Invertible Classes

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Abstract. This paper considers when one can invert general recursive operators which map a class of functions $\mathcal{F}$ to $\mathcal{F}$. In this regard, we study four different notions of inversion. We additionally consider enumeration of operators which *cover* all general recursive operators which map $\mathcal{F}$ to $\mathcal{F}$ in the sense that for every general recursive operator Ψ mapping $\mathcal{F}$ to $\mathcal{F}$, there is a general recursive operator in the enumerated sequence which behaves the same way as Ψ on $\mathcal{F}$. Three different possible types of enumeration are studied.

1 Introduction

In computational learning theory and Inductive Inference the main scenario is usually of the form

$$\text{Input} \rightarrow \textbf{?} \rightarrow \text{Output}$$

and the problem is then to find the rules that govern the "Black Box", represented by the question mark, from a known input and an observed output. Often however, we are in a different position. We know the black box and we can see the result, but we are interested in what caused the result. So, in some sense this paper starts where Inductive Inference ends – the process is already known and applied, but we need to reconstruct the input that was used. The diagram

$$\textbf{?} \rightarrow \text{Process} \rightarrow \text{Output}$$

represents this situation. The following is a small list of similar real life situations.

– *Cryptography.* Often the encryption algorithms are known, like the widely used "blowfish" algorithm [8] and we can intercept the encoded message, but can we get the message that resulted in the code?
– *Chemical analysis.* Many chemical processes are known. Assume we have the result of a chemical reaction. Can we find the ingredients that were used?
– *Customer modeling.* There are very good models of human motivation; cf. [6] for example. We can observe customer behaviour. But why did the customer actually buy or not buy the product? Where did he learn about the product and what advertisement measures were effective?

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There are more similar real-life situations. Furthermore there is the possibility that more than one input produces a given output. In such a situations, we might be interested in finding one – the first, the cheapest, the fastest – of the possible inputs which led to the given output.

It might not be reasonable to consider all inputs and outputs but only those which fit into a special context, thus a class $\mathcal{F}$ of such inputs is fixed. It is required that $\mathcal{F}$ contains only total functions and that all input and output are from this class. Therefore we consider mainly $\mathcal{F}$-preserving recursive operators Φ which map every $f \in \mathcal{F}$ to a total function in $\mathcal{F}$. In this paper Φ will mostly be general recursive, that is, map every total function to a total one, but in some special cases we investigate also $\mathcal{F}$-preserving operators which are not general recursive.

Following the above mentioned scenario, we are interested in studying when $\mathcal{F}$-preserving general recursive operators Φ can be inverted, that is, given $\Phi(f)$ as input, for $f \in \mathcal{F}$, when can we find a g such that $\Phi(g) = \Phi(f)$, via some *computable* mechanism? As the class $\mathcal{F}$ might be computationally very difficult to hit, we do not require that g belongs to $\mathcal{F}$ although this property is of course obtainable in the case of recursively enumerable classes. Furthermore, given $\Phi(f)$, possible methods for finding such a g usually work via trial and error, thus we would mostly be using limiting recursive functionals as methods for inverting Φ.

In Section 3 we study four different notions of inversion which form a hierarchy. In the following, let Φ be an $\mathcal{F}$-preserving general recursive operator and $\Psi = \lim_s \Psi_s$ be the limiting recursive operator to invert Φ:

- Ψ weakly inverts Φ iff, for all $f \in \mathcal{F}$, there exists a g such that $\Phi(f) = \Phi(g)$ and for all x, $\lim_s \Psi_s(\Phi(f))(x) = g(x)$;
- Ψ bounded weakly inverts Φ iff Ψ weakly inverts Φ and for all $f \in \mathcal{F}$, $\Psi(\Phi(f)) \leq_T \Phi(f)$;
- Ψ inverts Φ iff Ψ weakly inverts Φ and there are, for every $f \in \mathcal{F}$, only finitely many pairs (x, s) such that $\Psi_s(\Phi(f))(x) \neq \Psi(\Phi(f))(x)$;
- Ψ strongly inverts Φ iff Ψ inverts Φ and Ψ_0 is a general recursive operator.

In the formal Definition 4 (b) below, "strongly inverts" is defined equivalently but in slightly different form. Note that in the case of weakly inverting a function, the requirement $\Psi(\Phi(f)) \leq_T \Phi(f)$ is not automatically guaranteed as Ψ is a limiting process, it is indeed a restriction. The motivation for the requirement is the following: it is a natural constraint to say that one can compute the original input-function from the observed output-function; however one may not be able to perform these computations uniformly for all functions in the range of Φ and therefore may need a limiting-recursive process to invert the data of the observed output. A class $\mathcal{F}$ is called invertible (weakly invertible, strongly invertible, bounded weakly invertible), if one can invert (weakly invert, strongly invert, bounded weakly invert), every $\mathcal{F}$-preserving general recursive operator.

In this paper we will show that above notions of invertibility form a strict hierarchy. Theorem 5 shows that $\mathcal{R}$ is not weakly invertible. Proposition 6 shows that every bounded class is weakly invertible. However, Example 7 shows that the class of binary functions is weakly invertible but not bounded weakly invertible. Theorem 8 extends the result to the class of recursive binary functions. Example 14 gives a class which is bounded weakly invertible but not invertible. Example 12 gives a class which is invertible but not strongly invertible. Examples 10 and 11 show that strong invertibility is not trivial by giving interesting infinite classes of recursive functions which are strongly invertible. In Proposition 16 we show that every recursively enumerable class is strongly invertible.

The question of whether an operator is invertible also depends on the variety of operators that are available. Therefore one might ask how difficult an enumeration has to be so that all possible restrictions of mappings from $\mathcal{F}$ to $\mathcal{F}$, which can be done by general recursive operators, also occur in this enumeration. We call this notion coverability and study it in Section 4.

- An enumeration $\Phi_0, \Phi_1, \ldots$ weakly covers $\mathcal{F}$, iff for every $\mathcal{F}$-preserving general recursive operator Φ, there is an e such that Φ_e is general recursive and Φ_e, restricted to domain $\mathcal{F}$, is the same as Φ.
- Enumeration $\Phi_0, \Phi_1, \ldots$ covers $\mathcal{F}$, iff it weakly covers $\mathcal{F}$ and every Φ_e is total on $\mathcal{F}$.
- Enumeration $\Phi_0, \Phi_1, \ldots$ strongly covers $\mathcal{F}$, iff it weakly covers $\mathcal{F}$ and every Φ_e is general recursive.

$\mathcal{F}$ is (weakly, strongly) coverable, if some recursive enumeration of operators (weakly, strongly) covers $\mathcal{F}$. Note that the recursive enumeration of all recursive operators trivially weakly covers every class $\mathcal{F}$.

Example 23 shows that there is a class which is coverable but not strongly coverable. Coverable classes of recursive functions are quite restrictive: every coverable class of recursive functions is contained in a recursively enumerable class of recursive functions. Example 19 gives a class of binary functions which is strongly coverable, but not bounded weakly invertible. Remark 20 extends this to general classes of functions which are strongly coverable but not weakly invertible.

Proposition 21 shows that even the simple class $\{0^e 1 0^\infty : e \in \mathbb{N}\}$ is not coverable. On the other hand, Example 24 shows that any class of functions which recursively approximates a 1-generic set below the halting problem is coverable. Even though not every recursively enumerable class is coverable, Proposition 25 shows that every recursively enumerable class is covered by some K'-recursive enumeration of operators.

In Section 5 we pay special attention to the class of periodic functions, $\mathcal{F}_{per}$. Let $\Phi_0, \Phi_1, \ldots$ be an acceptable numbering of all recursive operators. Corollary 29 shows that the set $\{e : \Phi_e$ is $\mathcal{F}_{per}$-preserving$\}$ is Π_3-complete.

In Section 6 we consider variants of the notion of inverting. What happens if Φ is not general recursive? Furthermore, given an enumeration of operators, is it possible to invert all of the $\mathcal{F}$-preserving operators in this list on at least some of the functions in their range?

2 Basic Notation

In this section, some basic notation and definitions are introduced. Notation not explained here is standard and follows the textbooks of Odifreddi [7] and Soare [9].

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$ of natural numbers. Let $\varphi_0, \varphi_1, \ldots$ be an acceptable numbering of all partial-recursive unary functions and W_e be the domain of φ_e. $W_{e,s}$ denotes the set of all $x < s$ for which $\varphi_e(x)$ halts within s steps. K denotes the halting problem, $\{e : e \in W_e\}$. K' denotes the halting problem relative to K, that is $\{e : e \in W_e^K\}$ (where W_e^K is the set accepted by the e-th oracle Turing Machine using the oracle K).

Definition 1. Given a function f or a string σ of length at least n, $f[n]$ and $\sigma[n]$ denote the first n elements of f and σ, respectively. Furthermore, λ denotes the empty string which coincides with $f[0]$ and $\sigma[0]$ for all functions f and strings σ.

Remark 2. For several examples, an effective version of Ramsey's Theorem is needed. In particular the following notion is used. An A-recursive 2-*colouring* is an A-recursive function R with the domain $\{(x, y) : x < y\}$ and range $\{\text{false}, \text{true}\}$. The members of the range are called the colours. A set E is 2-*r-cohesive relative to* A iff, for all A-recursive 2-colourings R, there are an $e \in E$ and a colour u such that for all $x, y \in E$ with $e < x < y$, $R(x, y) = u$. One can prove by induction from Ramsey's Theorem, that for every A and infinite B, B has an infinite subset which is 2-r-cohesive relative to A. Hummel and Jockusch [3] give an overview on 2-r-cohesive sets and generalize these notion with respect to the involved parameters.

As in the present paper the main goal is to translate total functions into total functions, one can define recursive operators by the easiest approach and view them as oracle Turing machines following certain restrictions. Odifreddi [7, Section II.3] provides more information on recursive operators and introduces more variants of this model. The steps of the computation are basic units, but a bit more involved as normal Turing machine steps for getting the below convention on querying the oracle.

Definition 3. [9] A *recursive operator* Φ is an oracle Turing machine which takes functions as an oracle. So $\Phi(f)(x)$ is the value of the function computed by Φ at x with oracle f.

Without loss of generality, Φ asks $f(s)$ in the s-th step of its computation and nothing else. Therefore $\Phi(f[s])(x)$ is defined and y iff $\Phi(f)(x) = y$, $x < s$, the computation converges in less than s steps and the computation queries f only below s. Otherwise $\Phi(f[s])(x)$ is undefined.

Φ is a general recursive operator iff $\Phi(f)$, that is, the function $x \mapsto \Phi(f)(x)$, is total for every function f.

3 Inverting Operators

The mathematical model is taken from Inductive Inference which is the recursion-theoretic model of learning theory. In the following let $\mathcal{F}$ denote the class of functions under consideration. The involved agents can be viewed upon as Turing machines which, as they compute functions as a list of pairs of inputs and outputs, run for infinite time reading one input tape, using some computation tapes and writing one output tape. Given a general operator Φ, there are several degrees of inversion.

Definition 4. (a) A general recursive operator Φ is called $\mathcal{F}$-preserving iff it maps every function from $\mathcal{F}$ to $\mathcal{F}$.

(b) Ψ *strongly inverts* Φ if Ψ is a general recursive operator and for every $f \in \mathcal{F}$, there exists a g such that, g is a finite variant of $\Psi(\Phi(f))$ and $\Phi(g) = \Phi(f)$.

(c) Ψ *inverts* Φ if Ψ is a limit-recursive functional such that, for every $f \in \mathcal{F}$ and for all $x \in \mathbb{N}$, the limit $g(x) = \lim_s \Psi_s(\Phi(f))(x)$ exists, $\Phi(g) = \Phi(f)$ and there are only finitely many pairs (x, s) with $\Psi_s(\Phi(f))(x) \neq g(x)$.

(d) Ψ *weakly inverts* Φ if Ψ is a limit-recursive functional such that for every $f \in \mathcal{F}$, for all x, the limit $g(x) = \lim_s \Psi_s(\Phi(f))(x)$ exists and $\Phi(f) = \Phi(g)$.

(e) Ψ *bounded weakly inverts* Φ if Ψ is a limit-recursive functional such that for every $f \in \mathcal{F}$ the limit $g = \lim_s \Psi_s(\Phi(f))$ exists, $\Phi(f) = \Phi(g)$ and $g \leq_T \Phi(f)$.

(f) The class $\mathcal{F}$ is called *invertible*, *strongly invertible*, *weakly invertible* or *bounded weakly invertible* iff for every $\mathcal{F}$-preserving general recursive operator there is a Ψ such that Ψ inverts, strongly inverts, weakly inverts or bounded weakly inverts Φ, respectively.

Although part (d) has a certain interest on its own right, it is a limiting process where it is no longer possible to get g from $\Phi(f)$ by any effective means. Somehow, it might be natural also to consider the case where such a translation of $\Phi(f)$ into g at least exists, although it is not applied by Ψ. This additional requirement that $g \leq_T \Phi(f)$ is then considered in (e).

Note that (b) implies (c) as follows. The new operator is given by taking as the s-th approximation the first finite variant g_s, in some standard enumeration of the finite variants of $\Psi(\Phi(f))$, found for which $\Phi(g_s)[s] = \Phi(f)[s]$. For $f \in \mathcal{F}$, $\Psi(\Phi(f))$ is a finite variant of some g with $\Phi(g) = \Phi(f)$ and thus the g_s converge to this g or some other finite variant with the same property.

The implication from (c) to (e) comes from the following argument. Whenever Ψ converges to g on input $\Phi(f)$ according to (c), then $g = \Psi_s(\Phi(f))$ for large enough s. Since Ψ_s is a recursive operator for every s, $g \leq_T \Phi(f)$.

The implication from (e) to (d) is obvious since just one requirement on the process to get g from $\Phi(f)$ is dropped.

Theorem 5. *The class $\mathcal{R}$ of all recursive functions is not weakly invertible.*

Proof. Let W_e be the domain of the e-th partial recursive function with respect to some fixed underlying acceptable numbering. Now define an operator Φ by the equation

$$\Phi(f) = \begin{cases} (f(0))^\infty & \text{if } \forall s\ [|W_{f(0),f(s)}| \geq s] \text{ or } \forall s\ [|W_{f(0),s}| \leq f(1)]; \\ (f(0))^s(f(0)+1)^\infty & \text{if } s \text{ is the first positive number} \\ & \text{where the first case fails.} \end{cases}$$

For every e there is a recursive f with $\Phi(f) = e^\infty$. In the case that W_e is finite, such an f is $e|W_e|0^\infty$, in the case that W_e is infinite, such an f can be obtained by letting $f(x) = \min(\{s : |W_{e,s}| \geq x\})$. On the other hand, one can see that whenever W_e is finite then only functions f with $f(0) = e \wedge f(1) \geq |W_e|$ are mapped to e^∞, thus if Ψ inverts e^∞ in the limit, then the function $F(e) = \lim_s \Psi_s(e^\infty)(1)$ is K-recursive and satisfies $F(e) \geq |W_e|$ whenever W_e is finite. It follows that $\{e : W_e \text{ is finite}\} = \{e : |W_e| \leq F(e)\}$ where the first set is Σ_2^0-complete and the second is K-recursive, a contradiction. Therefore $\mathcal{R}$ is not weakly invertible. □

This result used the fact that the function $e \mapsto |W_e|$ restricted to the domain of all e, where W_e is finite, is not dominated by any K-recursive function. So although all involved functions are recursive, their initial growth from $f(0)$ to $f(1)$ cannot be captured even by a K-recursive function. One might ask what happens if growth-conditions cannot be exploited because all functions involved are bounded. The next result implies that every such class is weakly invertible. The following proposition is based on Kreisel's work [5] and is a uniform version of [7, Proposition V.5.31] relativized to $\Phi(f)$.

Proposition 6 (Based on Kreisel [5]). *For every constant c, the class $\{0, 1, \ldots, c\}^\infty$ is weakly invertible.*

Proof. Let Φ be a general recursive operator and $f \in \{0, 1, \ldots, c\}^\infty$. The set $T(\Phi(f)) = \{\sigma \in \{0, 1, \ldots, c\}^* : \forall x \leq |\sigma|\ [\text{if } \Phi(\sigma) \text{ is defined then } \Phi(\sigma)(x) = f(x)]\}$ forms an f-recursive tree and one can define a limit-recursive operator Ψ such that the function g given as $g(x) = \lim_s \Psi_s(\Phi(f))(x)$ exists and is in $\{0, 1, \ldots, c\}^\infty$ and is an infinite branch of $T(\Phi(f))$, that is, satisfies $\Phi(g) = \Phi(f)$. This is done by choosing $\Psi_s(\Phi(f))(x)$ be $\tau(x)$ for the lexicographic least string $\tau \in \{0, 1, \ldots, c\}^{x+s}$ which is in $T(\Phi(f))$. □

In the following, it is proven that the notions strongly invertible, invertible, bounded weakly invertible and weakly invertible form a strict hierarchy. The classes separating the levels of this hierarchy are subclasses of $\{0,1\}^\infty$.

Example 7. *The class* $\{0,1\}^\infty$ *is weakly invertible but not bounded weakly invertible.*

Proof. Let the partial-recursive function ψ be defined as

$$\psi(e) = \begin{cases} 0 & \text{if } \varphi_e(e)\downarrow \geq 1; \\ 1 & \text{if } \varphi_e(e)\downarrow = 0; \\ \uparrow & \text{if } \varphi_e(e)\uparrow. \end{cases}$$

Note that ψ is partial-recursive but has no total recursive extension, as the definition makes ψ inconsistent with all total recursive functions. Let ψ_s denote the finite part of ψ which is computed within s steps. Now define

$$\Phi(f)(s) = \begin{cases} 0 & \text{if } \psi_s \text{ and } f \text{ are consistent;} \\ 1 & \text{otherwise.} \end{cases}$$

This operator maps all total extensions of ψ to 0^∞ while it maps all recursive functions to $\{0^k1^\infty : k \in \mathbb{N}\}$. So some functions are mapped to 0^∞ but none of them is recursive relative to 0^∞ and thus the condition $g \leq_T \Phi(f)$ from Definition 4 (e) cannot be satisfied. □

This result was of course induced by the fact that the class contains nonrecursive functions. So one could ask whether there is a class containing only recursive functions which is not bounded weakly invertible. The following example shows that this is indeed true.

Theorem 8. *The class* $\mathcal{R}_{0,1}$ *consisting of all* $\{0,1\}$*-valued recursive functions is not bounded weakly invertible.*

Proof. As in Example 7, let ψ be a partial-recursive $\{0,1\}$-valued function without total recursive extension and ψ_s be the finite part of it computed in time s. Similarly, let ξ^K a corresponding partial K-recursive $\{0,1\}$-valued function without a total K-recursive extension. The function ξ^K has a $\{0,1\}$-valued recursive approximation $\xi_0, \xi_1, \ldots$ so that, for all e in the domain of ξ^K, $\lim_s \xi_s(e) = \xi^K(e)$. Now define for $e \in \mathbb{N}$ and $a \in \{0,1\}$ a function $\theta_{e,a}$ as follows:

$$\theta_{e,a}(x) = \begin{cases} 0 & \text{if } x < e; \\ 1 & \text{if } x = e \text{ or } x = e+1; \\ a & \text{if } x = e+2; \\ \psi_s(x) & \text{if } x > e+2 \text{ and } s \text{ is the first } t > x \text{ such that} \\ & \text{either } \psi_t(x) \text{ is defined or } \xi_t(e) = a. \end{cases}$$

Here, in the fourth case, $\theta_{e,a}(x)$ is undefined if either s is never found because the corresponding t does not exist or $\psi_s(x)$ is undefined. Now define Φ as follows:

$$\Phi(f) = \begin{cases} 0^\infty & \text{if } f = 0^\infty; \\ 0^e101^\infty & \text{if } f \text{ extends } 0^e10; \\ 0^e10^\infty & \text{if } f \text{ extends } \theta_{e,0} \text{ or } \theta_{e,1}; \\ 0^e10^{s+1}1^\infty & \text{if } f \text{ extends } 0^e11 \text{ but one finds} \\ & \text{in } s \text{ steps that the previous case fails.} \end{cases}$$

It is easy to verify that Φ is general recursive. For every e there is $a \in \{0,1\}$ such that $\xi_s(e) = a$ for infinitely many s. For such a, let $h_{e,a}$ be the function defined as follows: For $x \leq e+2$, $h_{e,a}(x) = 0^e 11a(x)$. For $x \geq e+2$, one finds the first $s \geq x$ such that $\xi_s(e) = a$. If $\psi_s(x)$ is defined then $h_{e,a}(x) = \psi_s(x)$ else $h_{e,a}(x) = 0$. Every total $h_{e,a}$ is a total and recursive extension of $\theta_{e,a}$ and for every e, either $h_{e,0}$ or $h_{e,1}$ is total.

Assume now by way of contradiction that Ψ weakly inverts Φ. Then $\Psi(0^e 10^\infty)$ converges to a function g_e and g_e extends $\theta_{e,g_e(e+2)}$. The function $e \mapsto g_e(e+2) = \lim_s \Psi_s(0^e 10^\infty)(e+2)$ is K-recursive. As ξ^K has no K-recursive total extension, there is an e such that $\xi^K(e)$ is defined and different from $g_e(e+2)$. Thus there is an s such that $\xi_t(e) = \xi^K(e)$ for all $t \geq s$. As a consequence, for all $x \geq s$ in the domain of ψ, $g_e(x) = \theta_{e,g_e(e+2)}(x) = \psi(x)$. Thus g_e is a finite variant of an extension of ψ. This contradicts the fact that ψ has no recursive extension, thus Ψ cannot exist and $\mathcal{R}_{0,1}$ is not bounded weakly invertible. □

Note that one could instead of $\mathcal{R}_{0,1}$ already use the class consisting of all functions $0^e 10^\infty$, $0^e 101^\infty$ and $h_{e,a}$ whenever the latter is total. The resulting class is finitely learnable, that is, there is a learner which outputs void hypotheses until it has seen enough data and outputs exactly one correct hypothesis from then on. Finite learning is one of the most restrictive learning criteria, but it still does not guarantee weak invertibility.

This contrasts with Proposition 16 below which says that all recursively enumerable classes are strongly invertible. The next section deals with recursively enumerable classes explicitly, but before that some further examples of invertible classes are presented.

Example 9. *Every class $\{f\}$ consisting of only one function is strongly invertible. The reason is that every $\{f\}$-preserving Φ maps f to itself and so one can take Ψ as the identity.*

One might ask whether this comes from the small cardinality of given class. It does not, as the following example of a similar class with cardinality $2^{\aleph_0}$ shows. The construction of a tree with the properties postulated in this example is well-known [7, Exercise V.2.18 (b)] and thus omitted.

Example 10. *There is a recursive tree T such that the class $\mathcal{F}$ of all its infinite branches satisfies that any two distinct members have incomparable Turing degrees. This class $\mathcal{F}$ is strongly invertible.*

Proof. If Φ is $\mathcal{F}$-preserving then it follows from the definition that $\Phi(f) = f$ for every infinite branch f of T. Thus one can choose Ψ to be the operator which maps every function to itself. □

Note that the same Ψ works for all $\mathcal{F}$-preserving Φ. Furthermore, the given example forms a Π^0_1 class, so Φ and Ψ can even detect eventually whenever their input is not from $\mathcal{F}$. The next example consists only of recursive functions but has a similar flavour.

Example 11. *Let $e_0, e_1, \ldots$ be an infinite sequence of minimal indices of total functions such that $\{e_0, e_1, \ldots\}$ is 2-r-cohesive relative to K'. Such a set exists by Remark 2. The class $\mathcal{F} = \{\varphi_{e_n} : n \in \mathbb{N}\}$ is strongly invertible.*

Proof. Given $\mathcal{F}$ and Φ, one defines R as a $\{\text{false}, \text{true}\}$-valued colouring by

$$R(i,j) \Leftrightarrow (\Phi(\varphi_i) = \varphi_j) \vee (\Phi(\varphi_j) = \varphi_i).$$

By Remark 2 there is a number k and a colour u such that for all i, j with $k < i < j$, $R(e_i, e_j) = u$. If $u =$ true then also $R(e_{k+1}, e_n) =$ true for almost all n. As $\Phi(\varphi_{e_{k+1}})$ takes only one value, one can conclude that $\Phi(\varphi_{e_n}) = \varphi_{e_{k+1}}$ for almost all n. If $u =$ false then $R(e_m, e_n)$ does not hold whenever $k < m < n$.

Thus in both cases, for all $n > k+1$, either $\Phi(\varphi_{e_n}) = \varphi_{e_n}$ or $\Phi(\varphi_{e_n}) \in \{\varphi_{e_0}, \varphi_{e_1}, \ldots, \varphi_{e_{k+1}}\}$ or $\varphi_{e_n} = \Phi(\varphi_{e_{k+1}})$. So there is a finite set $\mathcal{G} \subseteq \mathcal{F}$ such that every function in the range of $\mathcal{F}$ is either the image of itself or in $\Phi(\mathcal{G})$. Thus Φ is strongly invertible. □

Example 12. *Let $\Phi_0, \Phi_1, \ldots$ be an enumeration of all recursive operators and let G be the index set of the e where Φ_e is general recursive. Furthermore, let $F = \{2^n + \sum_{m<n} 2^m \cdot G(m)\}$ be a set of numbers coding initial parts of G by its binary digits. Now let $\{e_0, e_1, \ldots\}$ be a subset of F which is 2-r-cohesive relative to K'. Let $\mathcal{F}$ contain for every k the functions $0^{e_k}101^\infty$, $0^{e_k}10^\infty$ and θ_{e_k}. For any e, x, $\theta_e(x)$ is defined as follows:*

If $x < e+2$ then $\theta_e(x) = 0^e11(x)$. Otherwise find the $a < e$ with $x \equiv a$ modulo e. If $2^{a+1} \geq e$ then $\theta_e(x) = 0$. Otherwise determine the $a+1$-st least significant bit of e. If this bit is 0 then $\theta_e(x) = 0$ again. Otherwise

$$\theta_e(x) = \begin{cases} 0 & \text{if } \Phi_a(0^e10^\infty)(x)\downarrow > 0; \\ 1 & \text{if } \Phi_a(0^e10^\infty)(x)\downarrow = 0; \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that the function θ_e might be partial only if $e \notin F$. The class $\mathcal{F}$ is invertible but not strongly invertible.

Proof. Given $\mathcal{F}$ and Φ, there is a k such that for all $n, m > k$ and all $a, b, a', b' \in \{0, 1\}$,

- if f extends $0^{e_n}1ab$ then $\Phi(f)$ extends $0^{e_m}1$ only if $m = n$;
- if f extends $0^{e_n}1ab$ and f' extends $0^{e_m}1ab$ and $\Phi(f)$ extends $0^{e_n}1a'b'$ then $\Phi(f')$ extends $0^{e_m}1a'b'$.

These properties are obtained by considering appropriately chosen finitely many K'-recursive colourings and then taking the maximum of the corresponding k. For example, if for one $n > k$ it holds that $\Phi(0^{e_n}10^\infty) = 0^{e_n}101^\infty$ then this holds for all $n > k$. The indices of $0^{e_n}10^\infty$, $0^{e_n}101^\infty$ and θ_{e_n} can be computed from e_n and so an operator Ψ to invert Φ can handle the finitely many exceptions of the functions with a prefix $0^{e_n}1$ for some $n \leq k$ as in Example 11 and for the others read the prefix $0^{e_n}1ab$ and then conclude from a table which of the three functions in question has generated this input and then generate this function from its index. Note that this algorithm is sensitive to the fact that e_n has to be a member of F as it might be undefined on prefixes of other functions and thus the resulting operator is not general recursive. So it only witnesses that $\mathcal{F}$ is invertible.

To see that $\mathcal{F}$ is not strongly invertible, consider the operator Φ as follows. Φ maps 0^∞ to itself. For all e, Φ maps all functions beginning with 0^e10 to 0^e101^∞. For all e, Φ maps all functions beginning with 0^e11 to the following: if the function is consistent with θ_e, then it is mapped to 0^e10^∞; otherwise it is mapped $0^e10^{s+1}1^\infty$, where s is the number of steps needed to detect the inconsistency.

So 0^e10^∞ is only the image of total extensions of θ_e, which of course is the function θ_e itself in the case that θ_e is total. Now, if Φ_e is a general recursive operator and $e_n > 2^{e+1}$ then one has that $\Phi_e(0^{e_n}10^\infty) \neq \theta_{e_n}$, although θ_{e_n} is the only function with $\Phi(\theta_{e_n}) = 0^{e_n}10^\infty$. Thus no general recursive operator strongly inverts Φ and $\mathcal{F}$ is not strongly invertible. $\square$

For the separation of bounded weakly invertible from invertible, the following result of Kaufmann [4, Theorem 5.2.2] is crucial, which is formulated such that it fits conveniently into the setting of the present work; the proof is nevertheless almost the same as by Kaufmann and thus omitted. The original result of Kaufmann applies to the constructed tree indices of the $e+1$ partial-recursive functions $x \mapsto \Psi_{e',x}(0^e10^\infty)(x)$ with $e' \in \{0, 1, \ldots, e\}$ instead of invoking them directly. But a set of those indices could easily be generated from the parameter e and so Kaufmann's proof directly transfers to the current application.

Proposition 13 (Kaufmann [4]). *Let $\Psi_0, \Psi_1, \ldots$ be a recursive enumeration of all operators which are approximable in the limit. Let $\Psi_{e,s}$ be the s-th recursive approximation of Ψ_e such that $e, s \mapsto \Psi_{e,s}$ is effective. Then there is a uniformly recursive family $T_0, T_1, \ldots$ of trees such that for every e the following holds:*

- *each T_e is a subset of $0^e11\{0,1\}^* \cup \{0^e11[r] : r \leq e+2\}$.*
- *for each e, n, $|T_e \cap \{0,1\}^n| \leq e+2$, that is, T_e has bounded width;*
- *for each e and each infinite branch A of T_e and each $e' \leq e$, there are infinitely many x such that either $\Psi_{e',x}(0^e10^\infty)(x)$ is undefined or different from $A(x)$.*

Furthermore, each T_e has at least one infinite branch and all its infinite branches are recursive.

Using this result, one can now construct the separating class.

Example 14. *Let $\{e_0, e_1, \ldots\}$ be a set which is uniformly 2-r-cohesive relative to K' and satisfies, for every n and every $m \geq n$, $\varphi_n^{K'}(e_m) < e_{m+1}$. Let $\Psi_0, \Psi_1, \ldots$ be a recursive enumeration of all limit-recursive operators. Now let $\mathcal{F}$ contain the functions $0^{e_n}10^\infty$, $0^{e_n}101^\infty$ and the left-most infinite branch θ_{e_n} of T_{e_n} for all n. The class $\mathcal{F}$ is bounded weakly invertible but not invertible.*

Proof. The argumentation that $\mathcal{F}$ is bounded weakly invertible is parallel to the argumentation of the corresponding class being invertible in Example 12. Here note that θ_e is computable from e using the oracle K. Thus a limiting-recursive functional can compute θ_e.

To see that $\mathcal{F}$ is not invertible, consider the general recursive operator Φ with $\Phi(f)$ being determined by the following case distinction:

- $\Phi(f) = 0^\infty$ iff $f = 0^\infty$;
- $\Phi(f) = 0^e101^\infty$ iff f extends 0^e10;
- $\Phi(f) = 0^e10^\infty$ iff f extends 0^e11 and f on T_e;
- $\Phi(f) = 0^e10^s1^\infty$ iff f extends 0^e11 and s is the least number with $f[s] \notin T_e$.

It is easy to see that Φ is $\mathcal{F}$-preserving. Furthermore, $\Phi(\theta_{e_n}) = 0^{e_n}10^\infty$ for all n. Assume now by way of contradiction that Ψ_e inverts Φ and $n > e$. Then the function g_{e_n} given by $x \mapsto \Psi_{e,x}(0^{e_n}10^\infty)(x)$ differs from all infinite branches of T_{e_n} at infinitely many places. But for almost all s, the functions $\Psi_{e,s}(0^{e_n}10^\infty)$ are the same, thus their limit is a finite variant of g_{e_n} and not an infinite branch of T_{e_n}. As Φ maps only the infinite branches of T_{e_n} to $0^{e_n}10^\infty$, no finite variant of g_{e_n} is mapped to $0^{e_n}10^\infty$ and Ψ_e does not invert Φ in contradiction to the assumption. $\square$

4 Enumerating Operators and Functions

It is quite natural to deal with classes where there is an indexing for all the functions involved. Such classes are known as “indexed families”, “uniformly recursive classes” or “recursively enumerable classes” where the enumeration is now an enumeration of the involved functions and not of the elements of a set.

Definition 15. A class $\mathcal{F}$ is recursively enumerable iff there is a total recursive function $e, x \mapsto f_e(x)$ in two variables such that $\mathcal{F}$ equals the set of functions obtained by fixing the input e: $\mathcal{F} = \{f_0, f_1, \ldots\}$.

Such classes are quite easy to invert as given a general recursive operator Φ, the inverting Ψ can on input $\Phi(f)$ and x output $f_e(x)$ for the first e such that either $e = x$ or $\Phi_e(f)(y) = \Phi(f)(y)$ for all $y \leq x$. It is well-known that such an algorithm of “learning by enumeration” gives a general recursive operator which makes only finitely many errors.

Proposition 16. *Every recursively enumerable class of functions is strongly invertible.*

The question whether an operator can be inverted also depends on the variety of operators available. Therefore, one might ask how difficult an enumeration has to be so that all possible restrictions of mappings from $\mathcal{F}$ to $\mathcal{F}$ occur. This is formalized in the following definition.

Definition 17. (a) An enumeration $\Phi_0, \Phi_1, \ldots$ of operators *weakly covers* $\mathcal{F}$ if for every $\mathcal{F}$-preserving general recursive operator Ψ there is an e with Φ_e being general recursive and $\forall f \in \mathcal{F}\, [\Phi_e(f) = \Psi(f)]$.

(b) An enumeration $\Phi_0, \Phi_1, \ldots$ of operators *covers* $\mathcal{F}$ iff it weakly covers $\mathcal{F}$ and every $\Phi_e(f)$ is total for every $f \in \mathcal{F}$. Furthermore, $\mathcal{F}$ is *coverable* iff some recursive enumeration of operators covers $\mathcal{F}$.

(c) An enumeration $\Phi_0, \Phi_1, \ldots$ of operators *strongly covers* $\mathcal{F}$ iff it weakly covers $\mathcal{F}$ and every Φ_e is a general recursive operator. Furthermore, $\mathcal{F}$ is *strongly coverable* iff some recursive enumeration of operators strongly covers $\mathcal{F}$.

Note that every class is weakly covered by an acceptable numbering of all recursive operators. Clearly, $\{f\}$ is strongly coverable since an enumeration only needs to contain the identity operator in order to cover $\{f\}$. When considering inverting classes of recursive functions, coverable classes are restricted to be contained in enumerable ones.

Theorem 18. *Every coverable class of recursive functions is a subclass of a recursively enumerable class of recursive functions.*

Proof. Let $\mathcal{F}$ contain only recursive functions, $f \in \mathcal{F}$ and $\Phi_0, \Phi_1, \ldots$ be an enumeration covering $\mathcal{F}$. Now for every $g \in \mathcal{F}$ there is an operator Φ_e which maps every function to g and thus $\Phi_e(f) = g$. So $\mathcal{F} \subseteq \{\Phi_e(f) : e \in \mathbb{N}\}$ and the function $e, x \mapsto \Phi_e(f)(x)$ is total and recursive in both inputs. So $\mathcal{F}$ is a subclass of $\{\Phi_e(f) : e \in \mathbb{N}\}$, a recursively enumerable class of recursive functions. □

As a consequence, one has that every coverable class of recursive functions is also strongly invertible. One might therefore ask whether every strongly coverable class is also strongly invertible. This is unfortunately not the case.

Example 19. *Let ψ be a partial-recursive $\{0,1\}$-valued function without recursive total extension and f be a (nonrecursive) total extension of ψ. The class $\{0^\infty, 1^\infty, f\}$ is strongly coverable but not bounded weakly invertible.*

Proof. One just has to find an enumeration of general recursive operators which covers each of the finitely many possibilities how an operator can map the 3 functions inside this class. So the class is strongly coverable. It is not invertible as one might consider any operator which satisfies that $\Phi(0^\infty) = \Phi(1^\infty) = 1^\infty$, $\Phi(f) = 0^\infty$ and $\Phi(g) \neq 0^\infty$ for any recursive g. The proof of Example 7 essentially describes how to construct such a Φ. Then, one can conclude as in Example 7 that Φ is not bounded weakly invertible as $\Phi(f)$ is recursive, but the inverting algorithm cannot find any recursive g with $\Phi(f) = \Phi(g)$ as such a g does not exist. $\square$

Remark 20. A similar result can also be obtained for unbounded functions. Taking $T \subseteq \{0,1,\ldots\}^*$ to be a recursive tree which has infinite branches but no hyperarithmetic ones, one can select such a path f and then show by an argument similar to the previous one that $\{0^\infty, 1^\infty, f\}$ is strongly coverable but not weakly invertible.

Proposition 21. *Assume that $\mathcal{F}$ contains all functions of the form 0^e10^∞. Then $\mathcal{F}$ is not coverable.*

Proof. Given an enumeration $\Phi_0, \Phi_1, \ldots$, define a new operator Φ which maps 0^∞ to 0^∞ and every function starting with 0^e1 to 0^e10^∞ in the case that $\Phi_e(0^e10^\infty)$ does not extend 0^e1 and to $0^{e+1}10^\infty$ in the case that $\Phi_e(0^e10^\infty)$ extends 0^e1. It is easy to see that Φ is total. Furthermore, Φ is $\mathcal{F}$-preserving as every function different from 0^∞ is mapped to a function of the form 0^e10^∞ and the function 0^∞ is mapped to itself. Furthermore, $\mathcal{F}$ differs by construction from every Φ_e on $\mathcal{F}$ as it maps 0^e10^∞ to something different from $\Phi_e(0^e10^\infty)$. So Φ is not covered by the given enumeration. $\square$

Remark 22. A class $\mathcal{F}$ is said to be *finitely learnable* [1, 2] iff there is a general recursive operator mapping every function $f \in \mathcal{F}$ to a function of the form 0^*e^∞ such that $e > 0 \wedge \varphi_e = f$. Based on this notion, the above example can be generalized as follows. If $\mathcal{F}$ is recursively enumerable and $\mathcal{F}$ has an infinite finitely learnable subclass then $\mathcal{F}$ is not coverable.

Example 23. *Let $\Phi_0, \Phi_1, \ldots$ be an acceptable enumeration of recursive operators and let h be a strictly increasing function which grows so fast that $\Phi_e(f[h(n)])(x)$ is defined whenever $e, x \leq n$, $f \in \{0,1,2\}^\infty$ and Φ_e is a general recursive operator. Let H be the range of h. Then the class*

$$\mathcal{F} = \{f : \forall x\ [(x \notin H \Rightarrow f(x) = 0) \wedge (x \in H \Rightarrow f(x) \in \{1,2\})]\}$$

is coverable but not strongly coverable.

Proof. For any function f define $h_f(n) = \max(\{x : |\{y < x : f(y) \neq 0\}| \leq n\})$, that is, $h_f(n)$ is the n+1-st position x where $f(x)$ is different from 0. The function h_f is partial-recursive relative to the oracle f and total iff f is different from 0 infinitely often.

For showing that the class $\mathcal{F}$ is coverable, one defines an enumeration $\Psi_0, \Psi_1, \ldots$ covering $\mathcal{F}$ from the given enumeration $\Phi_0, \Phi_1, \ldots$ as follows.

To compute $\Psi_e(f)(x)$, one searches for the first s for which either $\Phi_e(f[s])(x)$ is defined or $s = h_f(x+e+1)$ and then defines that

$$\Psi_e(f)(x) = \begin{cases} \Phi_e(f[s])(x) & \text{if } s \text{ is found and } \Phi_e(f[s])(x) \text{ is defined;} \\ 0 & \text{if } s \text{ is found and } \Phi_e(f[s])(x) \text{ is undefined;} \\ \uparrow & \text{if } s \text{ is not found.} \end{cases}$$

Thus $\Psi_e(f)$ is total whenever either $\Phi_e(f)$ is total or $f(x) \neq 0$ for infinitely many x. In particular, $\Psi_e(f)$ is total for all $e \in \mathbb{N}$ and $f \in \mathcal{F}$.

If Φ_e is a general recursive operator then Ψ_e is also one since $\Phi_e(f)$ is total for every function f. For $f \in \mathcal{F}$, $h_f(e+x+1) = h(e+x+1)$ and by the choice of h and $\Phi_e(f[h(e+x+1)])(x)$ is defined. It follows that $\Psi_e(f)(x) = \Phi_e(f)(x)$. So the operators Ψ_e, Φ_e have the same behaviour on $\mathcal{F}$. Thus $\Psi_0, \Psi_1, \ldots$ covers $\mathcal{F}$.

Given a recursive enumeration of general recursive operators, there is, due to the Padding Lemma, a recursive set E of indices such that every operator in the enumeration is equal to some Φ_e with $e \in E$ and every Φ_e with $e \in E$ is general recursive. Now one defines a function $h'(n)$ to be the least number t such that for all $x \leq n$, for all $e \leq n$ with $e \in E$ and for all $f \in \{0,1,2\}^\infty$, $\Phi_e(f[t])(x)$ is defined. As all Φ_e with $e \in E$ are general recursive, h' is a recursive function. Furthermore $h'(n) \leq h(n)$ for all n. Now one defines

$$\Theta(f)(x) = \begin{cases} 0 & \text{if } x \neq h_f(n) \text{ for all } n \leq x; \\ 1 & \text{if } x = h_f(e) \text{ for some } e \in E \text{ with } e \leq x \text{ and } \Phi_e(f[h'(x+e+1)])(x)\downarrow \neq 1; \\ 2 & \text{otherwise.} \end{cases}$$

First, the operator Θ is general recursive as h' is a total function and all other tests apply to bounded search. Second, if $e \in E$ and $f \in F$ then $\Theta(f)(h(e)) = 2$ if $\Phi_e(f)(h(e)) = 1$ and $\Theta(f)(h(e)) = 1$ otherwise. Thus $\Theta(f) \neq \Phi_e(f)$ and Θ differs on $\mathcal{F}$ from every Φ_e with $e \in E$. Third, Θ is $\mathcal{F}$-preserving since, whenever $f \in \mathcal{F}$, $\Theta(f)(x) = 0$ for $x \notin H$ and $\Theta(f)(x) \in \{1,2\}$ for $x \in H$. Thus Θ is an $\mathcal{F}$-preserving general recursive operator different on $\mathcal{F}$ from all Φ_e with $e \in E$. So $\mathcal{F}$ is not strongly coverable. $\square$

Example 24. *Let F be a 1-generic set below K and let $f_0, f_1, \ldots$ be a sequence of recursive $\{0,1\}$-valued functions approximating the characteristic function of F. Then $\{f_0, f_1, \ldots\}$ is strongly coverable.*

Proof. Let $\Phi_0, \Phi_1, \ldots$ be the enumeration of all operators for which there is an n such that all f_m extending $F[n]$ are either mapped to themselves or all mapped to the same function f_k. As almost all f_m extend $F[n]$, one can obtain the enumeration of the Φ_e by changing, on finitely many input-functions, either the operator mapping all functions to f_k or the identity operator.

It remains to show that this enumeration covers $\{f_0, f_1, \ldots\}$. Given an operator Φ, the set A of all binary σ such that $\Phi(\sigma)$ is inconsistent with σ, is recursively enumerable.

In the case that no prefix of F is contained in A, one can find an n such that A does not contain any extension of $F[n]$. If f_m extends $F[n]$ then $\Phi(f_m) = f_m$, since otherwise there would be a prefix $f_m[y]$ for some y such that $\Phi(f_m[y])(x)$ is defined and different from $f_m(x)$ for some $x < y$. Then $f_m[y]$ would be in A in contradiction to the choice of n. So in this case, the above enumeration contains a Φ_e which behaves same as Φ on $\{f_0, f_1, \ldots\}$.

Otherwise there is a $\sigma \in A$ extended by F. There are only finitely many f_m such that f_m does not extend σ; thus there is a y such that $f_m[y] \neq f_k[y]$ whenever f_m, f_k do not extend σ and are different. Now one takes n so large that $\Phi(F[n])(x)$ is defined for all $x < y$. Then $\Phi(F[n])$ extends $f_k[y]$ for some unique function f_k and therefore $\Phi(f_m) = f_k$ for all f_m extending $F[n]$. Again, the above enumeration contains a Φ_e which behaves in the same way as Φ on $\{f_0, f_1, \ldots\}$. □

Although not every recursively enumerable class is coverable, the next result shows that it is at least coverable relative to K'. This relativized concept uses the notion of a K'-recursive enumeration of recursive operators. Here an enumeration $\Phi_0, \Phi_1, \ldots$ is K'-recursive iff there is a K'-recursive function h and an acceptable numbering $\Gamma_0, \Gamma_1, \ldots$ of operators with $\Phi_e = \Gamma_{h(e)}$ for all e.

Proposition 25. *If $\mathcal{F}$ is recursively enumerable then some K'-recursive enumeration of operators covers $\mathcal{F}$.*

Proof. Let $f_0, f_1, \ldots$ be an enumeration of $\mathcal{F}$ and $\Gamma_0, \Gamma_1, \ldots$ be an acceptable numbering of all recursive operators. Let $h(e)$ be the least number e' greater than all $h(e'')$ with $e'' < e$ such that

$$\forall x, y \; [\Gamma_{e'}(f_x)(y) \text{ is defined}].$$

This predicate is a Π^0_2 predicate as it is universally quantified over the Σ^0_1 condition whether a certain computation halts. Thus the predicate can be evaluated with the oracle K'. Furthermore, it just selects the first index e' after all indices $h(e'')$ with $e'' < e$ such that $\Gamma_{e'}$ is total on $f_0, f_1, \ldots$ and therefore all general recursive operators plus some others are covered. □

Proposition 26. *Let $\mathcal{F} = \{f_0, f_1, \ldots\}$ be recursively enumerable and $\Phi_0, \Phi_1, \ldots$ be any recursive enumeration weakly covering $\mathcal{F}$. The set*

$$E = \{e : \forall f \in \mathcal{F} \; [\Phi_e(f) \textit{ is total and } \Phi_e(f) \in \mathcal{F}]\}$$

of all operators which preserve $\mathcal{F}$ is a Π^0_3-set.

Proof. Given the enumerations $f_0, f_1, \ldots$ of functions and $\Phi_0, \Phi_1, \ldots$ of operators, the set E is defined by the following Π^0_3-formula:

$$\begin{aligned} e \in E \;\text{ iff }\; & \forall n \forall x \exists k \; [\Phi_e(f_n[k])(x) \text{ is defined}] \text{ and} \\ & \forall n \exists m \forall k, x \; [\text{if } \Phi_e(f_n[k])(x) \text{ is defined then } \Phi_e(f_n[k])(x) = f_m(x)]. \end{aligned}$$

So this formula says that e is in E iff for every n the function $\Phi_e(f_n)$ is total and there is an index m such that $\Phi_e(f_n)$ is consistent with f_m. The totalness from the first condition and the consistency from the second imply equality. □

Note that $\Phi_e(f)$ with $e \in E$ is only required to be total if $f \in \mathcal{F}$, so some of the indices $e \in E$ might belong to operators which are not general recursive. As the problem whether Φ_e is general recursive is Π^1_1-complete, one cannot check for an operator being general recursive with a Π^0_3-condition. Nevertheless, whenever Φ_e is general recursive then $e \in E$ iff Φ_e is $\mathcal{F}$-preserving.

5 Periodic Functions

A class of interest is the class $\mathcal{F}$ of all functions which are eventually periodic. Not so much because of its difficulty or richness, but because of its relation to the situation described in the introduction: an eventually periodic function could mean that the user repeats actions over and over again. One of the fundamental principles in Human-Computer interaction design is that the computer should behave consistently on user inputs. Hence it might be reasonable to expect that the computer answers the repeated inputs just the way it answered the previous ones.

From now on, "eventually" will be dropped from "eventually periodic" for the sake of simplicity of the notation.

Definition 27. The class $\mathcal{F}_{per}$ is the union of all $\mathcal{F}_n$ with period n; that is, the union of the classes defined by the condition $f \in \mathcal{F}_n$ iff $\forall^\infty m\ [f(m+n) = f(m)]$.

The class $\mathcal{F}_{per}$ is strongly invertible. Furthermore, it is not coverable as it contains an infinite finitely learnable subclass, namely $\{0^e 1^\infty : e \in \mathbb{N}\}$. Indeed, one can even code very difficult problems into any K'-recursive enumeration of operators covering $\mathcal{F}_{per}$ and the following theorem shows that this class is not coverable.

Theorem 28. *Given any K'-recursive enumeration $\Phi_0, \Phi_1, \ldots$ covering $\mathcal{F}_{per}$, the set $P = \{e : \Phi_e$ is $\mathcal{F}_{per}$-preserving$\}$ is not recursively enumerable relative to K'.*

Proof. In the following, let $F(e,x,s)$ be the first non-element of $W_{e,s}$ which is greater or equal than x. Now define Ψ_e to be the general recursive operator which maps every function f extending 0^x but not 0^{x+1} to the function

$$0^e 1 0^x 1 0^{F(e,x,0)} 1 0^{F(e,x,1)} 1 0^{F(e,x,2)} 1 \ldots$$

and 0^∞ to $0^e 1 0^\infty$. For every x the function $\Psi_e(0^x 1 0^\infty)$ is periodic iff there is a nonelement of W_e greater or equal than x.

Assume now by way of contradiction that there is a K'-recursive enumeration $\Phi_0, \Phi_1, \ldots$ of operators covering $\mathcal{F}_{per}$ such that the corresponding set P is recursively enumerable relative to K'. Then one can find given e using oracle K' an x such that one of the following two conditions holds.

(1) $x \in P$ and for all σ and y, if $\Phi_x(\sigma)(y)$ and $\Psi_e(\sigma)(y)$ are both defined then they are equal;
(2) for all $y \geq x$, $y \in W_e$.

If W_e is coinfinite then the search terminates with an x satisfying the first condition since Ψ_e is $\mathcal{F}_{per}$-preserving and there is a general recursive operator Φ_x having the same behaviour on all periodic functions as Ψ_e. In particular, $\Psi_e(\sigma 1 0^\infty)$ and $\Phi_x(\sigma 1 0^\infty)$ must be the same functions and thus the required consistency condition holds. On the other hand, the search obviously cannot terminate according to (2).

If W_e is cofinite then Ψ_e maps some periodic function f to nonperiodic ones. If $x \in P$ then $\Phi_x(f)$ is periodic and thus there is an n and a y with $\Phi_x(f[n])(y)$ and $\Psi_e(f[n])(y)$ are both defined and different. So the search cannot terminate by condition (1) although it terminates by condition (2) with x being the least upper bound of the finitely many nonelements of W_e.

So one gets that $\{e : W_e$ is coinfinite$\}$ is Turing reducible to K', a contradiction to the well-known fact that this set is Π^0_3-complete. $\square$

The above proof produced the family of Ψ_e in a uniform manner, so in the case that $\Phi_0, \Phi_1, \ldots$ is an acceptable numbering, one has a recursive function h with $\Phi_{h(e)} = \Psi_e$. Thus one can get Π^0_3-completeness in this case.

Corollary 29. *If $\Phi_0, \Phi_1, \ldots$ is an acceptable numbering of all operators then the set $P = \{e : \Phi_e$ is $\mathcal{F}_{per}$-preserving$\}$ is Π^0_3-complete.*

If Ψ strongly inverts Φ then Ψ produces a finite variant but not the correct output. One might ask whether this is necessary. Indeed there are only very few classes where one can avoid it. For example, if Ψ is permitted to be partial then one can invert every general recursive operator on the constant functions by the Ψ outputting on input x^∞ the function y^∞ for the first y found such that $\Phi(y^\infty)(0) = x$. Somehow, if one wants general recursive operators Ψ with this property, one has to go to a sufficiently small subclass. In the case of $\mathcal{F}_{per}$, there are operators Φ where every (even partial) Ψ inverting Φ makes finitely many errors.

Example 30. *Let ψ be a partial recursive $\{0,1\}$-valued function without recursive extension. Then every recursive operator Ψ inverting the following general recursive operator Φ makes errors on some inputs:*

$$\Phi(f) = \begin{cases} 0^e10^\infty & \text{if } (f \text{ extends } 0^e10 \text{ or } 0^e11) \text{ and } \psi(e) \text{ is undefined;} \\ 0^e10^\infty & \text{if } f \text{ extends } 0^e1\psi(e) \text{ and } \psi(e) \text{ is defined;} \\ 0^e10^s1^\infty & \text{if } f \text{ extends } 0^e1 \text{ but not } 0^e1\psi(e) \\ & \text{and } \psi(e) \text{ halts after exactly } s \text{ steps;} \\ f & \text{otherwise.} \end{cases}$$

If some Ψ would strongly invert Φ without errors then the recursive function $e \mapsto \Psi(0^e10^\infty)(e+1)$ would be a total extension of ψ in contradiction to its choice.

6 Other Notions of Inverting

It was already shown that there is a single general recursive operator Φ such that one cannot invert Φ on the class of all recursive functions. As recursive operators preserve recursiveness, it is not very interesting to deal with arbitrary classes for negative results. We now turn our attention to the following question: for every recursive operator Φ and every recursively enumerable class $\mathcal{F}$, is there an operator Ψ which inverts or at least weakly inverts Φ? The next result shows that the technique of inverting by enumeration can be kept as long as the operator to be inverted is total on the whole family $\mathcal{F}$.

Theorem 31. *If $\mathcal{F}$ is recursively enumerable and Φ $\mathcal{F}$-preserving although not necessarily general recursive, then there is a general recursive operator Ψ which strongly inverts Φ.*

Proof. Let $f_0, f_1, \ldots$ be a recursively enumerable class and Φ a recursive operator such that $\Phi(f_n)$ is total for all n. Then define Ψ as follows: $\Psi(f)(x)$ is $f_n(x)$ for the least n such that $\Phi(f_n[x-n])$ is consistent with f. Ψ is general recursive as it terminates on all inputs to some $f_n(x)$ with $n \leq x$ (as $n = x$ would qualify). Furthermore, if n is the first index with $\Phi(f_n) = f$ then for all sufficiently large x, every expression $\Phi(f_m[x-m])$ with $m < n$ is inconsistent with f and thus $\Psi(\Phi(f))(x) = f_n(x)$. □

This property is lost if one considers operators which might be partial on functions from the class.

Example 32. *Let $\mathcal{F} = \{0^e10^\infty, 0^e1^\infty : e \in \mathbb{N}\}$. Let ξ^K be a partial K-recursive $\{0,1\}$-valued function without a total K-recursive extension together with the approximations $\xi_0, \xi_1, \ldots$ as defined in the proof of Theorem 8. Now one defines the following recursive operator Φ: $\Phi(0^e1a^\infty)$ is the union of all 0^e10^s such that $\xi_s(e) = a$. One can without loss of generality assume that the approximation oscillates between 0 and 1 whenever $\xi^K(e)$ is undefined. So if $\xi^K(e)$ is undefined then both 0^e10^∞ and 0^e1^∞ are mapped to 0^e10^∞ but if $\xi^K(e)$ is defined and equals a then $\Phi(0^e1b^\infty) = 0^e10^\infty$ iff $b = a$. As a consequence, assuming that there would be a partial limit-recursive operator Ψ, approximated by $\Psi_0, \Psi_1, \ldots$, which would be weakly inverting Φ, $\lim_s \Psi_s(0^e10^\infty)(e+1)$ would exist for all e and coincide with $\xi^K(e)$ whenever $\xi^K(e)$ is defined. Thus the function $e \mapsto \lim_s \Psi_s(0^e10^\infty)(e+1)$ would be a total K-recursive extension of ξ^K which by choice does not exist. Thus no limit-recursive operator weakly inverts Φ.*

Another topic is whether given an enumeration $\Phi_0, \Phi_1, \ldots$ of operators, one can find an operator Ψ which inverts every $\mathcal{F}$-consistent operator Φ_e on at least one function. In the case that all Φ_e are total on $\mathcal{F}$ and $\mathcal{F}$ contains at least one recursive function f, this can be easily achieved: for all functions g, one defines $\Psi(g) = f$. Then one uses that every $\mathcal{F}$-preserving Φ_e satisfies $\Phi_e(f) \in \mathcal{F}$ and hence f is the inverse of some function $g \in \mathcal{F}$. The next example shows that this is no longer possible if a class consists of several recursive functions and operators may be undefined on some functions in $\mathcal{F}$ in the sense that these are mapped to partial functions which are not considered as a valid output.

Example 33. *Let $\mathcal{R}$ be the class of all recursive functions. There is an enumeration $\Phi_0, \Phi_1, \ldots$ of recursive operators which map at least one recursive function to a total one such that no $\Psi = \lim_s \Psi_s$ weakly inverts the operator Φ_e on some total $f \in \Phi_e(\mathcal{R})$, given e and f as input.*

Proof. To see this, one defines $\Phi_e(f)(x) = 0$ iff

- either $|W_{e,x}| \leq f(0)$;
- or for all $y \leq x$, $y \leq |W_{e,f(y)}|$.

If these two conditions do not hold then $\Phi_e(f)(x)$ is undefined. Clearly 0^∞ is the only function in $\Phi_e(\mathcal{R})$. Now let F be the index-set of the finite sets. The functions f_e given as

$$f_e(x) = \min(\{s : (e \in F \Rightarrow |W_e| \leq s) \wedge (e \notin F \Rightarrow x \leq |W_{e,s}|)\})$$

are all recursive since one needs only to know the cardinality of W_e in order to compute $f_e(x)$ for every x. It is easy to verify that $\Phi_e(f_e) = 0^\infty$ for all e.

But if there would be a limit-recursive $\Psi = \lim_s \Psi_s$ which weakly inverts all Φ_e using the parameter e in the limit, then

$$e \in F \Leftrightarrow |W_e| \leq \lim_{s\to\infty} \Psi_s(e, 0^\infty)(0)$$

and $F \leq_T K$ in contradiction to the well-known fact that F is a Σ_2^0-complete set. □

While partial operators might not be invertible, one can easily get the following uniform variant of Proposition 16 (the proof is omitted). For this, one should note that for a dense set $\mathcal{F}$ and a general recursive operator Φ it holds that whenever the range of Φ contains at least k functions so does $\Phi(\mathcal{F})$. Here a set $\mathcal{F}$ is called dense if it contains an extension of every initial segment over $\mathbb{N}$.

Proposition 34. *Let $\Phi_0, \Phi_1, \ldots$ be a recursive enumeration of all recursive operators and let $\mathcal{F}$ be recursively enumerable and dense. Then there is a recursive enumeration $\Psi_0, \Psi_1, \ldots$ of recursive operators with the following properties.*

- *If Φ_e is general recursive then Ψ_e is general recursive and strongly inverts Φ_e on $\mathcal{F}$.*
- *If Φ_e is general recursive and its range at most countable, then the cardinality of the functions $\Phi_e(f)$ such that $\Psi_e(\Phi_e(f))$ strongly inverts Φ_e on $\Phi_e(f)$ is the same as the cardinality of the range of Φ_e.*

Proposition 34 depends on the fact that the index e of the operator is supplied. If this index is not known, then there is an enumeration $\Phi_0, \Phi_1, \ldots$ of $\mathcal{F}_{per}$-preserving general recursive operators, all having at least two functions in the range, such that no Ψ inverts every operator Φ_e on at least two functions.

Example 35. *Let $\Phi_e(f) = 1^\infty$ if $f(0) = e$ and $\Phi_e(f) = 0^\infty$ otherwise. Given any Ψ, choose e such that $e \neq \Psi(1^\infty)(0)$. Then Ψ does not invert the operator Φ_e on the function 1^∞ and so Ψ inverts Φ_e on at most one function although the range of each Φ_e contains two functions.*

Theorem 36. *Let $\mathcal{F} = \{f_0, f_1, \ldots\}$ be a recursively enumerable class. Then there is a Ψ which inverts every general recursive operator Φ on infinitely many members of $\Phi(\mathcal{F})$ whenever Φ is $\mathcal{F}$-preserving and $\Phi(\mathcal{F})$ is infinite.*

Proof. The construction of Ψ needs several auxiliary ingredients. The overall goal is to construct sequences $i_0, i_1, \ldots$ and $j_0, j_1, \ldots$ of indices such that for every general recursive operator which maps $\mathcal{F}$ to infinitely many functions, there are infinitely many k such that the operator maps f_{i_k} to f_{j_k} and for these k, Ψ inverts the given operator on infinitely many f_{j_k}, although not all, to f_{i_k}.

Let $\Phi_0, \Phi_1, \ldots$ be an acceptable numbering of all recursive operators. Now one partitions the natural numbers in intervals I such that $|I| > \min(I)$ for each I in the partition. Let I_k denote the interval which contains k, thus each member of an interval is also an index of it and the indexing is not one-one. Furthermore, let $e_0, e_1, \ldots$ be a sequence of indices of operators such that

- for all k, k', if $I_k = I_{k'}$ then $e_k = e_{k'}$;
- for all e there are infinitely many k with $e_k = e$.

The mapping $k \mapsto i_k$ is a partial-recursive function such that for any k, if Φ_{e_k} is total on $\mathcal{F}$ and $|\Phi_e(\mathcal{F})| \geq |I_k|$ then the following holds:

- for all $k' \in I_k$, $i_{k'}$ is defined;
- for all different $k', k'' \in I_k$ there is an x for which $\Phi_{e_k}(f_{i_{k'}})(x), \Phi_{e_k}(f_{i_{k''}})(x)$ are defined and different.

Note that the above partial-recursive function can easily be implemented by a standard search and might also be defined for some e where Φ_e is not total on $\mathcal{F}$.

The indices j_k are found as limits of the following approximation $j_{k,s}$: if i_k is not yet defined at stage s then $j_{k,s} = 0$ else $j_{k,s}$ is the least ℓ such that either $\ell = s$ or f_ℓ extends $\Phi_{e_k}(f_{i_k}[s])$. This approximation is recursive and the $j_{k,s}$ converge to the least ℓ for which f_ℓ extends $\Phi_e(f_{i_k})$ whenever such an ℓ exist. Note that $j_{k,s} \leq j_{k,s+1}$ for all s and that the $j_{k,s}$ converge to ∞ if no f_ℓ extends $\Phi_e(f_{i_k})$.

The operator Ψ is given as the limit of Ψ_s where $\Psi_s(g)(x)$ is computed by the following algorithm.

1. Let ℓ be the least number such that either $f_\ell[x+s] = g[x+s]$ or $\ell \geq x+s$;
2. Let k be the least number such that either $j_{k,x+s} = \ell$ or $k \geq x+s$;
3. if i_k is defined at step $x+s$ then $\Psi_s(g)(x) = f_{i_k}(x)$ else $\Psi_s(g)(x) = 0$.

It is easy to see that every Ψ_s is a general recursive operator. Furthermore, the algorithm is uniform in s, so one can compute the value $\Psi_s(g)(x)$ from the input s, x effectively.

Assume now that Φ_e is an $\mathcal{F}$-preserving general recursive operator such that $\Phi_e(\mathcal{F})$ is infinite. Then for every k with $e_k = e$ the index i_k is defined as there are at least $|I_k|$ functions in $\mathcal{F}$ which are mapped to different images. Furthermore, as Φ_e is $\mathcal{F}$-preserving, all f_{i_k} are mapped to some f_{j_k} and the $j_{k,s}$ converge to j_k.

Now select any interval I such that $e_{k'} = e$ for all $k' \in I$. Note that the mapping $k' \mapsto j_{k'}$ is one-one on the domain I. Thus there is an index $k \in I$ such that $j_k \neq j_{k'}$ for all $k' < \min(I)$. Fix this k and let $x+s$ be so large that the following holds:

- $f_\ell[x+s] \neq f_{j_k}[x+s]$ for all $\ell < j_k$;
- i_k is defined at stage $x+s$ and $j_{k,x+s} = j_k$;
- $j_{k',x+s} > j_k$ for all $k' < k$ where $j_{k',0}, j_{k',1}, \ldots$ converges to a number larger than j_k or to infinity.

Then one can say the following about the algorithm to compute $\Psi_s(f_{j_k})(x)$.

- the ℓ in the algorithm to compute $\Psi_s(f_{j_k})(x)$ is j_k;
- the parameter k from the algorithm has the same value as the k considered here;
- $\Psi_s(f_{j_k})(x) = f_{i_k}(x)$ as i_k is already defined at stage $x+s$.

Thus every $\Psi_s(f_{j_k})$ is a finite variant of f_{i_k} and almost all $\Psi_s(f_{j_k})$ are equal to f_{i_k}. So Ψ inverts Φ_e on f_{j_k} to f_{i_k}. The function f_{j_k} selected in the interval I was not dealt with in smaller intervals $I_{k'}$ with $e_{k'} = e$, thus each such interval contributes a function on which Φ_e is inverted and therefore Φ_e is correctly inverted on infinitely many functions from $\Phi_e(\mathcal{F})$. □

In the previous proof, Ψ_0 is a general recursive operator which strongly inverts $\mathcal{F}$-preserving and every general recursive operator Φ_e with $|\Phi_e(\mathcal{F})| = \infty$ on infinitely many functions from this range. This is not put into the formulation of the theorem as in the case that the index e is unknown to the inverting operator, the implication "strongly inverts $\Rightarrow$ inverts" is no longer clear.

The next result states that although one can invert infinitely many functions, it can be impossible to invert uncountably many. Thus only a tiny fraction of the image of the operator can be inverted to its origin.

Proposition 37. *There is a general recursive operator Φ such that the range of Φ is uncountable but every Ψ weakly inverts at most countable many of the functions in the range of Φ.*

Proof. For a function f, let $O_f = \{x : f(x) \text{ is odd}\}$. Now one defines Φ as follows:

$$\Phi(f)(x) = \begin{cases} 1 & \text{if } x \in O_f \text{ and for all } e \leq x, \\ & \text{either } W^{O_f}_{e,f(y)} \text{ has at least } y \text{ elements for all } y \in \{e, e+1, \ldots, x\} \\ & \text{or } W^{O_f}_{e,x} \text{ has at most } f(e) \text{ elements;} \\ 0 & \text{otherwise.} \end{cases}$$

On one hand, for every set O, there is a fast growing function f such that (the characteristic function of) O is $\Phi(f)$. On the other hand, if O is infinite and $O = \Phi(f)$ for some f, then f has to grow so fast that $f(e) \geq |W_e^O|$ whenever the latter cardinality is finite. So the range of Φ is $\{0,1\}^\infty$ but it is impossible for any Ψ to invert the characteristic function of any infinite set O. Indeed if $f = \Psi(O)$ would satisfy $\Phi(f) = O$, then the equivalence

$$W_e^O \text{ is finite} \Leftrightarrow |W_e^O| \leq f(e)$$

would show that $\{e : W_e^O \text{ is finite}\} \leq_T O'$, a contradiction. Thus O is not inverted by Ψ and only the characteristic functions of some of the finite sets are inverted by Ψ. □

7 Conclusion

In this paper we considered how and when general recursive operators can be inverted. The research was motivated by the fact that in many situations in real life, one is interested in finding what caused a certain result. We also introduced the notion of coverability, which allows us to find and study simpler representative enumerations of operators which satisfy some desired properties.

The main results of the present paper might be summarised as follows: The four presented notions of inversion, as well as the three notions of coverability, form a strict hierachy. Furthermore, all of the given concepts are shown to contain non-trivial classes.

From a practical point of view, strong inversion is the most interesting type, since it allows us to get a finite variant of the original input uniformly from $\Phi(f)$. Getting the exact input is much harder, as shown at the end of Section 5 about periodic functions. It would be interesting to further explore partial inversion, that is, we might not be able to invert an operator completely, but on sufficiently many outputs. Another interesting topic might be the inversion in other special cases similar to periodicity.

Although we have separated the above notions and given – as we hope – interesting examples, there is more to learn about these concepts. One goal will be to find interesting sufficient and/or necessary conditions for classes to be invertible or coverable. Again from the practical side, the first candidates to look at should probably be strongly invertible and strongly coverable.

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