for all $n \in \mathbb{N}$, $L_n = B_{f(n)}$ and $H_n = B_{g(n)}$, and let M_1 and M_2 be exact confident learners for $\mathcal{L}$ and $\mathcal{H}$ respectively. An exact confident learner N for $\mathcal{L} \cup \mathcal{H}$ works as follows: on input σ, simulate M_1 and M_2 and let d and e be their respective outputs on σ. If $f(d) = g(e)$, then N outputs $f(d)$. If $f(d) \neq g(e)$, then N searches for an $s > |\sigma|$ such that there is a least x with $x \in B_{f(d),s} \triangle B_{g(e),s}$ (such s and x must exist because $B_0, B_1, B_2, \ldots$ is a one-one numbering and $f(d) \neq g(e)$). Set

$$N(\sigma) = \begin{cases} f(d) \text{ if } x \in \text{content}(\sigma) \cap B_{f(d),s} \text{ or } x \notin \text{content}(\sigma) \cup B_{f(d),s}; \\ g(e) \text{ if } x \in \text{content}(\sigma) \cap B_{g(e),s} \text{ or } x \notin \text{content}(\sigma) \cup B_{g(e),s}. \end{cases}$$

On any text T, both M_1 and M_2 will converge; suppose M_1 and M_2 converge to d_0 and e_0 respectively. If $f(d_0) = g(e_0)$, then N will converge to $f(d_0)$; if $f(d_0) \neq g(e_0)$, then N will converge to $f(d_0)$ (resp. $g(e_0)$) if the least x such that $x \in B_{f(d_0)} \triangle B_{g(e_0)}$ satisfies $x \in \text{content}(T) \cap B_{f(d_0)}$ or $x \notin \text{content}(T) \cup B_{f(d_0)}$ (resp. $x \in \text{content}(T) \cap B_{g(e_0)}$ or $x \notin \text{content}(T) \cup B_{g(e_0)}$). Furthermore, if T is a text for any $B_\ell \in \mathcal{L} \cup \mathcal{H}$, then at least one of M_1 and M_2 converges on T to a correct index for B_ℓ, so that $\ell \in \{f(d_0), g(e_0)\}$. If $f(d_0) = g(e_0)$, then $\ell = f(d_0) = g(e_0)$ and so N will output ℓ in the limit; if $\ell = f(d_0) \neq g(e_0)$, then there is a least x in $B_{f(d_0)} \triangle B_{g(e_0)}$ such that either $x \in \text{content}(T) \cap B_{f(d_0)}$ or $x \notin \text{content}(T) \cup B_{f(d_0)}$, and so N will again output ℓ in the limit; a similar argument applies if $\ell = g(e_0) \neq f(d_0)$. It follows that N exactly confidently learns $\mathcal{L} \cup \mathcal{H}$.

For explanatory learning, one notes that the classes $\mathcal{L}_1$ and $\mathcal{L}_2$ in both Cases 1 and 2 of the proof of Theorem 16 are η-families with only finitely much intersection, and so if $\mathcal{L}_1 = \{C_0, C_1, C_2, \ldots\}$ and $\mathcal{L}_2 = \{E_0, E_1, E_2, \ldots\}$, then, setting $F_{2i} = C_i$ and $F_{2i+1} = E_i$ for all $i \in \mathbb{N}$, $\{F_0, F_1, F_2, \ldots\}$ is almost a one-one uniformly r.e. numbering of $\mathcal{L}_1 \cup \mathcal{L}_2$ (finitely many languages may have two indices). Then $i \mapsto 2i$ and $i \mapsto 2i+1$ are recursive functions witnessing the one-one reductions of $\mathcal{L}_1$ to $\mathcal{L}_1 \cup \mathcal{L}_2$ and of $\mathcal{L}_2$ to $\mathcal{L}_1 \cup \mathcal{L}_2$ respectively. This can be easily modified to have a 1–1 indexing of $\mathcal{L}_1 \cup \mathcal{L}_2$. Furthermore, the classes $\mathcal{L}_1$ and $\mathcal{L}_2$ in both Cases 1 and 2 of the proof of Theorem 16 are explanatorily learnable (even exactly). Thus explanatory learning is not strongly closed under union. A similar argument based on the proof of Theorem 17 shows that finite learning is also not strongly closed under union. $\square$

6 Learnability of Families Closed Under Special Positive Equivalence Relations

So the general results up to now were that for every positive equivalence relation η, for each of the following conditions, there are η-families which satisfy it: (a) the family is explanatory learnable but not confidently learnable; (b) the family is behaviourally correctly learnable but not vacillatorily learnable; (c) the family is not behaviourally correctly learnable. The picture does not provide η-families which are confidently learnable and also not separate out the notion of vacillatory learning from explanatory learning. The first main result of this section is to construct a positive equivalence relation ϑ such that there is no confidently learnable ϑ-family and furthermore all vacillatorily learnable ϑ-families are explanatory learnable. Thus one cannot separate for all η the notions of vacillatory and explanatory learning and one also cannot show that every η has a confidently learnable η-family. On the other hand, one can show that there is an η for which some vacillatorily learnable η-family is not explanatorily learnable; furthermore, as seen in Theorem 14, there are infinitely many η such that some η-family is confidently learnable. The second main result shows that there is a positive equivalence relation ζ for which there are confidently learnable ζ-families but no finitely learnable ζ-families.

Theorem 20. *There is a positive equivalence relation ϑ satisfying:*

(1) There is only one ϑ-infinite r.e. set, namely $\mathbb{N}$.
(2) Every ϑ-family contains an infinite ascending chain $B_0 \subset B_1 \subset \cdots$ of ϑ-finite sets whose union is $\mathbb{N}$. In particular, no ϑ-family is confidently learnable and every behaviorally correctly learnable ϑ-family consists only of ϑ-finite languages.
(3) Every vacillatory learnable ϑ-family is explanatorily learnable.

Proof. Recall that W_n denotes the n-th recursively enumerable set, and $W_{n,s}$ is the finite part of W_n that is enumerated before the given step $s \in \mathbb{N}$. Recall also the definitions of $a_m \in \mathbb{N}$ and $A_n = [a_0, a_1, \ldots, a_{n-1}]$.

Construction of ϑ: Initially, let $a_{m,0} = m$ for all m — that is, to start, ϑ consists wholly of singleton equivalence classes. Then, for each $s = 0, 1, \ldots$ in turn, search for a triple $n, k, \ell < s$ satisfying:

- $n < \ell$ and $k < \ell$;
- $[a_{k,s}]_s \cap W_{n,s} = \emptyset$; and
- $[a_{\ell,s}]_s \cap W_{n,s} \neq \emptyset$.

If such a triple n, k, ℓ exists, then select one with ℓ as small as possible, merge the classes $[a_k]$ and $[a_\ell]$ together, that is, let $a_{k,s} \vartheta a_{\ell,s}$ (along with any of its implications based on ϑ being equivalence relation), and then set $a_{\ell',s+1} = a_{\ell',s}$, for $\ell' < \ell$ and $a_{\ell',s+1} = a_{\ell'+1,s}$, for $\ell' \geqslant \ell$, so that each $a_{m,s+1}$ is again the least element of the m-th class. If no such triple exists, then $a_{\ell',s+1} = a_{\ell',s}$ for all ℓ'. Then move on to the next value of s. This completes the construction.

Claim 21. *The equivalence relation ϑ thus produced has infinitely many classes (as demanded in the second last paragraph of Section 3).*

Proof: It is enough to verify that each $m \in \mathbb{N}$ serves the role of ℓ only finitely many times, and hence that each a_m reaches some limiting value. This is easily seen by induction on m: after each a_k, $k < m$ has reached its limiting value, a_m, changes its value for each pair $n, k < m$ at most once. Thus, each $a_m = \lim_{s\to\infty} a_{m,s}$, reaches its limiting value.

Claim 22. *For each W_n which is ϑ-closed, either $W_n \subseteq A_n$, or $W_n = A_m$ for some $m \geqslant n$, or $W_n = \mathbb{N}$. In particular, ϑ satisfies part (1) of the theorem.*

Proof: If a ϑ-closed W_n contains (the limiting value of) a given a_ℓ with $\ell \geqslant n$, then by construction W_n also contains $\{a_0, \ldots, a_{\ell-1}\}$. Hence if W_n is ϑ-closed and not contained in A_n, we have either $W_n = A_m$ for some m, or $W_n = \mathbb{N}$.

Claim 23. *If $\mathcal{L}$ is a ϑ-family then for every n there are infinitely many $L \in \mathcal{L}$ such that $A_n \subseteq L$.*

Proof: By induction on n. The base case, where $A_0 = \emptyset$, is trivial. For $n \geqslant 1$, let $U = \bigcup\{L \in \mathcal{L} : A_{n-1} \subseteq L\}$. Clearly U is ϑ-closed and r.e.; and by the induction hypothesis U has infinitely many different summands L, so U is ϑ-infinite by a pigeonhole argument. It follows by Claim 22 that $U = \mathbb{N}$, and in particular $a_{n-1} \in U$. Thus a_{n-1} is an element of one of U's summands L_0, so that $A_n \subseteq L_0$. Repeat the construction with L_0 omitted from $\mathcal{L}$ to get a second such L_1, then L_2, and so on.

Note that by Proposition 15 and part (1) of the theorem, every behaviourally correctly learnable ϑ-family contains only ϑ-finite sets. Thus, in light of Claims 22 and 23, part (2) of the theorem is true.

Claim 24. *Every vacillatorily learnable ϑ-family is explanatory learnable.*

Proof: Suppose $\mathcal{L}$ is a ϑ-family with vacillatory learner M; we know from part (2) of the Theorem (that is, from the previous claim) that $\mathcal{L}$ consists only of ϑ-finite sets. Given an enumeration of a language $L \in \mathcal{L}$, the learner M outputs at most finitely many indices, say with maximum n^*; and given this n^*, we know from an earlier claim that either $L \subseteq A_{n^*}$, or $L = A_m$ for some $m \geqslant n^*$. So the new explanatory learner N works by watching for the largest n output by M; then N watches to see if a_n is a member of L and if so, N conjectures that $L = A_m$ for the least m such that a_m does not belong to the input text; otherwise N conjectures that L is an appropriate subset of A_n (which consists of the elements a_i, $i < n$, which have appeared in the input text). It is easy to verify that N explanatorily learns $\mathcal{L}$. □

In the restricted setting of exact learning, vacillatory learnability is, by definition, the same as explanatory learnability whenever the family to be learnt is given as a one-one numbering of sets; in particular, for each η, this equivalence holds for every η-family.

Proposition 25. *If $\mathcal{L}$ is given as a one-one numbering $L_0, L_1, L_2, \ldots$, then $\mathcal{L}$ is exactly vacillatorily learnable iff it is exactly explanatorily learnable.*

It remains to show that for some positive equivalence relation η, there is an η-family that is vacillatorily learnable (though not with respect to the original numbering of the family) but not explanatorily learnable. This is already true when η is the equality relation, that is, in the usual setting of learning uniformly r.e. one-one families. For completeness, we provide an example of such a family that is vacillatorily learnable but not explanatorily learnable (as with Proposition 25, we do not know whether such an example has been explicitly given in the literature).

Theorem 26. *There is a class $\mathcal{L}$ of sets with a uniformly r.e. one-one numbering such that $\mathcal{L}$ is vacillatorily learnable but not explanatorily learnable.*

Proof. Let $\mathcal{L}$ be the class consisting of the following sets: (1) $\mathbb{N} - \{a\}$ for each a (cosingletons); (2) $\{e + x : x \in \mathbb{N}\}$ for each $e \geqslant 2$; (3) $\{e + x : x \leqslant |W_e|\}$ for each $e \geqslant 2$.

The class of all sets of the form $\{e + x : x \leqslant |W_e|\}$ or $\{e + x : x \in \mathbb{N}\}$ for some $e \geqslant 2$ has a uniformly r.e. numbering: for each e, one can define $L_{2e} = \{e + x : x \leqslant |W_e|\}$ and $L_{2e+1} = \{e + x : x \in \mathbb{N}\}$. Furthermore, the class of all cosingletons has a uniformly recursive one-one numbering and each finite set is contained in infinitely many cosingletons. Thus by Theorem 4, $\mathcal{L}$ has a uniformly r.e. one-one numbering.

$\mathcal{L}$ is vacillatorily learnable via the following learner M. On input σ, if content(σ) contains at least one of $\{0, 1\}$, then M conjectures $\mathbb{N} - \{a\}$ for the least $a \notin$ content(σ). Otherwise, M finds $e = \min(\text{content}(\sigma))$, $d = \max\{x : e + x \in \text{content}(\sigma)\}$, (where $\min(\emptyset)$ and $\max(\emptyset)$ are each defined to be 0); if $d > |W_{e,|\sigma|}|$, then M outputs a canonical index for $\{e + x : x \in \mathbb{N}\}$, else M outputs a canonical index for $\{e + x : x \leqslant |W_e|\}$.

Each text for a cosingleton contains 0 or 1, so M on such a text will, in the limit, correctly conjecture $\mathbb{N} - \{a\}$ for the least a not in the range of the text. Each text for a finite set of the form $\{e + x : x \leqslant |W_e|\}$, where $e \geqslant 2$, has range $\{e, e+1, \ldots, e + |W_e|\}$; thus, since $|W_e| = |W_{e,s}|$ for sufficiently large s, M on such a text will output a canonical index for $\{e + x : x \leqslant |W_e|\}$

in the limit. M on each text for an infinite set $\{e+x : x \in \mathbb{N}\}$ with $e \geqslant 2$ will, in the limit, output at most two different indices – one for $\{e+x : x \in \mathbb{N}\}$ or one for $\{e+x : x \leqslant |W_e|\}$; if $|W_e|$ is finite, then the range of almost every text segment contains a number larger than $|W_e|$, so M will converge to a canonical index for $\{e+x : x \in \mathbb{N}\}$, and if $|W_e|$ is infinite, then $\{e+x : x \in \mathbb{N}\} = \{e+x : x \leqslant |W_e|\}$ and so M will almost always output a correct conjecture, vacillating between at most two different indices.

If $\mathcal{L}$ were explanatorily learnable via some learner N, then one could use oracle K to decide for any given e whether or not W_e is finite: first, use K to find some minimum s and a corresponding $\tau \in \{e+x : x \leqslant |W_{e,s}|\}^*$ satisfying at least one of the following: (i) $|W_{e,s}| = |W_e|$; (ii) for each $\sigma \in \{e+x : x \in \mathbb{N}\}^*$, $N(\tau \circ \sigma) = N(\tau)$. If (i) holds, then $|W_e|$ is finite. If (ii) holds, then $W_{N(\tau)} = \{e+x : x \in \mathbb{N}\}$; otherwise, one could extend τ to a text for $\{e+x : x \in \mathbb{N}\}$ and therefore N would not explanatorily learn this set. Next, use K again to check whether $|W_e| > |W_{e,s}|$. If $|W_e| > |W_{e,s}|$, then W_e must be infinite, for otherwise one could extend τ to a text for $\{e+x : x \leqslant |W_e|\}$ and N would not explanatorily learn this set. If $|W_e| \leqslant |W_{e,s}|$, then W_e is finite. Consequently, $\{e : W_e \text{ is finite}\}$, which is Σ^0_2-complete [38, Theorem 13-VIII], would be Turing reducible to K, a contradiction.

We note that by Proposition 25, any vacillatory learner for $\mathcal{L}$ cannot use a one-one numbering of $\mathcal{L}$ as its hypothesis space. □

According to Theorem 14, for every positive equivalence relation η such that there is at least one finitely learnable η-family, there is also an η-family that is confidently but not finitely learnable. The next main result complements this theorem by showing that there is a positive equivalence relation ζ for which no finitely learnable ζ-family exists even though there are confidently learnable ζ-families.

Description 27. One defines a positive equivalence relation ζ using a dense simple set Z with $0 \notin Z$ as below; recall for this that a set is dense simple iff it is recursively enumerable, coinfinite and the sequence $a_0, a_1, \ldots$ of its non-elements in ascending order grows faster than every recursive function. It is known that such sets Z exist [35].

Now one defines that $x \,\zeta\, y$ iff there is an n with $a_n \leqslant \min\{x,y\} \leqslant \max\{x,y\} < a_{n+1}$. This relation is positive, as $x \,\zeta\, y$ is equivalent to

$$\forall z\, [\min\{x,y\} < z \leqslant \max\{x,y\} \Rightarrow z \in Z]$$

which is an r.e. condition. Furthermore, in coincidence with the notation used in this paper, each a_n is the least element of its equivalence class and the a_n are the ascending limits of the approximations $a_{n,t}$ which are the non-elements (in ascending order) of the set Z_t of the first t elements enumerated into Z, so that $Z_0 = \emptyset$ and $a_{n,0} = n$. As Z is coinfinite, there are infinitely many a_n's and so ζ induces infinitely many equivalence classes.

Theorem 28. *There is a confidently learnable ζ-family but no finitely learnable ζ-family.*

Proof. First one defines $B_n = \{x : x \,\zeta\, n \vee x \,\zeta\, a_n \vee n \leqslant x \leqslant a_n\}$. Note that $n \leqslant a_{n,0} \leqslant a_n$ and thus the minimum of B_n is the maximum of all a_m with $a_m \leqslant n$. Furthermore, each two different sets B_n and B_m have the different maxima a_n and a_m (among different a_i's), respectively, and therefore the so defined family is an r.e. one-one family of ζ-closed sets. So it is a ζ-family. A confident learner conjectures on an input sequence σ, whenever it exists, the least n such that $range(\sigma) \subseteq B_{n,|\sigma|}$; however, if such a set does not exist since none of the finitely many sets with $\min(B_{n,|\sigma|}) \leqslant \min(range(\sigma))$ contains $range(\sigma)$, then the learner conjectures B_0. Note that all B_n are η-finite and each number is only contained in finitely many B_n. Thus, whenever there is an n for which B_n contains all the data in the text then the learner converges to an index of B_n for the least such n and when no such n exists then the learner converges to an index of B_0. Hence the learner is confident and learns the family.

Second let $E_0, E_1, \ldots$ be any ζ-family and assume by way of contradiction that it is finitely learnable. Then for each E_n there is a finite subset $C_n \subseteq E_n$ such that the finite learner conjectures some index different from ? after having seen the elements C_n; the mapping, call it f, from n to an explicit list of the set C_n is recursive and so is g defined by $g(n) = \max\{\max(C_m) : m \leqslant 2^n\}$. Also the function g is recursive and by choice of Z, there is an n with $g(n) < a_n$. It follows that the $2^n + 1$ sets E_m with the indices satisfying $m \leqslant 2^n$ are all learnt based on data which is contained in $[a_0, a_1, \ldots, a_{n-1}]$ and therefore there are two distinct numbers $m, k \leqslant 2^n$ such that the ζ-closure of C_m and C_k are the same sets. It follows that both sets C_m and C_k must be subsets of both E_m and E_k and so the finite learner must err when learning the sets E_m and E_k. Hence the given ζ-family is not finitely learnable. □

7 Weakly Confident and Class-Preserving Learning

As Osherson, Stob and Weinstein [37] had already observed, confidence is a real restriction on explanatory or behaviourally correct learning; in particular, even the relatively simple family of all finite sets is not confidently behaviourally correctly learnable. In the present section, we examine a variant of confident learning known as *weak confident learning*, where the learner's sequence of hypotheses is required to converge only on each text whose range is contained in some language to be learnt. Such a learning property may be desirable in situations where a text for some target language may have missing data due to noise [9]. One can see, for example, that since the family of all finite sets is closed under taking subsets, it is weakly confidently explanatorily learnable.

We also study the effect of requiring *class-preservingness* of the learner's hypothesis space on the strength of confident as well as weakly confident learning. Our first observation is that there are positive equivalence relations η for which class-preserving learning is indeed a restriction on confidence with respect to the class of η-families.

Theorem 29. *There is a positive equivalence relation η for which there is a η-finite family that is confidently learnable but not class-preservingly confidently learnable.*

Proof. Take η to be the positive equivalence relation $x \,\eta\, y \Leftrightarrow x = y$. Now consider the class $\mathcal{L}$ comprising all 2-element sets $\{x, y\}$ with $x, y \in \mathbb{N}$, $x \neq y$, and all singletons $\{z\}$ with $z \in K'$. The class $\mathcal{L}$ has a uniformly r.e. one-one numbering $L_0, L_1, L_2, \ldots$, which is defined in stages as follows. Let $L_{e,s}$ denote the set of elements enumerated into L_e up to stage s.

– Stage 0. Set $L_{i,0} = \emptyset$ for all $i \in \mathbb{N}$.
– Stage $s+1$. For $i = 0, 1, \ldots, s+1$, $L_{i,s+1}$ is updated according to the following case distinction.
  Case 1: $i = \langle 2n, \langle t, u\rangle\rangle$ for some $n, t, u \in \mathbb{N}$. If $L_{i,s}$ contains two distinct elements, set $L_{i,s+1} = L_{i,s}$. Else, check whether (i) $\varphi_{n,s+1+u}^{K_{s+1+t}}(n)\downarrow$ holds, (ii) $L_{\langle 2n,\langle t',u'\rangle\rangle,s+1} \neq \{n\}$ for all $\langle t', u'\rangle < \langle t, u\rangle$, and (iii) $L_{\langle 2n,\langle t'',u''\rangle\rangle,s} \neq \{n\}$ for all $\langle t'', u''\rangle > \langle t, u\rangle$. If all three conditions (i), (ii) and (iii) are satisfied, then set $L_{i,s+1} = \{n\}$, else set $L_{i,s+1} = L_{i,s} \cup \{z, z'\}$ for the first pair $\{z, z'\}$ found such that $L_{i,s} \cup \{z, z'\}$ has size 2 and $L_{e,s+1} \neq L_{i,s} \cup \{z, z'\}$ for all $e < i$ and $L_{e,s} \neq L_{i,s} \cup \{z, z'\}$ for all $e > i$.
  Case 2: $i = \langle 2n + 1, t\rangle$ for some $n, t \in \mathbb{N}$. If $L_{i,s} = \emptyset$, let $L_{i,s+1}$ be the first pair $\{x, y\}$ found such that $x \neq y$, $L_{e,s+1} \neq \{x, y\}$ for all $e < i$ and $L_{e,s} \neq \{x, y\}$ for all $e > i$.

For each $n \in \mathbb{N}$ with $n \in K'$, there is at least one pair $\langle t, u\rangle$ such that $\varphi_{n,u'}^{K_{t'}}(n)\downarrow$ holds for all $t' \geqslant t$ and all $u' \geqslant u$, so there is a first pair $\langle t'', u''\rangle$ such that conditions (i), (ii) and (iii) in Case 1 are satisfied with respect to $\langle 2n, \langle t'', u''\rangle\rangle$ at almost all stages and $L_{\langle 2n,\langle t'',u''\rangle\rangle} = \{n\}$. The construction of $L_{i,s+1}$ in either Case 1 or Case 2 also ensures that there is no $e \neq \langle 2n, \langle t'', u''\rangle\rangle$ such that $L_e = \{n\}$. For each $n \in \mathbb{N}$ with $n \notin K'$ and every pair $\langle t, u\rangle$, $\varphi_{n,u'}^{K_{t'}}(n)\uparrow$ holds for infinitely many $t' \geqslant t$ and infinitely many $u' \geqslant u$, so according to Case 1, $L_{\langle 2n,t,u\rangle}$ will be defined as a pair of distinct elements. For each $n \in \mathbb{N}$ and $t \in \mathbb{N}$, according to Case 2, $L_{\langle 2n+1,t\rangle}$ will be defined as a pair of distinct elements. The final clauses in Cases 1 and 2 ensure that each pair of distinct elements occurs exactly once in the numbering.

Define a confident learner M as follows. On input σ, if $|\operatorname{content}(\sigma)| \leqslant 2$, then M conjectures content(σ); if $|\operatorname{content}(\sigma)| > 2$, then M outputs a default index. Then M will converge on any text T containing at most two elements to a canonical index for content(T), and so it explanatorily learns every language in $\mathcal{L}$; if, on the other hand, content(T) contains more than two elements, M will converge to a default index. Hence M confidently learns $\mathcal{L}$.

Assume that $\mathcal{L}$ has a class-preserving confident learner N. Then, given any $n \in \mathbb{N}$, one can check via oracle K whether or not $n \in K'$ as follows: feed N with the text $n \circ n \circ n \circ \ldots$ until N converges to some index e (this can be checked using oracle K). Then $x \in K'$ holds iff $W_e = \{n\}$ (the latter condition can also be checked using oracle K), and so $K' \leqslant_T K$, but this is impossible. ☐

The remainder of this section will consider a less stringent variant of confidence called *weak confidence*. It will be shown that for every positive equivalence relation η, there is an η-family witnessing the separation between explanatory learning and weak confident learning.

Definition 30. For any given uniformly r.e. family $\mathcal{L} = \{L_0, L_1, \ldots\}$, a text T is said to be *consistent with* $\mathcal{L}$ iff there is an e such that content(T) $\subseteq L_e$. A learner M is *weakly confident* iff M converges on every text that is consistent with the target class $\mathcal{L}$.

We first observe that for any positive equivalence relation η, weak confidence is indeed a less stringent learning constraint than confidence with respect to the class of η-families.

Theorem 31. *Let η be any given positive equivalence relation. Then $\mathcal{A}_\eta$ is weakly confidently learnable but not confidently learnable.*

Proof. It has been proven in Theorem 8 that the ascending family is not confidently learnable. Furthermore, the explanatory learner constructed in the proof of Theorem 8 is in fact weakly confident. To see this, note that for any text T such that $\mathrm{content}(T) \subseteq A_e$ for some $e \in \mathbb{N}$, there is a least a_n such that $a_n \notin \mathrm{content}(T)$, and so the explanatory learner constructed in the proof of Theorem 8 will, on T, converge to an index for A_n. ☐

The next result separates weak confident learning from its class-preserving variant.

Theorem 32. *Let η be any given positive equivalence relation. The class $\mathcal{L}$ consisting of all members of $\mathcal{A}_\eta$ and all $[a_{n+1}]$ with $n \in K'$ is an η-family that is weakly confidently learnable but not class-preservingly weakly confidently learnable.*

Proof. Note that $\mathcal{L}$ has a uniformly r.e. one-one numbering. First, a uniformly r.e. numbering of $\mathcal{L}$ may be obtained as follows: for all $n, s, t \in \mathbb{N}$, set

$$L_{\langle 2n,s,t\rangle} = A_n,$$
$$L_{\langle 2n+1,s,t\rangle} = \begin{cases} [a_{n+1}] \text{ if } \forall s' \geqslant s \, \forall t' \geqslant t \, [\varphi_{n,t'}^{K_{s'}}(n) \downarrow]; \\ A_{n+2} \ \text{ otherwise.} \end{cases}$$

Thus $\mathcal{L}$ is a uniformly r.e. superfamily of $\mathcal{A}_\eta$, and so by Proposition 9, $\mathcal{L}$ is also an η-family.

Let M be a learner that works as follows on input σ. If $a_0 \in \mathrm{content}(\sigma)$ and n is the least number for which $a_{n,|\sigma|} \notin \mathrm{content}(\sigma)$, M outputs a canonical index for A_n; if $a_0 \notin \mathrm{content}(\sigma)$ and $\mathrm{content}(\sigma) \neq \emptyset$, then M conjectures $[\min(\mathrm{content}(\sigma))]$; if $\mathrm{content}(\sigma) = \emptyset$, then M just outputs a default index. Then M will make infinitely many mind changes on a given text T only if, for all $n \in \mathbb{N}$, a_n appears in T; thus, since no language in $\mathcal{L}$ contains $\{a_n : n \in \mathbb{N}\}$, M will converge on every text for any $L \in \mathcal{L}$. Furthermore, given any text T for A_n, where $n \in \mathbb{N}$, there is an s large enough so that $a_{n,t} = a_n$, $a_m \in \mathrm{content}(T[t])$ and $a_{m,t} = a_m$ whenever $t \geqslant s$ and $m < n$, and so $M(T[t])$ is equal to a canonical index for A_n whenever $t \geqslant s$. Similarly, given any text T' for $[a_{n+1}]$, where $n \in \mathbb{N}$, a_0 will not appear in T' and so M will converge to an index for $[x]$, where x is the least element of $[a_{n+1}]$ to appear in T'. Thus M is a weakly confident learner of $\mathcal{L}$.

Assume, by way of contradiction, that $\mathcal{L}$ has a class-preserving weakly confident learner N. Then, given any $n \in \mathbb{N}$, one can check via oracle K whether or not $n \in K'$ as follows. Feed N with the text $a_{n+1} \circ a_{n+1} \circ a_{n+1} \circ \ldots$ until N converges to an index (since N is weakly confident and $a_{n+1} \in A_{n+2} \in \mathcal{L}$, N must converge on this text), say e. Then $n \in K'$ holds iff $a_{n+1} \in W_e \wedge a_0 \notin W_e$; the latter condition can be checked using oracle K. This implies $K' \leqslant_T K$, a contradiction. ☐

The next two results clarify the relationship between explanatory learning and weakly confident learning of one-one η-closed families for any given positive equivalence relation η.

Theorem 33. *Let η be any given positive equivalence relation. If there are at least two η-infinite r.e. sets, then there is an η-family that is explanatorily learnable but not weakly confidently learnable.*

Proof. Let B be any η-infinite r.e. set such that $B \neq \mathbb{N}$. Fix some $b \notin B$. Let η_s be the equivalence relation obtained by enumerating η-equivalent pairs of elements of B for s steps and keeping the relation closed transitively and reflexively. Furthermore, $b_{n,s}$ is the n-th element in the order of the enumeration satisfying that all elements before it in the enumeration are not η_s equivalent to $b_{n,s}$. Now set

$$B_0 = \mathbb{N},$$
$$B_{n+1} = \{x : \exists m < n\, \exists s\, [x\, \eta_s\, b_{m,s}]\}$$

for all $n \in \mathbb{N}$. As was shown in the proof of Theorem 16, $\mathcal{L} := \{B_n : n \in \mathbb{N}\}$ is explanatorily learnable. On the other hand, $\mathcal{L}$ is not weakly confidently learnable. For, if N were a weakly confidently learner for $\mathcal{L}$, then one could build a text T' that is the limit of sequences $\sigma_0 \circ \sigma_1 \circ \ldots \circ \sigma_s$ of strings, where, for all n, σ_n is the first string in B^*_{n+1} found such that N on input $\sigma_0 \circ \ldots \circ \sigma_n$ outputs an r.e. index for B_{n+1}; note that since $B_{n+i} \subset B_{n+i+1}$ for all i, such strings $\sigma_0, \sigma_1, \ldots$ must exist. Then T' is a text consistent with $\mathcal{L}$ (since $\mathbb{N} \in \mathcal{L}$) on which N makes infinitely many mind changes, which contradicts the fact that N is weakly confident. ☐

Theorem 34. *Let η be any given positive equivalence relation. If $\mathbb{N}$ is the only η-infinite r.e. set, then there is an η-family that is explanatorily learnable but not weakly confidently learnable.*

Proof. Let $\mathcal{L}$ be the class containing all η-closed sets A_{n+1} and $[a_1, a_{n+2}]$ for $n \in \mathbb{N}$. An explanatory learner M may work as follows. On input σ, M checks whether or not $a_0 \in \text{content}(\sigma)$. If $a_0 \in \text{content}(\sigma)$, then M determines the least n such that $a_{n+1,|\sigma|} \notin \text{content}(\sigma)$ and outputs a canonical index for A_{n+1}. If $a_0 \notin \text{content}(\sigma)$, then M determines the least n such that $a_{n+2,|\sigma|} \in \text{content}(\sigma)$ (if such an n exists) and outputs a canonical index for $[a_1, a_{n+2}]$; if no such n exists, then M just outputs any default index.

Now assume for the sake of a contradiction that $\mathcal{L}$ has a weakly confident learner N. Since $[a_1] \subseteq [a_1, a_{n+2}] \in \mathcal{L}$ for all $n \in \mathbb{N}$, N must (by the property of weak confidence) converge on any text for $[a_1]$. In particular, there exists a stabilising sequence σ for $[a_1]$, that is, a sequence $\sigma \in [a_1]^*$ such that for all $\tau \in [a_1]^*$, $N(\sigma) = N(\sigma\tau)$ (see Section 3). Set $A := \{x : \exists \tau \in ([a_1, x]^*\, [N(\sigma) \neq N(\sigma\tau)]\}$. Since N explanatorily learns $[a_1, a_{n+2}]$ for all $n \in \mathbb{N}$, A must contain a_{n+2} for all but at most one $n \in \mathbb{N}$. On the other hand, for all $x \in [a_1]$, $x \notin A$ by the choice of σ. Thus, taking $B = \bigcup_{x \in A}[x]$, B is an η-infinite r.e. set that is not equal to $\mathbb{N}$, contrary to the hypothesis of the theorem. ☐

8 Monotonic, Conservative and Class-Preserving Learning

In this section, we turn our attention to *monotonic* and *conservative* learning of η-families for any given positive equivalence relation η. Monotonicity and conservativeness are fairly natural learning constraints that have been well-studied in the context of uniformly recursive families [30] as well as more general classes of r.e. sets [26]. Monotonicity says that one mainly learns by generalisation, that is, the current concepts are replaced by more general ones. However, in its pure form as strongly monotonic, the principle is limited, but occurs in the various hull-concepts in mathematics where one learns the convex or linear hull of the data points seen so far and therefore behaviourally correctly learns all concepts which are the hull of finitely many data points [41]. Weak monotonicity deviates from this by allowing a nonmonotone step whenever the previous hypothesis is inconsistent with the data observed. Conservativeness is the principle to stick to a concept until it is proven to be wrong. Together with consistency, this is considered as an attempt to enforce natural behaviour of learners and to rule out coding tricks like padding, as they are considered cheating when aiming for learners of good complexity classes. However, Case and Koetzing [10] proved that this approach does not satisfy the expectations. Nevertheless, consistency and conservativeness remain two quite popular and well-studied notions in inductive inference. Also in the study of indexed families by Lange and Zeugmann [30], it was shown that for various properties like monotonicity, one can achieve these when allowing class-comprising hypothesis spaces to formulate intermediate hypotheses. For example the class of all empty, one-element and three-element sets do not have a consistent strong-monotonic and class-preserving learner, however, sacrificing any of these three properties allows such a learner.

We first revisit the ascending family, which was briefly studied in Section 4, and prove that this class is in fact conservatively explanatorily learnable for every positive equivalence relation.

Theorem 35. *For every positive equivalence relation η such that $[x]$ is infinite for each x, $\mathcal{A}_\eta$ is conservatively explanatorily learnable.*

Proof. It has already been shown in Theorem 8 that this family is explanatorily learnable. One may modify the explanatory learner constructed in the proof of this theorem to obtain a conservative explanatory learner N. The main idea is to use the fact that there are infinitely many equivalence classes and delay the change in hypothesis until the learner N can verify that the range of the current input contains an element not contained in the last conjecture copied from the explanatory learner in Theorem 8. In detail: Let the conjecture of the learner in Theorem 8 at step t be A_{n_t} and $f(n_t, t)$ be a recursive function such that

$$W_{f(n_t,t)} = \begin{cases} A_{n_t,s-1}, & \text{if there is a least step } s > t \text{ such that for some } m \leqslant n_t,\ a_{m,s} \neq a_{m,s+1}; \\ A_{n_t}, & \text{otherwise.} \end{cases}$$

We note that the programme $f(n_t, t)$ does not enumerate $a_{n_t,t}$ since $a_{n_t,t} \notin A_{n_t}$ and if there were a least $s > t$ and some $m < n_t$ such that $a_{m,t} \eta_s a_{n_t,t}$ holds, then the first clause in the definition of $W_{f(n_t,t)}$ would be satisfied at some step $s' \leqslant s$ and so $f(n_t, t)$ would not enumerate elements of $[a_{m,t}]_{t'} - [a_{m,t}]_{s'-1}$ for any $t' \geqslant s'$. At the initial step, the conservative explanatory learner N outputs $f(n_1, 1)$. Suppose the conjecture of N at step s is $f(n_t, t)$. At step $s + 1$, the conjecture of N is determined according to the following case distinction.

Case 1: The input contains $a_{n_t,t}$. Then N outputs $f(n_{s+1}, s + 1)$. Since $a_{n_t,t} \notin W_{f(n_t,t)}$, N maintains conservativeness.

Case 2: There is some $m \leqslant n_t$ such that $a_{m,t} \neq a_{m,s+1}$. Thus the first clause in the definition of $W_{f(n_t,t)}$ must be satisfied for some least $s' > t$ and some $m \leqslant n_t$; the first such pair

(s', m) can then be effectively found. If the input contains some element not in $A_{n_t,s'-1}$, then N outputs $f(n_{s+1}, s+1)$; otherwise, N repeats its previous conjecture $f(n_t, t)$. Hence N maintains conservativeness.

Case 3: Neither Case 1 nor Case 2. Then N repeats the conjecture $f(n_t, t)$; it maintains conservativeness in this case as well.

Suppose a text for the language A_n is presented to N. If N at some step outputs $f(n_t, t)$ for some $n_t < n$, then the input will eventually contain an element not in A_{n_t}; thus, N will output $f(n_{t'}, t')$ for some $t' > t$ according to Case 1 or Case 2. Since the sequence $\langle a_{n_t,t} : t \in \mathbb{N}\rangle$ is monotonically non-decreasing and converges to a_n, the sequence of conjectures of the learner in Theorem 8 will converge to a conjecture for A_n, and thus for large enough t, N will conjecture $f(n_t, t)$ for some t such that $n_t = n$. □

The next result shows that for any positive equivalence relation η, there is an η-subfamily of the ascending family for η that has no class-preserving conservatively behaviourally correct learner.

Theorem 36. *For every positive equivalence relation η, there is an η-family contained in $\mathcal{A}_\eta$ that is neither class-preservingly conservatively behaviourally correctly learnable nor class-preservingly strongly monotonically behaviourally correctly learnable.*

Proof. Let $\mathcal{L}$ contain all A_{2n} with $n \in \mathbb{N}$ and all A_{2n+1} where W_n is finite. This class is r.e., as one can choose $L_{\langle n,0\rangle} = A_{2n}$ and $L_{\langle n,m+1\rangle} = A_{2n+1}$ in the case that $|W_n| \leqslant m$ and A_{2n+2} in the case that $|W_n| > m$. This enumeration is clearly r.e. and one can conclude by Proposition 10 that there is an η-family for this class. Now assume that there is a class-preserving conservatively behaviourally correct learner for the family. If this learner does not output on any sequence from A^*_{2n+1} a hypothesis containing a_{2n} (which can be computed using K and which is in $A_{2n+1} - A_{2n}$), then the learner does not learn A_{2n+1} and thus W_n is infinite. If this learner outputs on some sequence from A^*_{2n+1} a hypothesis containing a_{2n} then one tests with K whether the hypothesis also contains a_{2n+1}. If so, again W_n is infinite, as due to class-preservingness of the learner, the only possible hypotheses issued are A_{2n+2} or a superset of this set and then the learner cannot learn A_{2n+1} by conservativeness. If not then the only hypothesis possible is A_{2n+1} by class-preservingness of the learner and therefore A_{2n+1} is in the class and W_n is finite. Thus the existence of a class-preserving conservatively behaviourally correct learner for $\mathcal{L}$ allows to decide with the halting problem whether a given set W_n is finite or infinite and that is impossible; thus such a learner cannot exist. Finally, we note that this proof by contradiction also works if one assumes that the learner is class-preservingly strongly monotonically behaviourally correct. □

Theorem 37. *For every positive equivalence relation η, there is an η-subfamily of $\mathcal{A}_\eta$ that is conservatively explanatorily learnable but not class-preservingly conservatively explanatorily learnable.*

Proof. Let f be a limit recursive function approximable from below that dominates all recursive functions (for an example of such a function f, see the proof of [39, Lemma 7.3(a)]); without loss of generality, assume that f is strictly monotonically increasing. Define, for all $n \in \mathbb{N}$,

$$B_n := \{x : \exists m \leqslant f(n)\, [x\, \eta\, a_m]\}.$$

By Theorem 35, $\mathcal{L} = \{B_n : n \in \mathbb{N}\}$, being a subclass of $\mathcal{A}_\eta$, is conservatively explanatorily learnable. Assume that M were a class-preserving conservative explanatory learner for $\mathcal{L}$. Let g be a recursive function such that for all $n \in \mathbb{N}$, $g(n)$ is one more than the maximum of the range of the first $\sigma \in \mathbb{N}^*$ found for which M makes n mind changes on σ and when $M(\tau_1) \neq M(\tau_2)$ and $M(\tau_2) \neq M(\tau_3)$ for prefixes τ_1, τ_2, τ_3 of σ with $\tau_1 \subseteq \tau_2 \subseteq \tau_3$, then $\text{content}(\tau_1) \subseteq W_{M(\tau_3)}$. That is, $g(n) = 1 + \max(\text{content}(\sigma))$, where σ is the first $\sigma \in \mathbb{N}^*$ found such that there exist n prefixes $\sigma[i_1], \ldots, \sigma[i_n]$ of σ (where $\sigma[j]$ denotes the prefix of σ of length j) such that $M(\sigma[i_j]) \neq M(\sigma[i_j + 1])$ for each $j \in \{1, \ldots, n\}$ and whenever M makes at least two mind changes between prefixes $\sigma[j_1]$ and $\sigma[j_2]$ of σ, then the conjecture of M after the second mind change includes the range of the prefix $\sigma[j_1]$. Since f dominates all recursive functions, there is some n such that $g(3n+3) \leqslant f(n)$. Note that all hypotheses that M issues on prefixes of σ as in the computation of $g(3n+3)$ are based on subsets of B_n. There are more than n different hypotheses output by M on prefixes of σ and each must be equal to some B_m with $m < n$, which is impossible. $\square$

The next theorem considers the question: given any positive equivalence relation η, can one always find an η-family witnessing the separation of monotonic behaviourally correct and explanatory learning? It turns out that this is indeed the case iff there are at least two distinct η-infinite r.e. sets.

Theorem 38. *Let η be any given positive equivalence relation. The following conditions are equivalent.*

(1) There is an η-family that is explanatorily learnable but not monotonically behaviourally correctly learnable.
(2) There is an η-family that is class-preservingly weakly monotonically explanatorily learnable but not monotonically behaviourally correctly learnable.
(3) There is an η-infinite r.e. set that is different from $\mathbb{N}$.

Proof. It is immediate that (2) $\Rightarrow$ (1). It will be shown that (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3). Suppose $\mathbb{N}$ is the only η-infinite r.e. set. Consider any η-family $\mathcal{L}$ that is explanatorily learnable, as witnessed by some learner P. First, Proposition 15 implies that for all $L \in \mathcal{L}$, L is the union of finitely many equivalence classes of η. Let Q be a learner such that on input σ, Q conjectures $\bigcup_{x \in \text{content}(\sigma)} [x]$. Note that Q is strongly monotonic by construction. Moreover, given any $L \in \mathcal{L}$, Q on any text for L will conjecture L in the limit because L is composed of only finitely many equivalence classes.

(3) $\Rightarrow$ (2). Suppose there is an η-infinite r.e. set B_0 such that for some $k \in \mathbb{N}$, $a_k \notin B_0$. Let $\mathcal{L}$ be the η-family comprising B_0 and $B_{n+1} := A_{n+k+1} = \bigcup_{m \leqslant n+k} [a_m]$ for all $n \in \mathbb{N}$.

The following learner M explanatorily learns $\mathcal{L}$ and is both weakly monotonic and class-preserving. On input σ:

Case 1: $a_k \notin \text{content}(\sigma)$. M conjectures B_0.

Case 2: $a_k \in \text{content}(\sigma)$.

Case 2.1: There is an $n \in \mathbb{N}$ such that $a_{n+k+1,|\sigma|} \notin \text{content}(\sigma)$ and $a_{m,|\sigma|} \in \text{content}(\sigma)$ whenever $m \leqslant n+k$. M outputs an index e such that

$$W_e = \begin{cases} \bigcup_{m \leqslant n+k} [a_{m,|\sigma|}] & \text{if } \{a_m : m \leqslant k\} \subseteq \\ & \{a_{m,|\sigma|} : m \leqslant n+k\}; \\ B_1 & \text{otherwise.} \end{cases}$$

Case 2.2: Neither Case 1 nor Case 2.1. M conjectures B_1.

The programme for W_e starts by enumerating all elements of $\bigcup_{m \leqslant n+k}[a_{m,|\sigma|}]$; if at some stage it is found that $\{a_{m,|\sigma|} : m \leqslant n+k\}$ does not contain all elements of $\{a_m : m \leqslant k\}$, then the programme enumerates all elements of B_1, which includes all elements of $\bigcup_{m \leqslant n+k}[a_{m,|\sigma|}]$ enumerated so far. Note that M always outputs an index for B_0 or for a set $\bigcup_{m \leqslant n}[a_m]$ with $n \geqslant k$, and hence it is class-preserving. By Case 1 and the fact that $a_k \notin B_0$, M on any text for B_0 will always conjecture B_0. In particular, M on any text for B_0 strongly monotonically explanatorily learns B_0. If M is presented with a text T for some B_{n+1}, then there is an s large enough so that $\{a_m : m \leqslant n+k\} \subseteq \text{content}(T[s])$ and for all $t \geqslant s$ and $m \leqslant n+k+1$, $a_{m,t} = a_m$. By Case 2, for all $t \geqslant s$, M on $T[t]$ will conjecture B_{n+1}. Hence M is a class-preserving explanatory learner of $\mathcal{L}$. To see that M is also weakly monotonic on every text for any set B_{n+1}, let τ_1 and τ_2 be any two sequences in $(B_{n+1} \cup \{\#\})^*$ such that τ_2 extends τ_1, and consider the following case distinction.

Case i: $a_k \notin \text{content}(\tau_2)$. Then M conjectures B_0 on both τ_1 and τ_2.

Case ii: $a_k \notin \text{content}(\tau_1)$ and $a_k \in \text{content}(\tau_2)$. Since M on τ_1 will conjecture B_0 but $a_k \notin B_0$, the weak monotonicity condition is satisfied vacuously.

Case iii: $a_k \in \text{content}(\tau_1)$. Suppose M on τ_1 conjectures B_1. As $a_k \in \text{content}(\tau_2)$, M on τ_2 must output a set B_i with $i \geqslant 1$, and therefore $W_{M(\tau_1)} \subseteq W_{M(\tau_2)}$. Suppose M on τ_1 conjectures some set B_{n+1} with $n \geqslant 1$. Then the first clause in the definition of W_e in Case 2.1 must have been applied, so there is some $n' \geqslant n$ and subset $\{i_0, \ldots, i_{n+k}\}$ of $\{0, \ldots, n'+k\}$ with $i_0 < \ldots < i_{n+k}$ such that

(1) For all $j \in \{0, \ldots, n+k\}$, $a_{i_j,|\tau_1|} = a_j$;
(2) $\{a_{m,|\tau_1|} : m \leqslant n'+k\} \subseteq \text{content}(\tau_1)$.

Note that for all $s, t \in \mathbb{N}$ with $s \leqslant t$, $\{a_m : m \in \mathbb{N}\} \subseteq \{a_{m,t} : m \in \mathbb{N}\} \subseteq \{a_{m,s} : m \in \mathbb{N}\}$. Thus, there is some $\ell \geqslant n+k$ such that $a_{\ell,|\tau_2|} = a_{n+k} = a_{i_{n+k},|\tau_1|}$, from which it follows that

$$\begin{aligned} \{a_m : m \leqslant n+k\} &\subseteq \{a_{m,|\tau_2|} : m \leqslant \ell\} \\ &\subseteq \{a_{m,|\tau_1|} : m \leqslant i_{n+k}\} \\ &\subseteq \text{content}(\tau_1) \\ &\subseteq \text{content}(\tau_2). \end{aligned}$$

Consequently, M on τ_2 will conjecture a set containing $\bigcup_{m \leqslant n+k}[a_m]$, and so $W_{M(\tau_1)} \subseteq W_{M(\tau_2)}$.

Now consider any behaviourally correct learner N of $\mathcal{L}$. It will be shown that N cannot be monotonic. Since N behaviourally correctly learns B_0, there is some $\sigma \in B_0^*$ such that $W_{N(\sigma)} =$

B_0. Pick some $n \in \mathbb{N}$ such that $a_{n+k+1} \in B_0$ and content$(\sigma) \subseteq B_{n+1}$; such an n exists because σ is finite and B_0 is an η-infinite r.e. set. As N also behaviourally correctly learns B_{n+1}, there is some $\tau \in B_{n+1}^*$ such that $W_{N(\sigma\circ\tau)} = B_{n+1}$. But $a_{n+k+1} \in (B_0 \cap B_{n+2}) - (B_{n+1} \cap B_{n+2}) = (W_{N(\sigma)} \cap B_{n+2}) - (W_{N(\sigma\circ\tau)} \cap B_{n+2})$ implies that $W_{N(\sigma)} \cap B_{n+2} \not\subseteq W_{N(\sigma\circ\tau)} \cap B_{n+2}$, and since content$(\sigma\tau) \subseteq B_{n+2} \in \mathcal{L}$, it may be concluded that N does not learn B_{n+2} monotonically – a contradiction. $\square$

9 Conclusion

The present work studied how the relations between the most basic inference criteria for learning from text are impacted when the only classes to be considered for learning are uniformly r.e. one-one families of sets which are closed under a given positive equivalence relation η. One considers the chain of implications finitely learnable $\Rightarrow$ confidently learnable $\Rightarrow$ explanatorily learnable $\Rightarrow$ vacillatorily learnable $\Rightarrow$ behaviourally correctly learnable which is immediate from the definitions. When choosing η as the explicitly constructed ϑ from Theorem 20, the implication from explanatorily learnable to vacillatorily learnable becomes an equivalence and the criterion of confidently learnable becomes void, that is, no ϑ-family satisfies it. For the positive equivalence relation ζ from Description 27, there is a ζ-family which is confidently learnable, but none which is finitely learnable. Furthermore, in the case that a finitely learnable η-family exists for some η, then there is also a confidently learnable η-family which is not finitely learnable. For all choices of η, the implications from confident to explanatory learning and from vacillatory to behaviourally correct learning cannot be reversed and the class of all η-closed r.e. set is an η-family which cannot be learnt behaviourally correctly. The learnability of η-families with additional constraints such as monotonicity and conservativeness was also investigated; one interesting finding was that the separability of learning criteria for η-families is sometimes contingent on the existence of at least two η-infinite sets.

Besides investigating the situation for further learning criteria, future work can investigate to which extent the results generalise to arbitrary uniformly r.e. families of η-closed sets. Here one would get that the family of all η-singletons is r.e. and finitely, thus confidently learnable: the learner generates an index of the η-equivalence class to be learnt from the first data-item observed and keeps this hypothesis forever. So one has one more level in the learning hierarchy. However, the collapse of vacillatory learning to explanatory learning for the constructed equivalence relation ϑ generalises to uniformly r.e. families.

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