Technical Report
on http://www.arxiv.org
Volume 00, Number 0, Pages 000–000
S ———(XX)0000-0

ON THE ORDER DIMENSION OF LOCALLY COUNTABLE PARTIAL ORDERINGS

KOJIRO HIGUCHI, STEFFEN LEMPP, DILIP RAGHAVAN, AND FRANK STEPHAN

Abstract. We show that the order dimension of any locally countable partial ordering $(P, <)$ of size κ^+, for any κ of uncountable cofinality, is at most κ. In particular, this implies that it is consistent with ZFC that the dimension of the Turing degrees under partial ordering can be strictly less than the continuum.

1. Introduction

This paper arose from a question posed by the first to the third author at the conference on Computability Theory and Foundations of Mathematics at Tokyo in 2015 regarding a set-theoretic property of a computability-theoretic structure:

Question 1.1 (Higuchi)**.** What is the order dimension of the Turing degrees regarded as a partial order?

Higuchi had already shown that this dimension must be uncountable and asked whether it is the continuum. This paper provides a partial answer: It is consistent with ZFC that the dimension is less than the continuum. But Higuchi's question raised a number of related questions to which we give some answers in this paper, all about the order dimension of locally countable partial orders. We start with some definitions:

Definition 1.2 (Dushnik, Miller [4], Ore [10])**.** Given a partial order $\mathcal{P} = (P, \prec)$, the *order dimension* (or simply *dimension*) of $\mathcal{P}$ is the smallest cardinality of a collection of linearizations of $\prec$ which intersect to $\prec$.

So, for example, the dimension of a linear order is clearly 1, and the dimension of an antichain is easily seen to be 2. It is also easy to see that the dimension of an infinite partial order $\mathcal{P}$ can be at most $|P|$: For each pair $x, y \in P$ with $y \not\preceq x$, fix a linearization $<_{x,y}$ of $\prec$ with $x <_{x,y} y$.

2010 *Mathematics Subject Classification.* Primary: 06A06, 03E04; Secondary: 03D28.

Key words and phrases. partial ordering, dimension, Turing degrees.

The first author was partially supported by Grant-in-Aid for JSPS Fellows. The second author was partially supported by NSF grant DMS-160022; he also wishes to thank the National University of Singapore for its hospitality during his sabbatical year when most of this research was carried out. The third author was partially supported by the Singapore Ministry of Education's research grant number MOE2017-T2-2-125. The fourth author was partially supported in part by the Singapore Ministry of Education Academic Research Fund Tier 2 grant MOE2016-T2-1-019 / R146-000-234-112. The authors would like to thank the referee for carefully reading the paper and for giving constructive comments.

Definition 1.3. Call a partial order $\mathcal{P} = (P, \prec)$ *locally finite* or *locally countable*, respectively, if for each $x \in P$, the set $\{y \in P \mid y \prec x\}$ is finite or countable, respectively.

Partial orders which are locally finite or locally countable, respectively are also often said to have the *finite predecessor property* or the *countable predecessor property.*

The order dimension of the Turing degrees can be thought of as a new cardinal invariant because it is between $\aleph_1$ and $2^{\aleph_0}$.

Definition 1.4. Let $\langle \mathcal{D}, <_T \rangle$ denote the class of Turing degrees equipped with the ordering of Turing reducibility. The cardinal $\mathfrak{dim}_T$ denotes the order dimension of $\langle \mathcal{D}, <_T \rangle$.

Since $\mathcal{D}$ has cardinality $2^{\aleph_0}$, $\mathfrak{dim}_T \leq 2^{\aleph_0}$, and by Higuchi's Proposition 4.3, $\aleph_1 \leq \mathfrak{dim}_T$. Thus the cardinal $\mathfrak{dim}_T$ sits between $\aleph_1$ and $2^{\aleph_0}$, like many of the standard cardinal invariants of the continuum such as $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{a}$, etc. The reader is referred to Blass [2] for a general survey of combinatorial cardinal characteristics of the continuum. Of course, under CH, $\mathfrak{dim}_T = \aleph_1 = 2^{\aleph_0}$. In this paper, we will show that $\mathfrak{dim}_T$ is smaller than $2^{\aleph_0}$ "most of the time". More precisely, we will show that there are only three circumstances under which $\mathfrak{dim}_T = 2^{\aleph_0}$ is possibly consistent: $2^{\aleph_0} = \aleph_1$, or $2^{\aleph_0}$ is a limit cardinal (either singular or weakly inaccessible), or $2^{\aleph_0}$ is the successor of a singular cardinal of countable cofinality.

We will now restate a result of Kierstead and Milner [6] on the dimension of locally finite partial orders in Section 2, state some results of ours on the dimension of locally countable partial orders and degree structures from computability theory in Section 3 and in Section 4, respectively, and close with some examples in Section 5.

2. The dimension of $[\kappa]^{<\omega}$

Kierstead and Milner [6] have determined the order dimension of $\left\langle [\kappa]^{<\omega}, \subsetneq \right\rangle$, which is universal among locally finite posets of cardinality κ, i.e., every locally finite poset $\mathcal{P} = (P, \prec)$ with $|P| = \kappa$ embeds into $\left\langle [\kappa]^{<\omega}, \subsetneq \right\rangle$ by assigning $a \in P$ to $\{f(b) : b \in P, b \preceq a\}$, where $f : P \to \kappa$ is any injection. Thus the result in this section provides an upper bound for every locally finite poset.

Definition 2.1. For any infinite cardinal κ,

$$\log_2(\kappa) = \min\{\lambda : 2^\lambda \geq \kappa\}$$

We state their theorem in this section for completeness:

Theorem 2.2 (Kierstead, Milner [6])**.** [1] *Let $\kappa \geq \omega$ be any cardinal. Then the order dimension of $\left\langle [\kappa]^{<\omega}, \subsetneq \right\rangle$ is $\log_2(\log_2(\kappa))$.*

It follows from Definition 2.1 that $\log_2(\log_2(\kappa))$ is the minimal λ such that $2^{2^\lambda} \geq \kappa$. Theorem 2.2 says in particular that $\left\langle [\omega_1]^{<\omega}, \subsetneq \right\rangle$ and indeed $\left\langle \left[2^{2^\omega}\right]^{<\omega}, \subsetneq \right\rangle$ have countable dimension. But $\left\langle \left[\left(2^{2^\omega}\right)^+\right]^{<\omega}, \subsetneq \right\rangle$ has uncountable dimension.

[1] The authors would like to thank Jing Zhang for pointing out this paper to us.

3. The dimension of locally countable partial orderings

The setting of locally countable partial orders (for which the Turing degrees and many other degree structures from computability theory form natural examples) is quite a bit more complicated.

Even though the following lemma is not needed for the proof of our main result, it provides information about arbitrary locally countable partial orders which is likely to be useful in their analysis. It proves the existence of a ranking function on them.

Lemma 3.1. *Suppose $\mathcal{P} = (P, \prec)$ is any locally countable partial ordering. Then there is a function $r : P \to \eta \cdot \omega_1$ such that for all $x, y \in P$, $x \prec y$ implies $r(x) < r(y)$. (Here, η is the order type of the rational numbers and $\eta \cdot \omega_1$ is the order product of these two order types under the antilexicographical ordering.)*

Proof. If P is empty, then there is nothing to prove. So we assume that P is nonempty. Let $\kappa = |P|$. From an enumeration $\langle y_\beta : \beta < \kappa \rangle$ of P, we construct a cofinal sequence $\langle x_\alpha : \alpha < \lambda \rangle$ of elements of P by recursion as follows: Let $x_0 = y_0$, and for $\alpha > 0$, let $x_\alpha = y_\beta$ for the least β such that $y_\beta \not\preceq x_{\alpha'}$ for any $\alpha' < \alpha$. (The recursion stops at the least ordinal λ when there is no such β.)

Now, for each $\alpha < \lambda$, let $A_\alpha = \{x \in P : x \preceq x_\alpha\}$ and recall that by local countability, each A_α is countable. Now define r by recursion on $\alpha < \lambda$: Let $r \restriction A_0$ map A_0 into $\eta \cdot \{0\}$ using any linearization of $\prec \restriction A_0^2$. For α with $0 < \alpha < \lambda$, assume that $r \restriction \left(\bigcup_{\alpha' < \alpha} A_{\alpha'}\right)$ has been defined. Find a countable set $B_\alpha \subseteq \alpha$ such that

$$A_\alpha \cap \bigcup_{\alpha' < \alpha} A_{\alpha'} = A_\alpha \cap \bigcup_{\alpha' \in B_\alpha} A_{\alpha'}.$$

Fix $\gamma < \omega_1$ such that

$$r'' \left(\bigcup_{\alpha' \in B_\alpha} A_{\alpha'} \right) \subseteq \eta \cdot \gamma.$$

Now extend the definition of r to $A_\alpha \setminus \left(\bigcup_{\alpha' \in B_\alpha} A_{\alpha'}\right)$ by mapping this set into $\eta \cdot \{\gamma\}$ using any linearization of $\prec$ on this set. It is clear that such countable γ must exist because $\bigcup_{\alpha' \in B_\alpha} A_{\alpha'}$ is countable, giving us the desired map r. $\square$

We will now establish a bound on the order dimension of locally countable partial orders of size θ in terms of certain cardinal characteristics of θ. The notion of a separating family of functions is first introduced.

Definition 3.2. Let X be a set and $\mathcal{F} \subseteq 2^X$. We say that $\mathcal{F}$ *separates countable subsets of X from points* if for any countable $B \subseteq X$ and $x \in X \setminus B$, there exists $f \in \mathcal{F}$ so that $f''B \subseteq \{0\}$ and $f(x) = 1$.

It will be shown below that there is a close connection between separating families and order dimension. Moreover separating families are related to almost disjoint families and to weak P-families.

Definition 3.3. Let $\kappa \geq \omega$ be a cardinal. $[\kappa]^{\mathrm{cf}(\kappa)}$ denotes $\{A \subseteq \kappa : |A| = \mathrm{cf}(\kappa)\}$. Sets $A, B \in [\kappa]^{\mathrm{cf}(\kappa)}$ are said to be *almost disjoint or a.d.* if $|A \cap B| < \mathrm{cf}(\kappa)$. A family $\mathscr{A} \subseteq [\kappa]^{\mathrm{cf}(\kappa)}$ is *almost disjoint* or *a.d.* if the members of $\mathscr{A}$ are pairwise a.d.

Let $\mathcal{P}(\kappa)$ denote the power set of κ. A family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is said to be a *weak P-family* if for any countable $\mathcal{B} \subseteq \mathcal{F}$ and $E' \in \mathcal{F} \setminus \mathcal{B}$, there exists a finite set $F \subseteq \kappa$ such that

$$\forall E \in \mathcal{B} \, [E \cap F \neq E' \cap F] \, .$$

Observe that Definition 3.3 deviates from the usual definition of an almost disjoint family on κ in that we do not require members of $\mathscr{A}$ to have size κ. The members of $\mathscr{A}$ have size precisely $\operatorname{cf}(\kappa)$. Observe also that a weak P-family is a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ which, when viewed as a subspace of 2^κ, has the property that every countable subset of $\mathcal{F}$ is relatively closed in $\mathcal{F}$. In other words, every member of $\mathcal{F}$ is a weak P-point in $\mathcal{F}$. This is a well-studied notion in topology.

Definition 3.4. For partial orders $\mathcal{P}_0 = \langle P_0, <_0 \rangle$ and $\mathcal{P}_1 = \langle P_1, <_1 \rangle$, we write $\mathcal{P}_0 \hookrightarrow \mathcal{P}_1$ when there exists an order embedding from $\mathcal{P}_0$ into $\mathcal{P}_1$. In other words, there exists $\psi : P_0 \to P_1$ so that $\forall p, p' \in P_0 \, [p <_0 p' \leftrightarrow \psi(p) <_1 \psi(p')]$.

Definition 3.5. Let θ be an uncountable cardinal. Define the following cardinals:

$$\mathfrak{la}(\theta) = \min \left\{ \lambda : \operatorname{cf}(\lambda) > \aleph_0 \text{ and } \exists \mathscr{A} \subseteq [\lambda]^{\operatorname{cf}(\lambda)} \, [|\mathscr{A}| \geq \theta \text{ and } \mathscr{A} \text{ is a.d.}] \right\};$$
$$\mathfrak{ls}(\theta) = \min \left\{ |\mathcal{F}| : \mathcal{F} \subseteq 2^\theta \text{ and } \mathcal{F} \text{ separates countable subsets of } \theta \text{ from points} \right\};$$
$$\mathfrak{lw}(\theta) = \min \left\{ \lambda : \exists \mathcal{F} \subseteq \mathcal{P}(\lambda) \, [|\mathcal{F}| \geq \theta \text{ and } \mathcal{F} \text{ is a weak P-family}] \right\};$$
$$\mathfrak{le}(\theta) = \min \left\{ \lambda : \text{there is an order embedding from } \left\langle [\theta]^{<\omega_1}, \subsetneq \right\rangle \text{ into } \langle \mathcal{P}(\lambda), \subsetneq \rangle \right\}.$$

It is obvious that $\log_2(\theta) \leq \mathfrak{lw}(\theta) \leq \theta$. Note that $\mathfrak{la}(\theta) \leq \theta^+$, and if $\operatorname{cf}(\theta) > \aleph_0$, then $\mathfrak{la}(\theta)$ is at most θ. We prove the following relationships.

Lemma 3.6. *Let θ be any uncountable cardinal. Then $\mathfrak{ls}(\theta)$ is the minimal λ so that there is a sequence $\langle g_\alpha : \alpha < \theta \rangle$ such that for each $\alpha < \theta$, $g_\alpha : \lambda \to 2$, and for every countable $B \subseteq \theta$ and every $\beta \in \theta \setminus B$, there exists $\xi < \lambda$ so that $\forall \alpha \in B \, [g_\alpha(\xi) = 0]$ and $g_\beta(\xi) = 1$.*

Proof. Set $\mu = \mathfrak{ls}(\theta)$. For one direction, suppose that $\mathcal{F} \subseteq 2^\theta$, $\mathcal{F}$ separates countable subsets of θ from points, and $|\mathcal{F}| = \mu$. Enumerate $\mathcal{F}$ as $\langle f_\xi : \xi < \mu \rangle$. Define a sequence $\langle g_\alpha : \alpha < \theta \rangle$ of functions from μ to 2 by stipulating that for each $\alpha < \theta$ and each $\xi < \mu$, $g_\alpha(\xi) = f_\xi(\alpha)$. If $B \subseteq \theta$ is countable and $\beta \in \theta \setminus B$, then since $\mathcal{F}$ separates countable subsets of θ from points, there exists $\xi < \mu$ with $f_\xi'' B \subseteq \{0\}$ and $f_\xi(\beta) = 1$. Now for any $\alpha \in B$, $g_\alpha(\xi) = f_\xi(\alpha) = 0$, while $g_\beta(\xi) = f_\xi(\beta) = 1$, as needed. This shows that the minimal λ as in the statement of the lemma is less than or equal to μ.

For the other direction, suppose λ and $\langle g_\alpha : \alpha < \theta \rangle$ are such that for each $\alpha < \theta$, $g_\alpha : \lambda \to 2$, and that for every countable $B \subseteq \theta$ and every $\beta \in \theta \setminus B$, there exists $\xi < \lambda$ so that $\forall \alpha \in B \, [g_\alpha(\xi) = 0]$ and $g_\beta(\xi) = 1$. For each $\xi < \lambda$ define $f_\xi : \theta \to 2$ by stipulating that for all $\alpha \in \theta$, $f_\xi(\alpha) = g_\alpha(\xi)$. Let $\mathcal{F} = \{f_\xi : \xi < \lambda\}$. Then $\mathcal{F} \subseteq 2^\theta$ and $|\mathcal{F}| \leq \lambda$. Also, if $B \subseteq \theta$ is countable and $\beta \in \theta \setminus B$, then there exists $\xi < \lambda$ so that $\forall \alpha \in B \, [g_\alpha(\xi) = 0]$ and $g_\beta(\xi) = 1$, whence $\forall \alpha \in B \, [f_\xi(\alpha) = 0]$ and $f_\xi(\beta) = 1$. So $\mathcal{F}$ separates countable subsets of θ from points. This shows that $\mu \leq |\mathcal{F}| \leq \lambda$, completing the proof. $\square$

Lemma 3.7. *For any uncountable cardinal θ, $\mathfrak{ls}(\theta) = \mathfrak{lw}(\theta) = \mathfrak{le}(\theta) \leq \mathfrak{la}(\theta)$.*

Proof. Set $\lambda = \mathfrak{lw}(\theta)$, $\mu = \mathfrak{ls}(\theta)$, and $\nu = \mathfrak{le}(\theta)$. By Lemma 3.6, there is a sequence $\langle g_\alpha : \alpha < \theta \rangle$ such that for each $\alpha < \theta$, $g_\alpha : \mu \to 2$, and that for every countable $B \subseteq \theta$ and every $\beta \in \theta \setminus B$, there exists $\xi < \mu$ so that $\forall \alpha \in B\, [g_\alpha(\xi) = 0]$ and $g_\beta(\xi) = 1$. This property implies in particular that $g_\alpha \neq g_\beta$ for any distinct $\alpha, \beta < \theta$. Therefore, letting $\mathcal{F} = \{A_\alpha : \alpha < \theta\}$, where $A_\alpha = \{\xi < \mu : g_\alpha(\xi) = 1\}$, we have that $\mathcal{F} \subseteq \mathcal{P}(\mu)$ and $|\mathcal{F}| = \theta$. Moreover, it is clear that $\mathcal{F}$ is a weak P-family. It follows that $\lambda \leq \mu$.

For the other direction, fix a family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$ so that $|\mathcal{F}| \geq \theta$ and $\mathcal{F}$ is a weak P-family. By Lemma 3.6, to prove $\mu \leq \lambda$, it is enough to prove that there is a sequence $\langle g_\alpha : \alpha < \theta \rangle$ such that for each $\alpha < \theta$, $g_\alpha : \lambda \to 2$, and that for every countable $B \subseteq \theta$ and every $\beta \in \theta \setminus B$, there exists $\xi < \lambda$ so that $\forall \alpha \in B\, [g_\alpha(\xi) = 0]$ and $g_\beta(\xi) = 1$. We noted earlier that $\log_2(\theta) \leq \lambda \leq \theta$. In particular, λ must be an infinite cardinal. Hence $|L| = \lambda$, where $L = \left\{\langle s, H \rangle : s \in [\lambda]^{<\omega} \wedge H \subseteq \mathcal{P}(s)\right\}$. So it suffices to find a sequence $\langle g_\alpha : \alpha < \theta \rangle$ of functions $g_\alpha : L \to 2$ satisfying the above property. Now $\mathcal{F}$ contains a subfamily of size equal to θ. Let $\langle E_\alpha : \alpha < \theta \rangle$ be a one-to-one enumeration of any such subfamily. Fixing $\alpha < \theta$, define $g_\alpha : L \to 2$ by stipulating, for all $\langle s, H \rangle \in L$, that $g_\alpha(\langle s, H \rangle) = 1$ if and only if $E_\alpha \cap s \in H$. To verify the required property, fix a countable $B \subseteq \theta$ and $\beta \in \theta \setminus B$. There is a finite set $F \subseteq \lambda$ such that $\forall \alpha \in B\, [E_\alpha \cap F \neq E_\beta \cap F]$ because $\mathcal{F}$ is a weak P-family. Let $H = \{F \cap E_\beta\}$. Note $\langle F, H \rangle \in L$ because $F \in [\lambda]^{<\omega}$ and $H \subseteq \mathcal{P}(F)$. Since $E_\beta \cap F \in H$, $g_\beta(\langle F, H \rangle) = 1$. On the other hand, for any $\alpha \in B$, $E_\alpha \cap F \notin H$ because $E_\alpha \cap F \neq E_\beta \cap F$. Hence $g_\alpha(\langle F, H \rangle) = 0$. So we have $\forall \alpha \in B\, [g_\alpha(\langle F, H \rangle) = 0]$, as required. So we have proved both $\mu \leq \lambda$ and $\lambda \leq \mu$, thus showing $\mu = \lambda$.

Next, we prove that $\mu = \nu$. For one direction, let $\mathcal{F} \subseteq 2^\theta$ be a family separating countable subsets of θ from points with $|\mathcal{F}| = \mu$. Define $\psi : [\theta]^{<\omega_1} \to \mathcal{P}(\mathcal{F})$ by setting $\psi(B) = \{f \in \mathcal{F} : \exists \alpha \in B\, [f(\alpha) = 1]\}$. Suppose $B_0, B_1 \in [\theta]^{<\omega_1}$. It is clear that if $B_0 \subseteq B_1$, then $\psi(B_0) \subseteq \psi(B_1)$. Suppose now that $B_0 \not\subseteq B_1$ and choose $\alpha \in B_0 \setminus B_1$. Since $\mathcal{F}$ separates countable subsets of θ from points, there is $f \in \mathcal{F}$ so that $f'' B_1 \subseteq \{0\}$ and $f(\alpha) = 1$, whence $f \in \psi(B_0) \setminus \psi(B_1)$. This proves that ψ is an embedding of $\left\langle [\theta]^{<\omega_1}, \subsetneq \right\rangle$ into $\langle \mathcal{P}(\mathcal{F}), \subsetneq \rangle$, which implies that $\nu \leq |\mathcal{F}| = \mu$.

For the other direction, fix an embedding $\psi : [\theta]^{<\omega_1} \to \mathcal{P}(\nu)$. For each $\xi \in \nu$, define a function $f_\xi : \theta \to 2$ by stipulating that $f_\xi(\alpha) = 1$ if and only if $\xi \in \psi(\{\alpha\})$, for all $\alpha \in \theta$. Then $\mathcal{F} = \{f_\xi : \xi \in \nu\} \subseteq 2^\theta$. We check that $\mathcal{F}$ separates countable subsets of θ from points. Indeed, let $B \subseteq \theta$ be countable and let $\alpha \in \theta \setminus B$. Since $\{\alpha\} \not\subseteq B$, $\psi(\{\alpha\}) \not\subseteq \psi(B)$. Choose $\xi \in \psi(\{\alpha\}) \setminus \psi(B)$. For each $\beta \in B$, $\{\beta\} \subseteq B$, implying that $\psi(\{\beta\}) \subseteq \psi(B)$, and thus $\xi \notin \psi(\{\beta\})$, whence $f_\xi(\beta) = 0$. Thus $\forall \beta \in B\, [f_\xi(\beta) = 0]$. On the other hand, $f_\xi(\alpha) = 1$ because $\xi \in \psi(\{\alpha\})$. This proves that $\mathcal{F}$ separates countable subsets of θ from points. Therefore $\mu \leq |\mathcal{F}| \leq \nu$, completing the proof that $\mu = \nu$.

Finally suppose that $\kappa = \mathfrak{la}(\theta)$. Then $\mathrm{cf}(\kappa) > \aleph_0$ and there is an a.d. family $\mathscr{A} \subseteq [\kappa]^{\mathrm{cf}(\kappa)}$ with $|\mathscr{A}| \geq \theta$. Then $\mathscr{A} \subseteq \mathcal{P}(\kappa)$, and we will check that $\mathscr{A}$ is a weak P-family. To this end, fix a countable subfamily $\mathcal{B} \subseteq \mathscr{A}$ and $E' \in \mathscr{A} \setminus \mathcal{B}$. For each $E \in \mathcal{B}$, $|E \cap E'| < \mathrm{cf}(\kappa)$ by almost disjointness. Since $\mathcal{B}$ is countable, and $\aleph_0 < \mathrm{cf}(\kappa)$ and $\mathrm{cf}(\kappa)$ is a regular cardinal, $\left|\bigcup_{E \in \mathcal{B}} (E \cap E')\right| < \mathrm{cf}(\kappa)$. We can choose $\xi \in E' \setminus \bigcup_{E \in \mathcal{B}} (E \cap E')$ because $|E'| = \mathrm{cf}(\kappa)$. Then $F = \{\xi\} \subseteq \kappa$ is finite and for any $E \in \mathcal{B}$, $E' \cap F = \{\xi\} \neq \emptyset = E \cap F$, as required. This proves that $\lambda \leq \kappa$. $\square$

Corollary 3.8. *For any uncountable cardinal θ, the order dimension of $\left\langle [\theta]^{<\omega_1}, \subsetneq \right\rangle$ is at most $\mathfrak{ls}(\theta)$.*

Proof. Lemma 3.7 shows that $\left\langle [\theta]^{<\omega_1}, \subsetneq \right\rangle \hookrightarrow \langle \mathcal{P}(\mathfrak{ls}(\theta)), \subsetneq\rangle$. As shown in Komm [7], it is well-known that the order dimension of the power set of A under set inclusion is exactly $|A|$. Therefore, the order dimension of $\left\langle [\theta]^{<\omega_1}, \subsetneq \right\rangle$ is at most the order dimension of $\langle \mathcal{P}(\mathfrak{ls}(\theta)), \subsetneq\rangle$, which is precisely $\mathfrak{ls}(\theta)$. □

We note that if θ is uncountable, then Lemma 4.2 in Section 4 tells us the order dimension of $\left\langle [\theta]^{<\omega_1}, \subsetneq \right\rangle$ is at least ω_1, since any family of ω_1-many singletons forms a strongly independent set. Now, let us turn to the order dimension of locally countable partial orders.

Theorem 3.9. *Let θ be an uncountable cardinal and $\mathcal{P} = (P, \prec)$ a locally countable partial order with $|P| = \theta$. Then the order dimension of $\mathcal{P}$ is at most $\mathfrak{ls}(\theta)$. In particular, $\mathfrak{dim}_T \leq \mathfrak{ls}(2^{\aleph_0})$.*

Proof. Let $\{x_\alpha : \alpha < \theta\}$ be a one-to-one enumeration of P. Define $\psi : P \to [\theta]^{<\omega_1}$ by stipulating that for each $x \in P$, $\psi(x) = \{\alpha < \theta : x_\alpha \preceq x\}$, which is a countable subset of θ because of the local countability of $\mathcal{P}$. It is easily checked that ψ is an embedding of $\mathcal{P}$ into $\left\langle [\theta]^{<\omega_1}, \subsetneq \right\rangle$. Therefore, the order dimension of $\mathcal{P}$ is at most the order dimension of $\left\langle [\theta]^{<\omega_1}, \subsetneq \right\rangle$, which is at most $\mathfrak{ls}(\theta)$. □

Theorem 3.9 says that $\mathfrak{dim}_T$ is bounded above by the minimal number of functions from $\mathbb{R}$ to 2 that are necessary for separating countable subsets of $\mathbb{R}$ from points. Recall the well-known fact that there is a countable family of functions from $\mathbb{R}$ to 2 which separates finite subsets of $\mathbb{R}$ from points. Kumar and Raghavan [8] have recently proved that it is consistent that $\mathfrak{dim}_T < \mathfrak{ls}(2^{\aleph_0})$, so the upper bound proved in Theorem 3.9 is not sharp. Kumar and Raghavan show that $\mathfrak{dim}_T$ can actually be characterized as the minimal number of linear orders on $\mathbb{R}$ that are necessary for separating countable subsets of $\mathbb{R}$ from points, which is a notion introduced in [8].

Lemma 3.10. *Suppose κ is any cardinal such that* $\mathrm{cf}(\kappa) > \omega$. *Then $\mathfrak{la}(\kappa^+) \leq \kappa$.*

Proof. By hypothesis $\mathrm{cf}(\kappa) > \aleph_0$. So it suffices to show that there exists an a.d. family $\mathscr{A} \subseteq [\kappa]^{\mathrm{cf}(\kappa)}$ with $|\mathscr{A}| \geq \kappa^+$. This actually follows from well-known results like Kunen [9, Theorem 1.2] in the case when κ is regular and some basic facts of PCF theory, like [1, Theorem 2.23], in the case when κ is singular. We give details for completeness.

Suppose first that κ is regular. By Theorem 1.2 in [9], there is a sequence $\langle E_\xi : \xi < \kappa^+\rangle$ such that:

(1) $\forall \xi < \kappa^+ \, [E_\xi \in [\kappa]^\kappa]$;
(2) $\forall \xi, \zeta < \kappa^+ \, [\xi \neq \zeta \implies |E_\xi \cap E_\zeta| < \kappa]$.

This is exactly as required.

Next suppose that κ is singular. Let $\mu = \mathrm{cf}(\kappa)$. By hypothesis, μ is an uncountable regular cardinal. By [1, Theorem 2.23], there exist sequences $\langle \lambda_\alpha : \alpha < \mu\rangle$ and $\langle f_\xi : \xi < \kappa^+\rangle$ satisfying the following conditions:

(3) for each $\alpha < \mu$, $\lambda_\alpha < \kappa$;
(4) for each $\xi < \kappa^+$, f_ξ is a function, $\mathrm{dom}(f_\xi) = \mu$, and $\forall \alpha < \mu \, [f_\xi(\alpha) \in \lambda_\alpha]$; and

(5) for all $\xi < \zeta < \kappa^+$, $\{\alpha < \mu : f_\xi(\alpha) \geq f_\zeta(\alpha)\}$ is bounded in μ.

Note that for each $\xi < \kappa^+$, $f_\xi \subseteq \mu \times \kappa$. Furthermore $|f_\xi| = \mu$, and if $\zeta \neq \xi$, then $|f_\zeta \cap f_\xi| \leq |\{\alpha < \mu : f_\zeta(\alpha) = f_\xi(\alpha)\}| < \mu$. Therefore $\{f_\xi : \xi < \kappa^+\} \subseteq [\mu \times \kappa]^{\mathrm{cf}(\kappa)}$ is an a.d. family and $|\{f_\xi : \xi < \kappa^+\}| = \kappa^+$. Since $|\mu \times \kappa| = \kappa$, this is as required. $\square$

We can now state and prove the main result of this section.

Theorem 3.11. *Suppose κ is any cardinal such that $\mathrm{cf}(\kappa) > \omega$ and $\mathcal{P} = (P, \prec)$ is any locally countable partial order of size κ^+. Then $\mathcal{P}$ has dimension at most κ.*

Proof. Since κ^+ is an uncountable cardinal, Theorem 3.9 applies and implies that the dimension of $\mathcal{P}$ is at most $\mathfrak{ls}(\kappa^+)$. By Lemma 3.10, $\mathfrak{la}(\kappa^+) \leq \kappa$, and by Lemma 3.7, $\mathfrak{ls}(\kappa^+) = \mathfrak{lw}(\kappa^+) \leq \mathfrak{la}(\kappa^+)$. Therefore the dimension of $\mathcal{P}$ is at most κ. $\square$

As mentioned earlier, Theorem 3.11 is not sharp in the sense that it is consistent to have locally countable partial orders of size κ^+ whose order dimension is strictly smaller than $\mathfrak{ls}(\kappa^+)$, and therefore strictly smaller than $\mathfrak{la}(\kappa^+)$. However, we do not know whether $\mathfrak{ls}(\kappa^+) = \mathfrak{la}(\kappa^+)$ for every cardinal κ with $\mathrm{cf}(\kappa) > \omega$. The following corollary is immediate from Theorems 3.11.

Corollary 3.12. *If the order dimension of some locally countable partial order of size continuum is $2^{\aleph_0}$, then either* CH *holds, or $2^{\aleph_0}$ is a limit cardinal, or $2^{\aleph_0}$ is the successor of a singular cardinal of countable cofinality.* $\square$

Kumar and Raghavan [8] have shown that the cases besides CH are also realized. In other words, they have produced models where $\mathfrak{dim}_T = 2^{\aleph_0} = \aleph_{\omega_1}$, $\mathfrak{dim}_T = 2^{\aleph_0} = \aleph_{\omega+1}$, and $\mathfrak{dim}_T = 2^{\aleph_0}$ where $2^{\aleph_0}$ is weakly inaccessible. Therefore Corollary 3.12 is sharp in the sense that ZFC does not eliminate any of the possibilities not covered by Corollary 3.12.

4. The dimension of some degree structures from computability theory

In this section, we state some results on the dimension of three degree structures from computability theory, the Turing degrees, the Medvedev degrees and the Muchnik degrees.

We start with the Turing degrees since the original motivation for our investigation was determining the dimension of the Turing degrees under the partial ordering, for which we obtain two partial results.

Definition 4.1. For a partial order $\mathcal{P} = (P, \prec)$ and a subset S of P, we say that S is a *strongly independent* antichain if for any subset T of S with $|T| < |S|$ and for any $x \in S \setminus T$, there is an upper bound $y \in P$ of T with $y \not\succeq x$.

Lemma 4.2. *Let $\mathcal{P} = (P, \prec)$ be a partial order with a strongly independent antichain S. Then the dimension of $\mathcal{P}$ is at least $|S|$.*

Proof. We provide a proof by contradiction. Suppose that the dimension ρ of $\mathcal{P}$ is less than $|S|$. Let $\hat{S} \subseteq S$ satisfy that $|\hat{S}| > \rho$ is a successor cardinal and let $\kappa = |\hat{S}|$. Choose linear extensions $\{<_\alpha\}_{\alpha<\rho}$ of $\prec$ whose intersection is $\prec$.

Fix $x \in \hat{S}$. Let $\{T_\alpha\}_{\alpha<\kappa}$ be any increasing sequence of subsets of $\hat{S}$ of cardinality $< |\hat{S}|$ such that $\bigcup_{\alpha<\kappa} T_\alpha = \hat{S} \setminus \{x\}$. By the strong independence, we can find a sequence $\{y_\alpha\}_{\alpha<\kappa}$ of upper bounds of T_α's such that for each $\alpha < \kappa$, $x \not\leq y_\alpha$, which means that there exists $\beta < \rho$ such that $y_\alpha <_\beta x$ by the choice of $\{<_\beta\}_{\beta<\rho}$. Since $\rho < \kappa$ and κ is regular (or finite), there must exist a fixed $\beta < \rho$ such that $y_\alpha <_\beta x$ holds for unboundedly many $\alpha < \kappa$. By the choice of $\{T_\alpha\}_{\alpha<\kappa}$, every element of $\hat{S}$ distinct from x is in almost all $\{T_\alpha\}_{\alpha<\kappa}$, and therefore, $y <_\beta x$ must hold for each $y \in \hat{S} \setminus \{x\}$. Hence we conclude that for any $x \in \hat{S}$, there exists $\beta < \rho$ such that $y <_\beta x$.

Since again $\rho < \kappa = |\hat{S}|$, there must exist a common $\beta < \rho$ and distinct $x_0, x_1 \in \hat{S}$ such that $y <_\beta x_i$ holds for each $y \in \hat{S} \setminus \{x_i\}$ and $i \in \{0, 1\}$. But this gives us $x_0 <_\beta x_1 <_\beta x_0$, and hence $x_0 <_\beta x_0$, a contradiction. □

We can now state two partial results about the dimension of the Turing degrees as a partial order:

Proposition 4.3 (Higuchi)**.** *The dimension of the Turing degrees is uncountable.*

Proof. By Sacks [12], every locally countable partial order of cardinality $\aleph_1$ is embeddable into the Turing degrees. Thus it is enough to find such a partial order whose dimension is at least $\aleph_1$. Let us consider the suborder $(P, \subsetneq)$ of $\mathcal{P}(\aleph_1)$ under set inclusion whose underlying set is

$$P = \{\{\alpha\} : \alpha < \omega_1\} \cup \{\{\gamma : \gamma < \beta\} \setminus \{\alpha\} : \alpha, \beta < \omega_1\}.$$

It is easy to see that $\{\{\alpha\} : \alpha < \omega_1\}$ is a strongly independent antichain of cardinality $\aleph_1$ in the partial order $(P, \subsetneq)$, and therefore, the dimension of $(P, \subsetneq)$ is at least $\aleph_1$ by Lemma 4.2. □

Theorem 4.4. *It is consistent with ZFC that the dimension of the Turing degrees is strictly less than the continuum. More precisely, the dimension of the Turing degrees can be continuum only if either* CH *holds, or* $2^{\aleph_0}$ *is a limit cardinal, or* $2^{\aleph_0}$ *is the successor of a singular cardinal of countable cofinality.*

Proof. The first part is a direct corollary of Theorems 3.11: Work in a model in which $2^{\aleph_0} = \aleph_2$ and apply the theorem with $\kappa = \aleph_1$. The second part follows from Corollary 3.12. □

It is worth noting that Corollary 3.12 actually implies that if $2^{\aleph_0} \leq \aleph_2$, then *every* locally countable partial order of size $2^{\aleph_0}$ has order dimension at most $\aleph_1$. In particular, under the Proper Forcing Axiom, $\mathfrak{dim}_T = \aleph_1$. This is rather unusual for a cardinal invariant.

We now turn our attention to the Medvedev and the Muchnik degrees. By a cardinality argument, the dimension of both can be at most $2^{2^{\aleph_0}}$.

Theorem 4.5 (Pouzet [11])**.** *Let* $\mathcal{P} = (P, \prec)$ *be a partial order. Then the dimension of* $(\text{IniSeg}(\mathcal{P}), \subset))$ *is the chain covering number of* $\mathcal{P}$*, where* $\text{IniSeg}(\mathcal{P})$ *is the set of initial segments of* $\mathcal{P}$ *and the* chain covering number *of* $\mathcal{P}$ *is the least cardinal* κ *such that there exists a set* $\mathcal{C}$ *of chains of* $\mathcal{P}$ *with* $|C| = \kappa$ *and* $\bigcup \mathcal{C} = P$.

It is known that the Muchnik degrees are isomorphic to the set of all final segments of the Turing degrees ordered by $\supset$. Note that the dimension of a partial order does not change if we reverse the order. Thus the dimension of the Muchnik degrees is the chain covering number of the Turing degrees, which is $2^{\aleph_0}$ since there are at most $2^{\aleph_0}$ many Turing degrees and the Turing degrees contain an antichain of size $2^{\aleph_0}$. Since the Muchnik degrees can be seen a suborder of the Medvedev degrees, the dimension of the latter is at least $2^{\aleph_0}$, and by a cardinality argument at most $2^{2^{\aleph_0}}$.

We can thus determine the dimension of the Muchnik degrees in ZFC but leave open the following questions:

Question 4.6. [2]

(1) Does ZFC determine the dimension of the Turing degrees?
(2) Does ZFC determine the dimension of the Medvedev degrees?

5. Some examples

One of the goals of our paper was to find out whether the order dimension of the structure of Turing degrees is $\aleph_1$ in all models of ZFC. More generally, we can pose the following

Question 5.1. [3] Is there a partially ordered set $(F, <)$ such that, in all models of ZFC, the following holds?

(1) The order $<$ is locally countable;
(2) for every at most countable subset $G \subseteq F$ there is an upper bound x of G, i.e., $y \leq x$ for all $y \in F$;
(3) the cardinality of F is $2^{\aleph_0}$; and
(4) the order dimension of $(F, <)$ is $\aleph_1$.

Each of the examples below will satisfy three of the properties but the fourth at most partially. Note that additional set-theoretical assumptions like $2^{\aleph_0} \in \{\aleph_1, \aleph_2\}$ might make the examples have all the desired properties.

Example 5.2. Let F be the set of all hereditarily countable sets and let $<$ be the transitive closure of the element-relation. As every set bounds only countably many other sets in a hereditary way, the partial order is locally at most countable, and it is well-founded. Furthermore, every at most countable ordinal α can be identified with the at most countable set $\{\beta : \beta < \alpha\}$, and these sets are hereditarily countable. The set $\{\{\{\alpha\}\} : \alpha \text{ is an at most countable ordinal}\}$ is an antichain of size $\aleph_1$ consisting of hereditarily countable sets; as each of its at most countable subsets is hereditarily countable, these sets witness that the antichain is indeed a strong antichain and that therefore the order dimension is at least $\aleph_1$. It is known that the set of hereditarily countable sets has cardinality $2^{\aleph_0}$.

Example 5.3. Let F be the set of all functions f from an ordinal $\alpha < \omega_1$ into ω_1; order F by letting $f < g$ iff there are ordinals $\alpha, \beta \in \mathrm{dom}(g)$ such that for all $\gamma \in \mathrm{dom}(f)$, $f(\gamma) = g(\alpha+1+\gamma)$, $g(\alpha) = g(\beta)$, $f(\gamma) < g(\alpha)$ and $\alpha+1+\mathrm{dom}(f) = \beta$. It is easy to see that $<$ is transitive and locally countable. Furthermore, one can easily see that for at most countably many functions $f_0, f_1, \ldots$, there is a common upper bound g by choosing an ordinal α strictly larger than all ordinals occurring in the f_k, considering an ω-power ω^γ larger than the domains of all f_k, letting the domain of h be $\omega^{\gamma+1}$, and setting $h(\omega^\gamma \cdot k + 1 + \gamma) = f_k(\gamma)$ for all $\gamma \in \mathrm{dom}(f_k)$ and $h(\delta) = \alpha$ for all δ in the domain of h where h is not yet defined. So every at most countable set G of members of F has a common upper bound, which strictly bounds

[2] Kumar and Raghavan [8] recently announced that Question 4.6(1) has a negative answer.

[3] Kumar and Raghavan [8] recently announced that Question 5.1 has a negative answer.

from above exactly the members of G and those members of F which are below a member of G. The cardinality of F is $2^{\aleph_0}$. Furthermore, the set of all functions with domain $\{0\}$ forms a strong antichain and therefore, the order dimension is at least $\aleph_1$.

Example 5.4. Let F be the set of all subsets of $\mathbb{R} \times \omega_1$ which are the unions of finitely many sets of the form $A_{x,y,z} = \{x\} \times \{u \in \omega_1 : y \leq u \leq z\}$ and $B_{x,y,z} = \{x\} \times \{u \in \omega_1 : y \leq u < z\}$, where $x \in \mathbb{R}$ and $y, z \in \omega_1$, and order this set F by set inclusion. The set $(F, <)$ satisfies all four conditions except for the second, which is weakened to the existence of common upper bounds of finitely many elements.

One can see from the definition that every set of the form $A_{x,y,z}$ or $B_{x,y,z}$ has at most countably many subsets of this form in F; furthermore each member of F is countable. Thus this is a locally countable partially ordered set.

Furthermore, as the finite union of any members of F is again a member of F, one has also that finitely many subsets have an upper bound. However, this does not extend to all countable subsets of F.

The cardinality of F is $2^{\aleph_0}$. The lower bound is seen by looking at all sets $A_{x,0,0}$ with $x \in \mathbb{R}$. The upper bound stems from the fact that there are $2^{\aleph_0}$ many countable subsets of $\mathbb{R}$.

Furthermore, let p, q be rational numbers with $p < q$ and $y \in \omega_1$. For each p, q, y, one defines a linear order $<_{p,q,y}$ as a linear extension of the partial order $<'_{p,q,y}$ defined by $A <'_{p,q,y} B$ iff B has more elements than A of the form (x, y) with $p \leq x \leq q$; note that each set has only finitely many such elements. Now if $B \not\subseteq A$ then there is an $(x, y) \in B \setminus A$ and there are rationals p, q such that x is the unique real number z with $p \leq z \leq q$ and $(z, y) \in A \cup B$. It follows that $A <_{p,q,y} B$. So $<$ is the intersection of all $<_{p,q,y}$ with $p, q \in \mathbb{Q}$ and $y \in \omega_1$ and $p < q$. It follows that the order dimension of $(F, <)$ is at most $\aleph_1$.

Now consider the set C of all $A_{0,y,y}$, which are all singletons. The set C forms a strong antichain, as for every $A_{0,x,x}$ and every at most countable set D of sets of the form $A_{0,y,y}$ with $y \neq x$, there is an upper bound z of all these y. For this upper bound z, now consider the set $E = B_{0,0,x} \cup A_{0,x+1,z}$, which is a superset of all $A_{0,y,y} \in D$ but not a superset of $A_{0,x,x}$. Now by Lemma 4.2, $(F, <)$ has order dimension at least $\aleph_1$, as this is the cardinality of C.

Example 5.5. Given F as in Example 5.4, the subset $G = \{A \in F : A \subseteq \{0\} \times \omega_1\}$ satisfies the first, second and last property, but differs from the third in all models of set theory where $\aleph_1 \neq 2^{\aleph_0}$.

Example 5.6. The set F of all countable subsets of ω_1 with the order of inclusion satisfies the property that every countable subset of F has an upper bound in F and the cardinality is $2^{\aleph_0}$. Furthermore, $(F, <)$ has order-dimension $\aleph_1$.

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(Higuchi) College of Engineering, Nihon University, 1 Nakagawara, Tokusada, Tamuramachi, Koriyama, Fukushima Prefecture 963-8642, JAPAN

Email address: higuchi.koujirou@nihon-u.ac.jp

URL: http://kenkyu-web.cin.nihon-u.ac.jp/Profiles/130/0012920/prof_e.html

(Lempp) Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706-1325, USA

Email address: lempp@math.wisc.edu

URL: http://www.math.wisc.edu/~lempp

(Raghavan, Stephan) Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076, Republic of Singapore

Email address: matrd@nus.edu.sg

URL: https://dilip-raghavan.github.io/math/

Email address: fstephan@comp.nus.edu.sg

URL: http://www.comp.nus.edu.sg/~fstephan/