Implementing Fragments of ZFC within an r.e. Universe*

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Abstract. The present work addresses the question: to what extent do natural models of a sufficiently rich fragment of set theory exist? Such models, here called Friedberg models, are built as a class of sets of natural numbers together with the element-relation "x is in y" given by $x \in A_y$, where A_0, A_1, A_2, $\ldots$ is a Friedberg numbering of all r.e. sets of natural numbers. A member A_x of this numbering is considered to be a set in the given model iff the transitive closure of the induced membership relation starting from x is well-founded. Furthermore, for all k, the set $B_k = \{x : A_x \text{ has size } k\}$ must be recursive. It will be examined whether the axioms of set theory and some basic set-theoretic properties hold in such a model. Because they do not hold in full generality, comprehension and replacement need to be properly adapted. The validity of the axiom of power set depends on the Friedberg model under consideration. The other axioms hold in every Friedberg model.

Keywords: recursively enumerable model; Friedberg numbering; axioms of Zermelo Fraenkel set theory (ZFC).

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1 Introduction

The present work starts with the question: which axioms of set theory, possibly suitably weakened, can hold in an r.e. model of set theory? An r.e. model of set theory has as domain the set $\mathbb{N}$ of natural numbers. The numbers represent the sets of the model, some of which can be ill-founded. The model has a binary r.e. relation. A pair (x, y) belongs to that relation iff the set represented by x is an element of the set represented by y. The relation is chosen in such a way that the sets A_0, A_1, A_2, $\ldots$ represented by 0, 1, 2, $\ldots$, respectively, are pairwise different.

Rabin [14, 17] showed that a certain conservative extension of ZFC (today known as NBG referring to von Neumann, Bernays and Gödel) has no r.e. model. The idea of the proof is that

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if such a model existed, basic functions (for example, the function that assigns to each $n \in \mathbb{N}$ the n^{th} ordinal) would be recursive. One could then find two recursively inseparable r.e. subsets A and B of $\mathbb{N}$ such that some first-order definable set X would separate the representatives of A and B in the model. But the subset of $\mathbb{N}$ which X is a representative of would then be recursive, which is impossible by the choice of A and B. To obtain his result, Rabin permitted the existence of proper classes in the model, defined as the members of the domain which are not element of any member of the domain. However, this is not essential for the argument: the proof could be modified to let proper classes be elements of proper classes, or to preclude proper classes.

On the basis of Rabin's result, the current study only aims at investigating how computable structures can capture parts of set theory, without satisfying all classical axioms. Other work along the same lines is in relation to Kripke-Platek (KP) set theory [1]. The theory of admissible sets restricts comprehension and replacement to Δ_0 formulas, and drops the axiom of power set. It has as a trivial recursive model the collection of hereditarily finite sets. The existence of computable models of fragments of KP set theory has been investigated, in particular, by Mancini and Zambella [10]. A totally different line of research deals with constructive or intuitionistic set theories — see for example Myhill [11] —, that can have recursive models — see for example Beeson [2]. Cantone, Ferro and Omodeo [3] give an overview of computable aspects and models of set theory.

In the present work, the approach is to exhibit a natural class of models (the Friedberg models), and then see how much of Zermelo Fraenkel set theory [4, 8, 19] can be recovered. Although most of the axioms are satisfied, Friedberg models have the following nonstandard features.

- Set-theoretic operations like union and intersection exist in a Friedberg model, but they are not effective. For example, one cannot compute the index of the union of two sets from their indices.
- Because the class of recursively enumerable sets is not closed under complementation, the difference of two sets might fail to belong to a Friedberg model.
- Full comprehension is not valid: one cannot use a first-order formula of arbitrary complexity to define a subset of a given set. What holds instead is the restricted variant that every r.e. subset of a set is a set in the model. This could be called comprehension along enumeration reducibility.
- The axiom of power set is satisfied in some and only some models, and power sets cannot be obtained in a uniform way (the index of the power set of X cannot be computed effectively from the index of X).
- All sets in the von Neumann universe are well-founded (in the sense that they satisfy the axiom of foundation, or equivalently, have an ordinal rank). In contrast, in the present work, some sets are not well-founded, as outlined in Definition 1 below. The ill founded sets of a model are referred to as its (proper) classes. Note that this use of the term "class" is totally different to its standard acception, that of a collection of sets which is not itself a set — sometimes, but not always, because it is "too large" to be a set. So what is called "set" here

is consistent with what is called "set" in classical theory, making the relationships between this framework and ZFC easier to express.

To summarise, the overall aim of this paper is to investigate the extent to which the problems identified by Rabin [14] can be overcome, and introduce a natural class of models, called Friedberg models, whose key properties are those outlined above. The general idea of a Friedberg model is formally captured as follows.

Definition 1. A *Friedberg numbering* A_0, A_1, A_2, ... is a one-to-one numbering of all r.e. subsets of $\mathbb{N}$ [5]. Such a numbering is called a *Friedberg model* if for all $k \in \mathbb{N}$, the set B_k defined as $\{x : |A_x| = k\}$ is recursive.

Let R be the unique partial mapping from the von Neumann Universe V into a Friedberg model $A_0, A_1, A_2, \ldots$ defined by transfinite induction as follows. Given a member X of V, $R(X)$ is defined and takes the value $x \in \mathbb{N}$ iff for all $Y \in X$, $R(Y)$ is defined and $A_x = \{R(Y) : Y \in X\}$.[3] Such a mapping R is called a *realisation*. The members A_x, $x \in \mathbb{N}$, of the Friedberg model are called *classes*,[4] and for a given $x \in \mathbb{N}$, A_x is said to be a *set* (*in the model*) iff x is in the range of R. A class that is not a set is called a *proper class*. Given $X \in V$ and $x \in \mathbb{N}$, if $x = R(X)$ then one says that A_x *realises* X.

Realisations define a partial order between Friedberg models as follows. A first model is smaller than or equal to a second model iff every set realised in the first model is also realised in the second model. This definition makes it possible to talk about smallest, largest, minimal, and maximal Friedberg models (with respect to set realisation).

It might be very difficult to identify the sets within a Friedberg model. Still, it will be shown below that there are Friedberg models for which the set of indices for the proper classes is recursively enumerable. The set C of indices for sets is not recursively enumerated, for then C would be a set whose index is not in the collection, a contradiction. For most proofs in this paper, it would be sufficient to assume that B_1 and B_2 are recursive; the postulate that B_3, B_4, ... are also recursive is a natural generalisation and it simplifies the setting. Theorem 4 below shows that there is a Friedberg model. The next example provides various sets and constructions that are realised in all Friedberg models.

An important question is: to what extent does a Friedberg model resemble the von Neumann universe of Zermelo Fraenkel set theory? The fundamental property of the von Neumann universe is that it consists of a hierarchy of sets built from sets defined on lower levels of the hierarchy. For example, the first ordinals (natural numbers) are

- 0, represented by $\emptyset$;
- 1, represented by $\{0\} = \{\emptyset\}$;
- 2, represented by $\{0, 1\} = \big\{\emptyset, \{\emptyset\}\big\}$;

[3] Note the analogy with Mostowski's collapsing lemma, where the relationship $x \in A_y$ on $\mathbb{N}^2$ plays the role of an arbitrary binary relation over a set in V.

[4] A class A_x is a set iff there exists no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = x$ and $\forall n\, [f(n+1) \in A_{f(n)}]$.

– 3, represented by $\{0,1,2\} = \big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\big\}$.

The way the hierarchy is captured in a Friedberg model is based on the following equivalence:

y is an element of x, viewed as a model of (some fragment of) set theory, iff $y \in A_x$.

For an illustration, 0 is represented by the index x_0 of the empty set, 1 by the index x_1 of the set $\{x_0\}$, 2 by the index x_2 of the set $\{x_0, x_1\}$ and 3 by the index x_3 of the set $\{x_0, x_1, x_2\}$. The first infinite ordinal would then be represented by the index x_ω of the set $\{x_0, x_1, x_2, \ldots\}$, provided that $\{x_0, x_1, x_2, \ldots\}$ is r.e., that is, exists as a set in the model. In general, a set X is realised in a Friedberg model iff every element of X is realised in the model and the collection of indexes of the realising subsets of $\mathbb{N}$ is r.e. Note that a set X exists as a class in a Friedberg model iff it exists as a set in the model. Here are some examples of sets which are realised in every Friedberg model.

Example 2. *Let b_0 be the index of the empty set. Inductively define b_{n+1} to be the index of the set $\{b_n\}$. Then $\{b_0, b_1, \ldots\}$ is r.e. and has an index x; in particular, A_x realises the infinite set $\big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\{\{\emptyset\}\}\}\}, \ldots\big\}$ (*the Zermelo integers*) from the von Neumann universe.*

Every hereditarily finite set is realised in every Friedberg model. This can be shown by induction over the hereditarily finite sets. More generally, if the sets X_1, X_2, …, X_n are realised in a Friedberg model, so is $\{X_1, X_2, \ldots, X_n\}$.

For all $x \in \mathbb{N}$, $\{e : (e \in B_1 \wedge A_e \subseteq A_x) \vee (e \in B_2 \wedge A_e \subseteq A_x)\}$ is recursively enumerable and represents the set of all unordered pairs in A_x. So the set of all unordered pairs over a set in a Friedberg model exists in the model.

If a set X is realised in a Friedberg model, and if given a subset Y of X, $\{R(Z) : Z \in Y\}$ is r.e., then Y is also realised in the model.

If A_x, A_y are two sets in a Friedberg model and $f : A_x \to A_y$ is partial recursive, then the function f is represented as $\big\{\{\{u\}, \{u, f(u)\}\} : u \in A_x\big\}$ and is recursively enumerable, thus f is realised in the model. This shows in particular that in a Friedberg model, there exists a bijection in the model between any pair of infinite sets in the model.

Every Friedberg model contains infinite sets. Given two infinite sets X and Y realised in the model, the Cartesian product $X \times Y = \big\{\{\{x\}, \{x, y\}\} : x \in X, y \in Y\big\}$ is realised in the model. Furthermore, if $\sqsubset$ recursively orders X, then $\big\{\{\{x\}, \{x, y\}\} : x, y \in X \wedge x \sqsubset y\big\}$ is realised in the model.

The reader is expected to be familiar with the fundamental constructions of recursion theory. Recall that the recursively enumerable sets of natural numbers are closed under union and intersection, but not under set difference. Recall that the image of an r.e. set is an r.e. set. There are special numberings of r.e. sets, most notably acceptable numbering and Friedberg numberings. Both enumerate all r.e. sets, but whereas the equivalence problem is Π^0_2-complete for an acceptable numbering, it is trivial for a Friedberg numbering $A_0, A_1, A_2, \ldots$, since $A_x = A_y$ iff $x = y$. Friedberg numberings [5] have many applications in recursion theory, and they play a key role in the present work. One often needs to refer to the set $A_{x,s}$ of elements enumerated into A_x within s steps. While there is no uniform decision procedure for A_x, there is one for $A_{x,s}$, whose maximal element is usually assumed to be at most s. Constructions are often carried out in stages, and in some cases during the construction, one might anticipate future events and so, for example, refrain from acting to make two sets explicitly different in a Friedberg numbering if one already knows that they will eventually be different thanks to further constraints in the construction.

There is a comprehensive body of research on recursion theory. The interested reader is referred to the classical textbooks by Odifreddi [12, 13], Rogers [15] and Soare [18] for background and further reading.

Section 2 gives an overview of the extent to which the axioms of Zermelo Fraenkel set theory are satisfied in the framework that is the object of study of this paper. The existence of Friedberg models is shown, and Theorem 6 provides an example of a model where all recursive ordinals exist in the model. On the other hand, the set V_ω of all hereditarily finite sets does not exist in any Friedberg model. Some axioms are satisfied only partly, which will be explained in detail; this is the case, in particular, for the comprehension schema.

Section 3 deals with the complexity of the predicate $Set(x)$ which is true iff A_x is a set. It will be shown there that the complexity of this predicate ranges from Π^0_1 to Π^1_1.

In Section 4, the existence of a smallest Friedberg model will be shown. It consists precisely of the sets of the von Neumann universe which exist in all Friedberg models. The smallest Friedberg model can also be characterised as consisting of the members of the von Neumann universe that exist in some Friedberg model, and have the additional property that all but finitely many members of their transitive closure have a cardinality bounded by some constant k. It will also be shown that there is no largest Friedberg model: for every Friedberg model M, there is a set of the von Neumann universe which exists in some Friedberg model but not in M.

Section 5 provides a methodology to study the theory of a Friedberg model and its complexity. It is shown that two natural approaches to define such a theory can be translated one into the other. Therefore, the complexity in terms of Turing degrees is the same for both approaches. The actual complexity is mainly determined by the complexity of the predicate Set.

Section 6 deals with the axiom of power set. It is the only axiom whose validity depends on the underlying Friedberg model. While many Friedberg models do not satisfy the axiom, the proof of Theorem 32 exhibits a specific Friedberg model for which the axiom does hold. In this setting, the power set of a set A can only be the set of all r.e. subsets of A, which is not large enough to preserve Cantor's result that the cardinality of the power set of a set A is greater than that of A. As already noted, given two infinite sets A_x and A_y in a Friedberg model, there exists in the Friedberg model itself a recursive permutation from A_x into A_y. As a consequence, whenever the power set of some infinite set A in a Friedberg model belongs to the model, it has the same cardinality as A (both outside and within the model).

2 Friedberg models and the axioms of ZFC

The beginning of this section provides an overview of the extent to which a Friedberg model satisfies the axioms of Zermelo Fraenkel set theory. For some axioms, Friedberg models satisfy only certain of the standard variants. Moreover, variants of an axiom that are equivalent in the presence of the other axioms of ZFC may not be equivalent here because the axioms that are needed to prove the equivalence fail to hold.

1. **Axiom of extensionality**. If $x \neq y$ then $A_x \neq A_y$. This property follows directly from the fact that A_0, A_1, $\ldots$ is a Friedberg numbering.
2. **Axiom of regularity**. Well foundedness distinguishes sets from classes in this setting, where the axiom of regularity is captured indirectly. Recall that a set in a Friedberg model is defined as the image of a set in the von Neumann universe under the realisation functor R. It follows that whenever A_x is a nonempty set in a Friedberg model, there exists $y \in A_x$ such that no $z \in A_x$ satisfies $z \in A_y$. It should be noted that some classes also satisfy the axiom of regularity, so this axiom is not strong enough to separate the sets from the classes. (Similarly, a set in the von Neumann universe can fail to be well-founded from the outside, though it satisfies the axiom of regularity.) The axiom has some interesting consequences, such as the fact that $x \notin A_x$ for all sets in a Friedberg model. In some constructions, one can use regularity in order to "force" some numbers to be indices of classes, using for instance a particular enumeration technique.
3. **Axiom schema of comprehension**. As noted already, a class is a set iff all its members are sets. Recall that the element-relation on the transitive closure of a class is not well-founded if the class is proper, and is well-founded if it is a set. So if one intersects a set and a class, then the "bad" elements of the class do not belong to the intersection: whenever A_x is a set and A_y is a class, $A_x \cap A_y$ is a set, that is, one can have a set as a subset of a class. Still this is a weak form of comprehension, as the difference of two sets in a Friedberg model does not always exist, neither as a set nor as a proper class (on the other hand, arbitrary unions — see below — and finite intersections of sets exist as sets). Furthermore, formulas with universal quantification fail to define sets (on the other hand, existential quantification is safe as it defines a new set in a positive way). A more general comprehension property can be derived from the fact that whenever E is e-reducible to A_x, then there exists y with $A_y = E$. This allows one to perform some operations expressed in terms of e-reducibility.
4. **Axiom of pairing**. Given two sets A_x and A_y in a Friedberg model, there is $z \in \mathbb{N}$ such that $A_z = \{x, y\}$. So the axiom of pairing is satisfied in an effective way: there is a recursive function f such that $A_{f(x,y)} = \{x, y\}$ for all $x, y \in \mathbb{N}$.
   This has consequences for various constructions based on the axiom of pairing: the class of all ordered pairs defined as $\{\{x\}, \{x, y\}\}$ exists in every Friedberg model, and thus, one can define Cartesian products and functions, since a function f on a domain A_x is the set $\big\{\{\{y\}, \{y, f(y)\}\} : y \in A_x\big\}$. Note that f exists in the Friedberg model iff it is partial-recursive. Furthermore, whenever a function is defined on a class, its restrictions to sets which are subclasses of the given class in the Friedberg model exist in the model.

Note that the requirement of partial-recursiveness can be quite restrictive: while there is always a function f satisfying $A_{f(x)} = \{x\}$ for all x (even for indices of classes), there is also for every infinite r.e. set A_x an infinite subset A_z of A_x such that the function g with domain A_x satisfying $A_{g(y)} = A_z \cap A_{f(y)}$ is not partial-recursive.

5. **Axiom of union**. Since a Friedberg numbering covers all r.e. subsets of $\mathbb{N}$, if A_x is a set in a Friedberg model, then A_y defined as $\{z : \exists u \in A_x\,[z \in A_u]\}$ is a set in the model. Still, y cannot be effectively computed from x.
6. **Axiom of replacement**. For every set A_x, and for every partial function f in the Friedberg model that is defined at least on A_x and satisfies that $A_{f(y)}$ is a set for all $y \in A_x$, there exists $z \in \mathbb{N}$ such that $A_z = \{f(y) : y \in A_x\}$ and A_z is a set in the Friedberg model.
The axiom is mainly true, since any function which is a class in the Friedberg model can be identified with a partial-recursive function. If such a partial function is defined on every member of a set A_x in the Friedberg model, then $\{f(y) : y \in A_x\}$ is an r.e. set and has an index. Furthermore, it is well-founded, and it is a set in the Friedberg model.
This is why this extra condition is needed. Also, though Jech [8] and other authors use functional classes to formulate the axiom schema of replacement, it would be misleading to adopt the same approach and have the axiom of replacement satisfied as a consequence of this latitude. Indeed, what defines a class in the present setting is totally different to what defines a class in classical set theory. Thus, it would be more useful to require that every first-order definable function maps sets to sets. Still, this is too strong a requirement, as one can first-order define a function that maps some infinite set to one of its subsets that is not represented in the Friedberg model.
If one imposed that replacement be satisfied for every partial function which (a) is represented in the model as a class and (b) maps sets to sets, then that form of replacement would be satisfied in a Friedberg model. However, such an interpretation would be a bit too restrictive and fail to account for the intuition behind the schema of replacement. It seems more fair to say that the axiom is only partially satisfied.
7. **Axiom of infinity**. There is $x \in \mathbb{N}$ such that A_x is an infinite set. This set can be constructed by searching inside B_1. More precisely, let b_0 be such that $A_{b_0} = \emptyset$, and let b_{n+1} be inductively defined as the unique element of B_1 with $b_n \in A_{b_{n+1}}$ — see Example 2 above. Note that this procedure shows the existence of A_x, but it does not give its index x effectively.
A more traditional approach would be to postulate the existence of an inductive set. So, one would postulate that there is a set A_x in the Friedberg model with the following properties: the index of the empty set is in A_x, and for every $y \in A_x$, there is $z \in A_x$ such that for all u, $u \in A_z \Leftrightarrow u \in A_y \vee u = y$. This version of the axiom of infinity is satisfied only by some Friedberg models, not all of them. In particular, it is not satisfied by the smallest Friedberg model constructed in Theorem 17 below, whereas it is satisfied by the Friedberg model constructed in Theorem 12 below. As in ZFC, assuming the existence of an infinite ordinal in the model is sufficient to satisfy this variant of the axiom of infinity.
8. **Axiom of power set**. It takes the following form, and it holds only in some but not all Friedberg models: for every set A_x in the model, there is a set A_y in the model such that

$A_y = \{z : A_z \subseteq A_x\}$. In any case, the power set cannot be formed in a uniform way, that is, there is no function f defined at least on all indices of sets such that $A_{f(x)} = \{z : A_z \subseteq A_x\}$. If such an f would exist, there would be $x \in \mathbb{N}$ with $A_x = V_\omega$, by uniformly taking $\emptyset$, the power set of $\emptyset$, the power set of the power set of $\emptyset$, ... Then, V_ω, as the union of those sets, would be represented by an r.e. set, and therefore, should be in the model, which would then contradict Theorem 7 below. But it will be shown in Theorem 32 below that there exists a Friedberg model in which every set has a power set, and every class a power class.

9. **Axiom of choice**. There is a partial-recursive function ch which maps every x with $A_x \neq \emptyset$ to some member of A_x (just looking for such an element); ch is undefined only on the index of the empty set. As the graph of ch consists of the ws with $A_w = (v, u)$ and $u = ch(v)$, it is an r.e. set. Therefore, it exists as a class in the Friedberg model. Furthermore, for every set A_x in the model, A_y defined as $\{w : A_w = (v, u) \wedge v \in A_x \wedge ch(v) = u\}$ exists as a set in the model. It is the graph of ch restricted to A_x. This means that the axiom of choice is realised in the strongest possible form in every Friedberg model, in a way which is not required to hold in models of ZFC.

To prove the existence of a Friedberg model, and for other constructions, the following preliminary notions, conventions and background knowledge will be useful.

Background 3. In a Friedberg model, the sets A_x are uniformly recursively enumerable, pairwise distinct, and their constructions are performed in stages; $A_{x,s}$ denotes that part of A_x which is constructed by stage s, consisting of elements which are usually at most equal to s, and that might encompass some information that allows the construction to proceed further. The constructions are always monotone, so $A_{x,s} \subseteq A_{x,s+1}$, and A_x is the union of all $A_{x,s}$, $s \in \mathbb{N}$.

Often, the constructions use some enumeration E_0, E_1, E_2, ... of some family of recursively enumerable sets as a parameter. This enumeration might itself be constructed in a particular way. Again, $E_{d,s}$ denotes the set of elements, usually all at most equal to s, enumerated into E_d within s stages. The stages might take the form of a basic step in case there are no constraints on E_d. Also, $E_{d,s}$ might consist of elements enumerated within t steps for some $t \leq s$, with some explicit side-condition being imposed at stage t that is still valid at stage s. In all constructions, $E_{d,0}$ is the empty set.

Various constructions explicitly mention the d^{th} finite set; it is always the unique finite set X satisfying $d = \sum_{a \in X} 2^a$.

The next notion is particularly important. The ordinal height of a Friedberg model is the smallest ordinal α that is not realised in the model, that is, the smallest ordinal α for which there exists no set A_x in the model that is isomorphic to α as a well-ordering. It is clear that the ordinal height of a Friedberg model cannot be larger than the Church Kleene ordinal, ω_1^{CK}. It will be shown below that the ordinal height can take both possible extreme values ω and ω_1^{CK}. Moreover, it will be shown that not all values in between are possible; in particular, the ordinal height must be a limit ordinal.

A set E is enumeration-reducible to a set F in case there is an r.e. set G of conditions of the form $x_1 \wedge x_2 \wedge \ldots \wedge x_n \rightarrow y$ such that E consists of all y for which there is a condition

$x_1 \wedge x_2 \wedge \ldots \wedge x_n \rightarrow y$ in G with $x_1, x_2, \ldots, x_n$ in F. Enumeration-reducibility is not essential to this paper. Still, it is a convenient notion used to express some closure-properties of Friedberg models.

Theorem 4. *There is a Friedberg model, that is, a Friedberg numbering $A_0, A_1, A_2, \ldots$ of all r.e. subsets of $\mathbb{N}$ such that $B_k = \{x : |A_x| = k\}$ is a recursive subset of $\mathbb{N}$ for each k.*

Proof. Friedberg [5] showed the existence of a one-to-one numbering of all r.e. subsets of $\mathbb{N}$. It is easy to modify his construction to get a one-to-one numbering $E_0, E_1, E_2, \ldots$ of all r.e. subsets of $\mathbb{N}$ with at least 2 nonelements. Now, let $A_0 = \emptyset$, $A_1 = \mathbb{N}$, and define A_x for x equal to 2, 3, $\ldots$ according to which case below applies and for which the priority is highest, that is, has the lowest numerical value. Below, one says that A_x *follows* Y (*at stage* s) if all elements of Y_s are in A_x at stage s, and A_x has been explicitly declared to be a *follower* of Y. Note that at a later stage s', A_x may stop following Y, and at such a stage s', only finitely many elements of Y will have been enumerated into A_x. Furthermore, one can always check in the construction which target A_x is currently following.

- Priority $3d$: This case applies if $A_{y,x} \neq X$ for all $y < x$ where X is the d^{th} finite subset of $\mathbb{N}$. If this case is selected, then A_x follows X as long as $E_{e,s} \neq X$ for all $e < |X|$. In case there is $e < |X|$ and a stage $s > x$ with $E_{e,s} = X$, A_x stops following X and starts to follow $\mathbb{N} \setminus \{u\}$ forever, where u is the least number such that $u \notin X$ and $\mathbb{N} \setminus \{u\}$ does not currently have a follower.
- Priority $3d + 1$: This case applies if $|E_{d,x}| > d$ and E_d does not yet have a follower. If this case is selected, then $A_x = E_d$; that is, A_x follows E_d forever.
- Priority $3d + 2$: This case applies if no A_y with $1 < y < x$ is currently following $\mathbb{N} \setminus \{d\}$. If this case is selected, then $A_x = \mathbb{N} \setminus \{d\}$; that is, A_x follows $\mathbb{N} \setminus \{d\}$ forever.

Note that the priorities of different choices are different. Therefore, it is always determined which choice is taken to initialise A_x. It follows from the priorities that every subset of $\mathbb{N}$ of the form $\mathbb{N} \setminus \{u\}$ eventually receives exactly one follower A_x. Furthermore, each E_d with at least $d + 1$ elements eventually receives exactly one follower. Since whenever $c \neq d$, $E_c \neq E_d$, no infinite subset of $\mathbb{N}$ eventually has two followers. But if E_d is finite, there might be some A_x initially following E_d. This A_x will then stop following E_d, and instead start to follow some $\mathbb{N} \setminus \{u\}$. Hence, all the E_d eventually have a unique follower A_x, and all A_y with $y \neq x$ are different from E_d. Furthermore, in case there is no $d < |X|$ with $E_d = X$, the protocol for the finite A_x created by the first case makes sure that each X exists in the model. Hence, the resulting numbering is one-to-one, that is, a Friedberg numbering. Furthermore, one can see that for every $k \in \{1, 2, 3, \ldots\}$, there are only finitely many x such that initially, A_x has the cardinality k, and later, A_x receives more elements. Hence, each B_k is a finite variant of $\{x : A_x$ has at the initialisation k elements$\}$, and so each B_k is recursive. This completes the proof that the constructed model is a Friedberg model. ∎

Harrison [7] showed that there is a recursive linear ordering with an initial segment of type ω_1^{CK}, where ω_1^{CK} is the first nonrecursive ordinal, named after Church and Kleene. The next proposition, due to Harrison, is subsequently used to show that the ordinal height of a Friedberg model can be ω_1^{CK}.

Proposition 5 (Harrison [7]). *There is a recursive linear ordering $\sqsubset$ on the set of natural numbers with the following properties.*

- *For every recursive ordinal α, there is an x such that $\sqsubset$ well-orders $\{y : y \sqsubset x\}$, and makes this set isomorphic to α.*
- *If W is an r.e. bounded set (or even a hyperarithmetic bounded set), then W has a least upper bound b in this ordering.*

Proof. This recursive linear ordering $\sqsubset$ is given as the Kleene-Brouwer ordering on a tree $T \subseteq \mathbb{N}^*$, which has an infinite branch but no hyperarithmetic infinite branch.

For the second item, Harrison [7] showed that for every hyperarithmetic set W, which one can, without loss of generality, assume to be closed downward under $\sqsubset$, a node is an upper bound for W iff it is outside W. In order to prove the existence of a least upper bound, let a_n be the least upper bound of W among all nodes up to height n in the tree. Note that all a_n exist. Indeed, a_0 is the root of T and an upper bound of W; and the set of nodes up to height n is well-ordered by the Kleene-Brouwer ordering (as the subtree of all nodes up to height n has no infinite branch). For each n, one has either $a_{n+1} = a_n$ or $a_{n+1} \sqsubset a_n$. In the latter case, the height of a_{n+1} is $n + 1$, and a_{n+1} is a successor of a_n as a node in T. (Otherwise, a_{n+1} would be a successor of a node a'_n of height n with $a'_n \sqsubset a_n$, in contradiction to the choice of a_n as the least upper bound of W within that subset of T that contains the nodes up to height n.) Note that either almost all a_n are the same node b, or they describe an infinite branch F in the tree. In the first case, the node b is a least upper bound of W. In the second case, one can compute F inductively using the halting problem W' relative to W, as one would have that a_{n+1} is the least successor of a_n which is not in W. Since T has no infinite hyperarithmetic branch, and since every set recursive in W' is hyperarithmetic, the first case applies, and the downward closure of W under $\sqsubseteq$ is $\{x : x \sqsubset b\}$.

For the first item, let α be the least ordinal that is not isomorphic to an initial segment of $\sqsubseteq$. This α cannot be a recursive limit ordinal, for one would otherwise (a) obtain a recursive representation of α, (b) see that every initial segment is isomorphic to a unique initial segment of the Harrison ordering, and (c) then conclude that the limits of both initial segments are isomorphic. If $\alpha = \beta + 1$ and β is a recursive ordering that is isomorphic to an initial segment of $\sqsubset$, one can then take a representation, and see that this initial segment is hyperarithmetic: It can be computed relative to every f which is an order-preserving mapping from the ordering representing the ordinal β onto an initial segment of $\sqsubset$. It can also be computed relative to such an f, as such an f must exist. So one could either existentially or universally quantify over f, and formalise the properties of f with additional number quantifiers, to express that the initial segment of $\sqsubset$ isomorphic to β is hyperarithmetic. It is bounded and has an upper bound b, which could then be joined to β, in order to get an initial segment isomorphic to α. Hence, α must be ω_1^{CK}. ∎

Theorem 6. *There is a Friedberg model in which every recursive ordinal is represented, that is, has the ordinal height* ω_1^{CK}.

Proof. Let $\sqsubset$ be the recursive linear ordering from Proposition 5, and define $A_{2x} = \{2y : y \sqsubset x\}$ for all x. Furthermore, let $A_1 = \{0, 2, 4, 6, \ldots\}$ and $A_3 = \mathbb{N}$.

For the remaining members of the Friedberg model, one adapts the proof of Theorem 4 in order to turn this construction into a Friedberg model. Let E_0, E_1, E_2, $\ldots$ be a one-to-one numbering of all r.e. subsets of $\mathbb{N}$ with at least 2 nonelements. The main modification in the construction is that one has to add some witnesses in order to keep the new members of the numbering different to those with an even index that have been built so far. Now, A_{2x+5} is selected from the following options. Let X be the d^{th} finite subset of $\mathbb{N}$.

- Priority $3d$: This case applies if $A_{2y+5,x} \neq X$ for all $y < x$, and either X contains an odd element, or there are v, w found within time x with $v \sqsubset w$, $v \notin X$ and $w \in X$. If this case is selected, then A_{2x+5} follows X as long as $E_{e,s} \neq X$ for all $e < |X|$. If at stage $s > x$, there is an index $e < |X|$ with $E_{e,s} = X$, then A_{2x+5} starts to follow $\mathbb{N} \setminus \{u\}$ for the least u such that $u \notin X$ and $\mathbb{N} \setminus \{u\}$ has no other follower so far.
- Priority $3d + 1$: This case applies if $|E_{d,x}| > d$, E_d does not have a follower yet, and either an odd element has been found in E_d at time x, or there are v, w found within time x with $v \sqsubset w$, $v \notin E_{d,x}$ and $w \in E_{d,x}$. In the latter case, v is kept fixed from now on. If this case is selected, then A_{2x+5} follows E_d, until a stage s is reached with $\{v\} \subseteq E_{d,s} \subseteq \{0, 2, 4, \ldots\}$. If such a stage s is reached, then A_{2x+5} starts to follow $\mathbb{N} \setminus \{u\}$ for the least u such that u is not yet enumerated into A_{2x+5}, and $\mathbb{N} \setminus \{u\}$ does not have another follower so far.
- Priority $3d + 2$: This case applies iff no A_{2y+5} with $y < x$ is currently a follower of $\mathbb{N} \setminus \{d\}$. If this case is selected, then A_{2x+5} follows $\mathbb{N} \setminus \{d\}$ forever.

First, it should be noted that all r.e. subsets of $\mathbb{N}$ which are closed downward along $\sqsubset$ are either $\mathbb{N}$ itself, or of the form $\{v : v \sqsubset w\}$. Hence, in the construction, all r.e. proper subsets W of $\{0, 2, 4, \ldots\}$ which are not covered by A_0, A_2, A_4, $\ldots$ satisfy: there exists v, w with $v \sqsubset w$, $2v \notin W$ and $2w \in W$. Hence, each such W will eventually qualify with priority $3d$ or $3d + 1$ when the corresponding witnesses are found. There might be prior witnesses that are wrong, and lead to the follower to be cancelled, but eventually, the right witnesses will be found. If (i) W is r.e., (ii) W is a proper subset of $\{0, 2, 4, \ldots\}$, and (iii) $\{w : 2w \in W\}$ is closed downward under $\sqsubset$, then W is equal to some A_{2x}. Eventually, since a correct pair v, w of witnesses cannot be found, all attempts to follow W by priority $3d + 1$ will fail.

Second, the remaining verification is similar to the verification in Theorem 4. To see that the numbering is one-to-one, note that explicit collisions can only happen between finite subsets X of $\mathbb{N}$ and E_d, where the follower of X then gets destroyed. Each subset E_d of $\mathbb{N}$ can destruct only one follower for each of the cardinalities $d + 1$, $d + 2$, $\ldots$ Hence, for each k, it happens at most k times that a follower of a subset of $\mathbb{N}$ with k elements is destroyed. Therefore, A_0, A_1, A_2, $\ldots$ is a one-to-one numbering of all r.e. subsets of $\mathbb{N}$. Furthermore, each B_k is a finite variant of the corresponding class $\{2x + 5 : A_{2x+5}$ was initialised as a follower of an X with k elements$\}$, and, as a consequence, each B_k is recursive. ▮

Recall that in set theory, V_α is the set of all sets of rank less than α, where the rank function ρ is defined inductively as: $\sup(\emptyset) = 0$, and for every nonempty set x, $\rho(x) = \sup\{\rho(y) + 1 : y \in x\}$. In particular, V_ω captures a natural concept: it consists of all hereditarily finite sets, that is, all sets whose transitive closure is finite. (The transitive closure of a set x is the set of all y for which there exists k and $z_0, z_1, \ldots, z_k$ such that $z_0 = x$, $z_k = y$, and for all $h < k$, $z_{h+1} \in A_{z_h}$.) In a Friedberg model, a set x belongs to V_ω iff x is acyclic and the transitive closure of x is finite. The next result shows that V_ω does not exist in any Friedberg model. In particular, there is no function representing the rank of sets in a Friedberg model.

Theorem 7. *There is no Friedberg model in which the set V_ω of all hereditarily finite sets exists in the model.*

Proof. Essentially, this statement says that in every Friedberg model, V_ω is a Σ^0_2 set, but not an r.e. set. Therefore, V_ω is not represented in a Friedberg model. For a contradiction, suppose otherwise, that is, $V_\omega = \{x : A_x \text{ is hereditarily finite}\}$ is recursively enumerable. Furthermore, let E be an infinite recursive subset of $\{b_0, b_1, b_2, \ldots\}$ which is defined by b_0 being the unique index of the empty set, and b_{n+1} being the unique index of $\{b_n\}$. As B_1 is recursive, one can find b_{n+1} by searching in B_1 for the unique member of B_1 which contains b_n.

Now, one constructs inductively the following infinite subset $\widetilde{E} = \{e_0, e_1, e_2, \ldots\}$ of E: e_y is the first element $s \in E$ strictly greater than all e_z with $z < y$ such that at least one of the following two conditions holds:

- y is enumerated into V_ω within s steps;
- there is an $u \in A_{y,s} \setminus \{e_z : z < y\}$ with $u < s$.

The search for s terminates. Indeed, E is infinite, and one of the following conditions holds for all sufficiently large $s \in E$:

- A_y is hereditarily finite and y shows up in V_ω within s steps;
- A_y is finite but not hereditarily finite and A_y contains a member outside E, which shows up within s steps and is strictly below s;
- A_y is infinite and contains some element $u \notin \{e_z : z < y\}$, which shows up within s steps and is strictly below s.

So $\widetilde{E}$ is an infinite and recursive subset of E. Furthermore, $\widetilde{E}$ differs from all infinite subsets A_y of E. Indeed, there is a $u < e_y$ such that $u \notin \{e_z : z < y\}$, while $\widetilde{E}$ contains exactly the e_z with $z < y$ as elements below e_y. It follows that $\widetilde{E}$ is an infinite recursive subset of $\mathbb{N}$ not occurring in the list $A_0, A_1, A_2, \ldots$, in contradiction to the assumption on the model. ∎

This result establishes that one cannot enumerate all indices of sets of finite rank. Hence, the rank of sets is not represented in this model at all. It is natural to ask whether one can access cardinality. In other words, one might ask whether there is a function ψ such that $\psi(x)$ is defined and equal to $|A_x|$ when A_x is finite, but undefined when A_x is infinite. The answer is "no". Indeed, assuming that the answer was "yes", one could enumerate V_ω inductively: first put the index of the empty set into V_ω, and then add at every stage s all x such that $\psi(x)$ has converged within s steps to some value k, and all k members of A_x are already enumerated into A_x and into V_ω. Hence ψ cannot exist.

Corollary 8. *There is no Friedberg model in which there is a partial-recursive function ψ such that $\psi(x) = k$ iff $|A_x| = k$. So, the cardinality even of finite sets (and classes) cannot be computed, and B_0, B_1, B_2, $\ldots$ cannot be uniformly recursively enumerable.*

Remark 9. A key difference with the traditional setting is that in a Friedberg model, all infinite sets have the same cardinality. This follows from the fact that for any two infinite r.e. sets A_x and A_y, there is a partial-recursive bijection from A_x onto A_y with domain A_x.

It has been mentioned above that the union and the power set cannot be formed effectively. The same is true for the intersection, even if it is intersection with singletons. Note that one can, for each z, compute an index $f(z)$ of the set $\{z\}$, by searching for that $b \in B_1$ with $z \in A_b$.

Proposition 10. *For every infinite set A_x, there is an infinite subset A_y of A_x such that the mapping g given by $z \mapsto A_y \cap \{z\}$ for all $z \in A_x$, does not exist in the Friedberg model. In particular, there is no partial-recursive function which realises the intersection of sets.*

Proof. Given an infinite r.e. set A_x, A_x has an infinite recursive subset R, and R has an infinite r.e. subset A_y that is not recursive. Now, assume, by way of contradiction, that the function g that maps z to an index for $A_y \cap \{z\}$, is recursive. One can then determine whether $z \in A_y$, by checking whether $z \in R$ and, if so, whether $g(z)$ differs from the fixed index of the empty set. This gives a contradiction.

Since every Friedberg model has an infinite set A_x, and since the mapping $z \mapsto f(z)$ with $A_{f(z)} = \{z\}$ is recursive, the nonrecursiveness of the above g implies: it is also impossible to, from the indices x and y, find an index for $A_x \cap A_y$, even if one wants this mapping only for sets and not for all classes. ∎

3 The complexity of recognising the sets

In this section, a Friedberg model is constructed such that the indices of the sets in the model are co-r.e., and thus, much below the maximal possible complexity Π^1_1. The formalisation makes use of a predicate *Set* such that *Set*(x) is true iff A_x is a set. The proof of Theorem 12 below will use the following result of Kummer.

Theorem 11 (Kummer [9]). *Assume that $\mathcal{A}$ has a numbering, $\mathcal{B}$ has a one-to-one numbering, $\mathcal{A}$ and $\mathcal{B}$ are disjoint, and every finite set has a superset in $\mathcal{B}$. Then $\mathcal{A} \cup \mathcal{B}$ has a one-to-one numbering.*

Theorem 12. *For each recursive ordinal α, there is a Friedberg model of ordinal height strictly above α where Set is of complexity Π^0_1, that is, $\{x : A_x \text{ is a proper class}\}$ is recursively enumerable.*

Proof. Let α be an infinite recursive ordinal. Let $\sqsubset$ be a recursive well-ordering on the set of even numbers of order type α with least element 0. Set $A_{2x} = \{2y : 2y \sqsubset 2x\}$. Let E_0, E_1, E_2, ... be a Friedberg numbering of all r.e. subsets of $\mathbb{N}$ that are different from $\{0, 2, 4, \ldots\}$, different from A_0, A_2, A_4, ..., and that have at least 2 nonelements. The existence of such a Friedberg numbering follows from the result of Kummer cited above, Theorem 11. The classes $\mathcal{A}$ and $\mathcal{B}$ mentioned in the theorem are going to be defined as follows. The class $\mathcal{A}$ contains all r.e. sets of the form $W \setminus \{u, v, w\} \cup \{t\}$ where W is r.e., u, v, w, t are distinct, and either u, t are even and $u \sqsubset t$, or t is odd. The class $\mathcal{B}$ contains all subsets of $\mathbb{N}$ with exactly two nonelements. This class clearly has a one-to-one numbering. Thus, also their union $\mathcal{A} \cup \mathcal{B}$ has the desired one-to-one numbering E_0, E_1, E_2, ... by Kummer's result [9].

For the following construction, let U be the set of all indices x of sets A_x such that the transitive closure of their the element relation is not well-founded in the following sense: there are a function f and numbers n, m with $n > m$ such that $f(0) = x$, $f(k+1) \in A_{f(k)}$ for all $k < n$ and $f(m) = f(n)$. Note that U is recursively enumerable. An enumeration of U will be used when defining A_x for an odd x. Let U_s be the set of elements enumerated into U within s steps.

So, let $A_1 = \mathbb{N}$, let $A_3 = \{0, 2, 4, \ldots\}$, and for each odd $x > 3$, let A_x be initialised according to the entry of highest priority (where the high priorities have low numerical value and vice versa).

- With priority $3d$, make A_x a follower of the d^{th} finite set X if $x > \max(X)$ and for all $y < x$, all sets $A_{y,x}$ are different from X, and $X \neq A_z$ for all even z.
- With priority $3d + 1$, make A_x a follower of E_d provided that $|E_{d,x} \cap \{0, 1, 2, \ldots, x - 1\}| > d$ and E_d has no current follower.
- With priority $3d + 2$, let $A_x = \mathbb{N} \setminus \{d\}$ provided that $\mathbb{N} \setminus \{d\}$ has not already been assigned a follower.

After choosing the A_x according to the priority, one updates A_y for odd y with $3 < y < x$ provided that one of both following cases applies.

- If A_y is initialised as the d^{th} finite set X and there is some e with $E_{e,x} = X$ and $|E_{e,x}| > e$, then let A_y follow $\mathbb{N} \setminus \{u\}$ for the first u found such that $u \notin X$ and $\mathbb{N} \setminus \{u\}$ does not yet have a follower.
- If A_y is initialised as E_d for some d, there are an odd $z < x$ and $d' \neq d$ with A_z following $E_{d'}$, $A_{y,x} \cap U_x = \emptyset$, and there are a function f and $n > 0$ with $f(0) = y$, $f(n) = z$, and $f(m+1) \in A_{f(m)}$ for all $m < n$. Then, let A_y, A_z follow $\mathbb{N} \setminus \{v\}$ and $\mathbb{N} \setminus \{w\}$, respectively, where $\mathbb{N} \setminus \{v\}$ and $\mathbb{N} \setminus \{w\}$ have not yet a follower, v is not yet enumerated into A_y, w is not yet enumerated into A_z, and $v \neq w \Leftrightarrow y \neq z$.

First, it will be shown that each r.e. set W appears exactly once in the numbering.

- One assigns for each set $\mathbb{N} \setminus \{u\}$ exactly one follower which never abandons this set. Hence, every set $\mathbb{N} \setminus \{u\}$ equals exactly one A_x, $\mathbb{N}$ equals A_1, and the set of even numbers equals A_3.

– Let W be given such that W has at least two nonelements, and there is a d with $E_d = W$ and $|E_d| > d$. There is then a stage x such that $|E_{d,x} \cap \{0, 1, 2, \ldots, x\}| > d$. Hence, for stage $y \geq x$, the set A_y is made equal to E_d with priority $3d + 1$, and, hence, E_d will eventually appear in the list A_0, A_1, A_2, $\ldots$ previously defined. It can happen that the first instantiations of E_d will be transformed into sets of the form $\mathbb{N} \setminus \{u\}$. But then, some element of E_d goes into U, and this happens only finitely often, so E_d will eventually have an index. Furthermore, if E_d is finite and there is an A_y initialised as a finite set X equal to E_d, then A_y will eventually abandon X and follow the set $\mathbb{N} \setminus \{u\}$ for some u. Hence, no such A_y ends up being equal to E_d. If E_d is infinite, then its complement has at least two elements, and it cannot be equal to any $E_{d'}$ by the choice of the numbering E_0, E_1, E_2, $\ldots$ from which the infinite members in the numbering A_0, A_1, A_2, $\ldots$ are built.

– Assume the last case, where W is finite and not equal to any E_d instantiated in the numbering. Let $d = \sum_{a \in W} 2^a$. A follower will then eventually be assigned to W with priority $3d$. Furthermore, each of the sets $E_{d'}$ with $d' < |W|$ can force the follower to abandon W and to follow a set of the form $\mathbb{N} \setminus \{u\}$ instead. Still, at some later stage, some further element will be enumerated into $E_{d'}$, and a new set will start to follow W. From the way the numbering A_0, A_1, A_2, $\ldots$ is constructed, it follows that W shows up there exactly once.

Second, it will be shown that all B_k are recursive. Note that almost all sets which initially follow a finite set X of cardinality k will never abandon X. Only finitely many initialisations of these sets get modified, temporarily clashing with some $E_{d'}$ with $d' < k$. Only finitely many of them coincide with some $E_{d'}$, which does not get its index at the first attempt, but might be moved when the first of its elements goes into U. Furthermore, there is only one even x such that A_x has exactly k elements. So, B_k is a finite variant of the set of all odd indices x where A_x is initialised as a set of k elements, and A_x is defined first. Hence B_k is recursive.

Third, it will be shown that U contains all x such that A_x is a proper class. It is obvious that all $x \in U$ are indices of proper classes. It has already been noted that U is recursively enumerable. Obviously, U is also infinite. Now, assume that $x \notin U$ but there is a function f such that $f(0) = x$ and for all k, $f(k+1) \in A_{f(k)}$. It follows from $x \notin U$ that there are no distinct m, n with $f(m) = f(n)$. Furthermore, there is also no m with $A_{f(m)}$ being cofinite: otherwise, $A_{f(m)}$ would contain an element of U, and so $x \in U$, in contradiction to the assumption. Similarly, $f(m) \neq 1$ for all m. Assume for a contradiction that there are m, n, d, d' such that $A_{f(m)} = E_d$, $A_{f(n)} = E_{d'}$, $m < n$, $d < d'$, and no element of E_d is in U. Then, the second condition to modify the set $A_{f(m)}$ would eventually be satisfied, and $A_{f(m)}$ and $A_{f(n)}$ would both eventually become sets of the form $\mathbb{N} \setminus \{v\}$ and $\mathbb{N} \setminus \{w\}$, respectively. This would cause x to be enumerated into U, in contradiction to the assumption on x. Hence, there are only finitely many k such that $A_{f(k)} = E_d$ for some d. Furthermore, there is no k such that $f(k)$ is even, since every A_x with even x contains only even numbers, and it is a set in the Friedberg model. Hence, there is an m such that $f(n)$ is odd and $f(n)$ is an index of a finite set with elements only below $f(n)$ for all $n \geq m$. It follows that $f(n) \leq f(m) + m - n$ for all $n \geq m$ and $f(f(m)+1) < 0$, a contradiction.

Hence, A_x is a set, and the element-ordering is well-founded on the transitive closure of A_x iff $x \notin U$. ∎

Proposition 13. *There is no Friedberg model in which Set is a recursively enumerable predicate.*

Proof. Assume, by way of contradiction, that there is an index x with $A_x = \{y : Set(y)$ is true$\}$. Then, A_x is a set, since every element of A_x is a set. Hence, $x \in A_x$, contradicting the characterisation of sets in the Friedberg model as those classes for which the element-relation is well-founded on their transitive closure. ∎

Note that this construction follows exactly the lines of Russel's antinomy of naive set theory [16]. The following classes of ill-defined objects which are similar to Russel's construction exist: the class of all classes which contain themselves (in every Friedberg model), and the class of all proper classes (in some Friedberg model, as shown in Theorem 15).

Proposition 14. *There is no Friedberg model of ordinal height ω_1^{CK} where the predicate Set is hyperarithmetic.*

Proof. Assume that there is a Friedberg model A_0, A_1, A_2, ..., with the corresponding numbering B_0, B_1, B_2, ... as given in Definition 1, such that the ordinal height is ω_1^{CK}, and the predicate *Set* is hyperarithmetic. Then, also the collection O of all ordinals is hyperarithmetic. Indeed, $x \in O$ if and only if

1. $Set(x)$,
2. A_x is transitive (that is, $z \in A_x$ for all $y \in A_x$ and all $z \in A_y$), and
3. all different elements y, z of A_x are comparable (that is, either $y \in A_z$ or $z \in A_y$).

Let $<_0$, $<_1$, $<_2$, ... be an acceptable numbering of all r.e. partial orders on $\mathbb{N}$. Here, a partial ordering $<_e$ is r.e. iff $\{\langle x, y\rangle : x <_e y\}$ is an r.e. subset of $\mathbb{N}$, where $\langle x, y\rangle$ is the usual Cantor pairing function given by $x + (x+y) \cdot (x+y+1)/2$. The relation $<_e$ is a well-ordering iff there are a function f and an $x \in O$ such that f is an order preserving isomorphism from $(\mathbb{N}, <_e)$ to A_x, where the ordering on A_x is the element-relation — y is below x iff $y \in A_x$.

Since O is hyperarithmetic, and hyperarithmetic sets are Σ_1^1, the expression above, characterising the set $\{e\colon <_e$ is a well-ordering on $\mathbb{N}\}$, is Σ_1^1. However, this set is Π_1^1-complete. Therefore, the assumption that O is hyperarithmetic is false. Therefore, if a Friedberg model has ordinal height ω_1^{CK}, then the predicate *Set* is not hyperarithmetic. ∎

The following shows that the predicate *Set* can be quite complex.

Theorem 15. *There is a Friedberg model in which the predicate Set is Π_1^1-complete.*

Proof. Let $A_{3x} = \{3y : y < x\}$ for all x. For the construction of sets of the form A_{3x+1}, one codes into them recursive trees in a certain way. First, some facts about trees will be given. There is a numbering S_0, S_1, S_2, ... of trees such that each $S_e \subseteq \mathbb{N}^*$ and $\{e : S_e$ is well-founded$\}$ is Π_1^1-complete [12, Theorem IV.2.15]. Here, $\mathbb{N}^*$ is the set of finite strings over $\mathbb{N}$ where for all

$a \in \mathbb{N}$, σa is the a^{th} successor of σ. A tree is well-founded iff it does not have an infinite branch. Now, one constructs from each S_e the tree T_e given as

$$T_e = \big\{(2a_0, 2a_1, 2a_2, \ldots, 2a_n), (2a_0, 2a_1, 2a_2, \ldots, 2a_n, b) : \\ (a_0, a_1, a_2, \ldots, a_n) \in S_e \wedge b \text{ is odd } \big\} \cup \big\{ \text{ the root of } S_e \big\}.$$

It is clear that the nodes ending in an odd number are the leaves. Also, every node ending in an even number has infinitely many successors. Furthermore, T_e has an infinite branch if S_e has, that is, T_e is well-founded iff S_e is well-founded. Note that the set of leaves is uniformly recursive, every node is either a leaf or has an infinite set of successors, and $\{e : T_e \text{ is well-founded}\}$ is Π^1_1-complete.

There is a recursive one-to-one function g from $\{1, 4, 7, 10, \ldots\}$ onto the disjoint union of T_0, T_1, T_2, ... Let $F_e = g^{-1}(T_e)$ and $r_e = g^{-1}(\text{the root of } T_e)$. So, g induces a recursive partition of $\{1, 4, 7, 10, \ldots\}$ into the sets F_0, F_1, F_2, ..., and the set $\{r_0, r_1, r_2, \ldots\}$ is recursive. Furthermore, the set $\{3y+1 : g(3y+1) \text{ is a leaf}\}$ is recursive. Let e be the index with $3x+1 \in F_e$. If $g(3x+1)$ is a leaf, then let $A_{3x+1} = \{6y, 6y+3\}$ for the least y such that the set $\{6y, 6y+3\}$ is not already used by some leaf $3z+1$ with $z < x$. Otherwise, let $A_{3x+1} = \{3y+1 \in F_e : g(3y+1)$ be a successor of $g(3x+1)$ in $T_e\}$. Note that there is a function f with $f(0) = 3x+1$ and $f(n+1) \in A_{f(n)}$ for all n iff $g(3x+1)$ lies on an infinite branch of T_e. This branch is then defined by f. Hence, A_{3x+1} is a set in the Friedberg model iff there is no infinite branch of T_e through $g(3x+1)$. Now, it has been achieved that — independent of the further constructions — $Set(r_e)$ is true iff T_e is well-founded. So, Set is Π^1_1-complete.

It only remains to define the sets A_{3x+2} such that the resulting numbering defines a Friedberg model. Let $A_2 = \mathbb{N}$, and let $A_5 = \{0, 3, 6, 9, 12, \ldots\}$. For the sets of the form A_{3x+8}, one adapts the proofs of Theorems 4 and 6 in order to complete this construction to a Friedberg model.

Let E_0, E_1, E_2, ... be a one-to-one numbering of all r.e. subsets of $\mathbb{N}$ with at least 2 nonelements. The main modification in the construction is that one has to add some witnesses in order to keep the new sets different to those, built so far, with index of the form $3y, 3y+1$. In the following, $A_{y,s}$ denotes the elements enumerated into A_y within s steps, and X denotes the d^{th} finite subset of $\mathbb{N}$. Now, A_{3x+8} is selected from the following options.

- Priority $3d$: This case applies if the following conditions hold:
  - $A_{3y+8,x} \neq X$ for all $y < x$,
  - X is not of the form $\{3y : y < z\}$ for any z, and
  - X is not of the form $\{6y, 6y+3\}$ for any y.

  If this case is selected, then A_{3x+8} follows the set X as long as $E_{e,s} \neq X$ for all $e < |X|$. If, at stage $s > x$, there is an index $e < |X|$ with $E_{e,s} = X$, then A_{3x+8} follows $\mathbb{N} \setminus \{u\}$ from now on forever, where u is the least number such that $u \notin X$ and no set is following $\mathbb{N} \setminus \{u\}$ so far.
- Priority $3d+1$: This case applies if there are v, w such that the following conditions hold:
  - $|E_{d,x}| > d$,
  - no set A_{3y+8} with $y < x$ is currently a follower of E_d,

- if $E_{d,x} \subseteq \{0, 3, 6, 9, 12, \ldots\}$, then v is the least number with $v \notin E_{d,x} \wedge v + 3 \in E_{d,x}$, and
- if there is a y with $E_{d,s} \subseteq A_{3y+1}$, then $w = \min(A_{3y+1} \setminus E_{d,x})$.

Note that if the fourth condition applies, then, since $E_{d,x}$ is not empty by the first condition, the y, w there are unique. So, there can be at most one y with $E_{d,x} \subseteq A_{3y+1}$. If this case is selected, then v and w are kept fixed as parameters of the construction for this set A_{3x+8} from now on, and A_{3x+8} follows E_d until each of the following cases occurs at some stage $s > x$:

- v is enumerated into $E_{d,s}$ and $E_{d,s} \subseteq \{0, 3, 6, 9, 12, \ldots\}$, and
- there is a unique y with $\{w\} \subseteq E_{d,s} \subseteq A_{3y+1}$.

If this happens, then A_{3x+8} starts to follow $\mathbb{N} \setminus \{u\}$ forever, where u is the least number that is not yet enumerated into A_{3x+8}, and no other set is currently following $\mathbb{N} \setminus \{u\}$.

- Priority $3d + 2$: This case applies iff no A_{3y+8} with $y < x$ is currently a follower of $\mathbb{N} \setminus \{d\}$. If this case is selected, then A_{3x+8} follows $\mathbb{N} \setminus \{d\}$ forever.

First, one has to note that the algorithm for the entry of priority $3d+1$ can be executed effectively, that is, one can always check whether there is a y with $E_{d,x} \subseteq A_{3y+1}$. Indeed, the sets A_1, A_4, A_7, $\ldots$ have been selected such that they, together with $\{3z + 1 : g(3z + 1)$ is the root of some tree $T_e\}$, form a partition of $\{0, 1, 3, 4, 6, 7, 9, 10, \ldots\}$. Hence, one can for every finite set $E_{d,x}$ and later $E_{d,s}$, determine all y with A_{3y+1} intersecting $E_{d,x}$ and $E_{d,s}$, respectively, and then see whether any of the A_{3y+1} actually does not only intersect, but also contains the finite set. If this is so, then this set A_{3y+1} is unique. Furthermore, from the property of being a partition of $\{0, 1, 3, 4, 6, 7, 9, 10, \ldots\}$, it follows that the sets A_{3y+1} are uniformly recursive.

Second, note that by the choice of the trees T_e, each node in T_e has either none or infinitely many successors. In case it has infinitely many successors, the corresponding A_{3y+1} representing the set of these successors is also infinite. Hence, it is different to X in the entry for priority $3d$. Hence, the conditions given there are enough to make sure that the set A_{3x+8}, if selected by this case, is different from all the sets A_0, A_1, A_3, A_4, A_6, A_7, A_9, A_{10}, $\ldots$, and it is also different to the infinite sets A_2 and A_5.

Now, consider the case where a set A_{3x+8} follows some set E_d from some time on. Note that $A_{3x+8} = E_d$. One has to show that A_{3x+8} differs from all other sets in the numbering. Since E_d has at least two nonelements, A_{3x+8} differs from A_2, which is $\mathbb{N}$. It also differs from all sets of the form $\mathbb{N} \setminus \{u\}$. If $A_{3x+8} \subseteq A_5 = \{0, 3, 6, 9, 12, \ldots\}$, then the parameter v is selected such that $v \in A_5 \setminus A_{3x+8}$. Furthermore, $v \in A_{3y} \setminus A_{3x+8}$ for all y with $A_{3x+8} \subseteq A_{3y}$. If there is a set A_{3y+1} with $A_{3x+8} \subseteq A_{3y+1}$, then this set is unique, and $w \in A_{3y+1} \setminus A_{3x+8}$. For all $y \neq x$, it holds that A_{3y+8} is different to A_{3x+8} by the following case distinction.

- If A_{3y+8} follows some finite set X according to the case of priority $3d$, then the argument above shows that $A_{3x+8,s} \neq X$ for almost all stages s. For otherwise, A_{3y+8} would have been redirected to some set of the form $\mathbb{N} \setminus \{u\}$.
- If there is an e such that A_{3y+8} follows E_e forever, then $E_e \neq E_d$. Indeed, the given numbering E_0, E_1, E_2, $\ldots$ is one-to-one. Hence, $A_{3y+8} = E_e \neq E_d = A_{3x+8}$.

– If A_{3y+8} is eventually equal to some set $\mathbb{N} \setminus \{u\}$, then $A_{3y+8} \neq A_{3x+8}$. Indeed, A_{3x+8} has at least two nonelements.

Furthermore, the protocols governing the process of making a set equal to $\mathbb{N} \setminus \{u\}$ ensure that for each u, there is at most one index $3x+8$ such that $A_{3x+8} = \mathbb{N} \setminus \{u\}$.

Third, one has to verify that every r.e. subset of $\mathbb{N}$ occurs in the numbering. If X is finite and differs from all sets A_0, A_1, A_3, A_4, A_6, A_7, A_9, A_{10}, $\ldots$, and also from all sets E_e with $e < |X|$, then the case of priority $3d$ with $d = \sum_{c \in X} 2^c$ will eventually hold, and produce a set A_{3x+8}. Note that finitely many of these trials might be redirected to sets of the form $\mathbb{N} \setminus \{u\}$ for some u. Indeed, the set X is temporarily equal to some $E_{e,s}$ with $e < |X|$, but the case with priority $3d$ will eventually qualify, and get a follower which will no longer be redirected.

Now, consider any set E_d such that $|E_d| > d$ and E_d is different to all sets A_0, A_1, A_3, A_4, A_6, A_7, A_9, A_{10}, $\ldots$, and well as to the sets A_2 and A_5. If $E_d \subseteq \{0, 3, 6, 9, 12, \ldots\}$, then either v will, for all sufficiently large x, take value equal to the least multiple of 3 such that $v \notin E_d \wedge v+3 \in E_d$, or $E_{d,x} \not\subseteq \{0, 3, 6, 9, 12, \ldots\}$ for all large enough x. If $E_d \subseteq A_{3y+1}$ for some y, then either w as defined in the description of priority $3d+1$ will be the minimum of $A_{3y+1} \setminus E_d$ for all sufficiently large stages x, or $E_{d,x} \not\subseteq A_{3y+1}$ for any y for all sufficiently large x. Hence, if x is sufficiently large and E_d has just abandoned its follower, there will again be a new follower $A_{3x'+8}$, for some $x' \geq x$. The parameters v, w will be chosen such that this follower will not be abandoned. Hence, E_d will appear in the numbering.

For the sets of the form $\mathbb{N} \setminus \{d\}$, the case with priority $3d+2$ assures that these sets eventually show up in the numbering. Furthermore, $A_2 = \mathbb{N}$. Hence, all sets with at most one nonelement occur in the numbering.

It remains to show that every B_k is recursive. The case $k = 2$ is special, as B_2 is a finite variant of

$$\begin{aligned}\{6\} \cup \{3y+1 : g(3y+1) \text{ is a leaf of some tree } T_e\} \cup \{3y+8 :\\ A_{3y+8} \text{ is initialised by case } 3d \text{ as an } X \text{ with 2 elements }\}.\end{aligned}$$

For $k \neq 2$, B_k is a finite variant of

$$\{3k\} \cup \{3y+8 : A_{3y+8} \text{ is initialised by case } 3d \text{ as an } X \text{ with } k \text{ elements }\}.$$

The verification is as in Theorems 4 and 6. ▮

4 A smallest model

Recall from Definition 1 that there is a partial mapping from V to a Friedberg model which maps every set X realised in the model to the $x \in \mathbb{N}$ such that A_x represents X. Now, one can compare two Friedberg models A and B as follows: A is smaller than or equal to B iff every set (from V) realised in A is also realised in B. For this partial ordering (where different models can be equivalent, as proper classes are not taken into account), a Friedberg model A

is (i) a smallest one iff it is smaller than or equal to every Friedberg model; (ii) minimal iff it is either incomparable or smaller than or equal to every Friedberg model; (iii) a largest one iff every Friedberg model is smaller than or equal to A; (iv) maximal iff every Friedberg model is either smaller than or equal to or incomparable to A. The goal of this section is to establish the existence of a smallest Friedberg model with respect to this partial ordering.

Definition 16. A set X in V is *hereditarily k-bounded* iff the element-relation on the transitive closure of X is well-founded, and only finitely many sets in the transitive closure of X have more than k elements.

This definition is used in the construction of a smallest Friedberg model, and the sets in this model are characterised in the following theorem.

Theorem 17. *There exists a Friedberg model A_0, A_1, A_2, ... such that for every set X — given up to isomorphism by the transitive structure of the elements of X, and elements of elements, and so on — the following conditions are equivalent.*

- *X is isomorphic to some A_x in this model.*
- *X is hereditarily k-bounded, for some k, and has an isomorphic copy in some Friedberg model.*
- *X has an isomorphic copy in every Friedberg model.*

Furthermore, in this model, the collection of all indices of proper classes is recursively enumerable.

Proof. The proof is similar to that of Theorem 12. First, the smallest model will be constructed. Then, the equivalence of the three statements above will be shown. Let E_0, E_1, E_2, ... be a Friedberg numbering of all r.e. subsets of $\mathbb{N}$ which have at least 2 nonelements. Let $A_0 = \emptyset$ and $A_1 = \mathbb{N}$.

In the following construction, let U be the r.e. set of all indices x of sets A_x which are easily seen not to be well-founded, in the sense that there are a function f and numbers n, m with $n > m$ such that $f(0) = x$, $f(k+1) \in A_{f(k)}$ for all $k < n$, and $f(m) = f(n)$. Note that U is r.e., and that an enumeration of U is used when defining A_x for an odd x. Let U_s be the elements enumerated into U within s steps.

For each $x > 1$, let A_x is initialised according to the entry of highest priority (where high priorities correspond to low numerical values).

- With priority $3d$, try to make A_x the d^{th} finite set, provided that $x > d$ and the d^{th} finite set currently differs from all $A_{y,x}$ with $y < x$.
- With priority $3d+1$, let A_x follow E_d, provided that $|E_{d,x} \cap \{0, 1, 2, \ldots, x-1\}| > d$ and E_d has no current follower.
- With priority $3d+2$, let A_x follow $\mathbb{N} \setminus \{d\}$ forever, provided that this set does not already have a follower.

After choosing the A_x according to the priority, one updates A_y for y with $1 < y < x$, provided that one of the following three cases applies.

– If A_y is initialised as the d^{th} finite set, there is some e with $E_{e,x}$ equal to this set, and $|E_{e,x}| > e$, then let A_y, from now on, follow $\mathbb{N} \setminus \{u\}$, for the first u found such that u is not already in A_y and no other set is currently following $\mathbb{N} \setminus \{u\}$.
– If there are numbers k, n, z and a function f such that $A_{y,x}$ has at least k elements, $z < x$, A_z follows E_k, A_y does not follow $E_0, E_1, E_2, \ldots, E_k$, $A_{z,x} \cap U_x = \emptyset$, $0 < n < x$, $f(0) = z$, $f(n) = y$, and for all $m < n$, $f(m+1) \in A_{f(m),x}$, then let A_y follow $\mathbb{N} \setminus \{u\}$ for the first u found such that $u \notin A_{y,x}$ and $\mathbb{N} \setminus \{u\}$ does not currently have a follower.
– If A_y is initialised as E_d for some d, there is a $z < x$ such that A_z follows $E_{d'}$ with $d' \geq d$, $A_{y,x} \cap U_x = \emptyset$, and there are a function f and $n > 0$ with $f(0) = y$, $f(n) = z$, and for all $m < n$, $f(m+1) \in A_{f(m),x}$, then let A_y, A_z follow $\mathbb{N} \setminus \{v\}$ and $\mathbb{N} \setminus \{w\}$, respectively, where $\mathbb{N} \setminus \{v\}$, $\mathbb{N} \setminus \{w\}$ have not yet any followers, v is not yet enumerated into A_y, w is not yet enumerated into A_z, and $y \neq z \Rightarrow v \neq w$.

As in the proof of Theorem 12, one can show the following.

– Every r.e. set W appears exactly once in the enumeration of the A_x.
– Every $B_k = \{x : |A_x| = k\}$ is recursive.
– U contains all x where the element-relation is not well-founded on the transitive closure of A_x.

Now, it will be shown that every set in the model is hereditarily x-bounded for some x. Consider any E_k, and assume by way of contradiction that E_k is a set and not hereditarily k-bounded. Let x be the index of E_k. It will turn out that this x can also serve as the bound. Let y be an index of a set in the transitive closure of E_k. In case $|A_y| > k$ and $A_y \notin \{E_0, E_1, E_2, \ldots, E_k\}$, the second condition in the modifications of sets A_y to sets of the form $\mathbb{N} \setminus \{u\}$ is activated, unless E_k has already some other element which is enumerated into U. Hence, E_k becomes a proper class, and is no longer a set. Now, consider any set A_x in the model. From x, one can, by following the element-relation, either reach indices $y < x$ by not going through any index of an E_k, or go through an index of an E_k with $k \leq x$. In the first case, the set contains only elements below y, and is thus x-bounded. In the second case, E_k is hereditarily k-bounded as outlined above, and thus, the y reached is also the index of an x-bounded set. So, A_x is hereditarily x-bounded.

The arguments up to now together give: all sets A_x in the Friedberg model are hereditarily k-bounded for some k, and every proper class has already an index in U. Hence, every set that has an isomorphic copy in all Friedberg models is hereditarily k-bounded. Now, it will be shown, by induction over the finitely many exceptions, that every hereditarily k-bounded set has a representative in every Friedberg model.

Assume that X is a hereditarily k-bounded set and X has a representation in some Friedberg model $\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2, \ldots$ with $\widetilde{B}_k = \{x : |\widetilde{A}_x| = k\}$. Let $\widetilde{y}_0, \widetilde{y}_1, \widetilde{y}_2, \ldots, \widetilde{y}_n$ be the indices of sets with more than k elements occurring in the construction of X (with $\widetilde{y}_n$ being the index of X). One can, without loss of generality, assume that the $\widetilde{y}_m$ are ordered in such a way that, in the transitive closure downwards, only $\widetilde{y}_h$ with $h < m$ occur below $\widetilde{y}_m$. It will be shown, by induction, that for each set $\widetilde{A}_{\widetilde{y}_m}$, there is a counterpart A_{y_m} in any given numbering $A_0, A_1, A_2, \ldots$ of a Friedberg model. Let $f(\widetilde{y}_h) = y_h$ for $h < m$. Map the unique element of $\widetilde{B}_0$ to the unique element of B_0, that is, the index of the empty set, to the index of the empty set. For every $\widetilde{x} \in \widetilde{B}_1 \cup \widetilde{B}_2 \cup \ldots \cup \widetilde{B}_k$, find the cardinality $c = |\widetilde{A}_{\widetilde{x}}|$ and the c elements $\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_c$ of $\widetilde{A}_{\widetilde{x}}$. In case f is defined on these elements, search in B_c for the unique index x such that A_x contains $f(\widetilde{a}_1), f(\widetilde{a}_2), \ldots, f(\widetilde{a}_c)$, and let $f(\widetilde{x}) = x$. With this method, one can define f uniformly on all elements of $\widetilde{A}_{\widetilde{y}_m}$, and gets that the set $\{f(\widetilde{x}) : \widetilde{x} \in \widetilde{A}_{\widetilde{y}_m}\}$ is recursively enumerable. Hence, it has an index y_m, and one can define $f(\widetilde{y}_m) = y_m$. By this induction, one can get an isomorphic copy of X in the Friedberg model $A_0, A_1, A_2, \ldots$ One derives that an isomorphic copy of X exists in the given Friedberg model. As this induction did not use any property which does not hold in some Friedberg model, one has that X has an isomorphic copy in every Friedberg model. ∎

Remark 18. Analysing the argument of the last paragraph of the proof, one can see that the isomorphism f can be built iff there is an effective representation of X which allows one to compute, for each index occurring in the transitive closure of the membership relation, the cardinality of the corresponding set. If this is impossible, then X does not have any isomorphic copy in any Friedberg model. An example would be the following set X. First, X contains an inductive list $x_0, x_1, x_2, \ldots$ of elements such that x_0 represents the empty set, and x_{n+1} represents $\{x_n\}$. Second, X contains also a list y_n such that y_n represents $\{x_n, x_{n+2}\}$ if $n \notin K$, and y_n represents $\{x_n, x_{n+1}, x_{n+2}\}$ if $n \in K$, where K is the halting problem. Then, the cardinalities cannot be computed, although X is hereditarily 4-bounded. Hence, not every r.e. structure on hereditarily bounded sets can be realised in a Friedberg model.

Note that the Friedberg model constructed in Theorem 17 does not contain any infinite ordinal α, since such an α contains for every $\ell \in \mathbb{N}$ an index of the set representing ℓ, and, thus, is not hereditarily k-bounded. So, the ordinal height of this Friedberg model is ω. However, the Friedberg model contains, for every recursive ordinal α, a well-ordered set of order type α. The only difference is that this well-ordered set is not isomorphic to the structure of the relation induced by $x \in A_y$, as it is in the von Neumann universe.

One could ask whether a model can be embedded properly into another model via a partial recursive function. The answer is negative.

Theorem 19. *If there is a partial-recursive one-to-one function f which maps every set in a Friedberg model $A_0, A_1, A_2, \ldots$ to an isomorphic copy in a Friedberg model $\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2, \ldots$, then f is surjective. Moreover, both models realise exactly the same sets from the von Neumann universe.*

Proof. Assume that $A_0, A_1, A_2, \ldots, \widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2, \ldots$ and f are as in the statement of the theorem. Let X be the collection of all indices of sets in the first model. Let $Y = \{f(x) : x \in X\}$. Note that every member of Y is an index of a set in the second model. Assume, by way of contradiction, that there is a set $\widetilde{A}_z$ with $z \notin Y$. Let Z be the transitive closure of $\widetilde{A}_z$. Since the element-relation on Z is well-founded, there is some $y \in Z$ with $y \notin Y$ and $\widetilde{A}_y \subseteq Y$. Then, $\{v : f(v) \in \widetilde{A}_y\}$ is r.e. Let x be an index, so $A_x = \{v : f(v) \in \widetilde{A}_y\}$. For each $w \in \widetilde{A}_y$, there is exactly one v with

$f(v) = w$; this v is in X. Hence, A_v is a set in the first model, and $\widetilde{A}_{f(v)}$ is a set in the second model. Since every $v \in A_x$ is the index of a set, A_x is a set, and so is $\widetilde{A}_{f(x)} = \{f(v) : v \in A_x\}$. It follows that $y = f(x)$ and $y \in Y$, in contradiction to the choice of y. Hence, z cannot exist, and Y is the collection of indices of sets in the second model. The restriction of f to X is then an isomorphism from the sets in the first model to the sets in the second model. So, both models contain representatives of the same sets. ∎

More generally, one can wonder which sets from the von Neumann universe can be represented in some Friedberg model. Though the authors have not found any characterisation so far, the results above shed some insight. First, although the system of canonical indices is a representation of all finite sets in a uniformly recursive way, Theorem 7 shows that the set V_ω of all hereditarily finite sets cannot be represented in any Friedberg model. Second, Theorem 6 shows that some Friedberg model realises every recursive ordinal.

While there is a smallest Friedberg model, one might ask whether there is also a largest Friedberg model which realises all sets from the von Neumann universe that are realised in some Friedberg model. The answer is negative.

Theorem 20. *There is no Friedberg model realising all sets from the von Neumann universe that are realised in some Friedberg model.*

This result will be a corollary to Propositions 22 and 23 below. To express those results, one first needs to define a set $E[F]$ with F a K-recursive function which is approximable from below, that is, $F(x) = \max_s F_s(x)$ for some recursive approximation F_s to F, and $F(x)$ is defined for all x. Note that only some, but not all, K-recursive functions are approximable from below. Proposition 22 shows that for every Friedberg model, there is such a function F for which $E[F]$ does not exist in the model, while Proposition 23 shows that for every such function F, there is a Friedberg model which contains the set $E[F]$.

Definition 21. Let F be approximable from below, and let $A_0, A_1, \ldots$ be a given Friedberg model. As in Example 2, define inductively $b_0, b_1, b_2, \ldots$ such that $A_{b_0} = \emptyset$ and $A_{b_{n+1}} = \{b_n\}$ for all n. Note that the function $n \mapsto b_n$ is recursive. Now, $E[F]$ is the set of all tuples of indices of $\{b_n, b_m, z\}$ satisfying $n < m \leq F(n)$, $\{b_n, b_m\} \subseteq A_z \subseteq \{b_n, b_{n+1}, \ldots, b_m\}$, and $|A_z| > n + 1$.

Note that there are no distinct n, m with $b_n = b_m$. Indeed, $b_0 \neq b_n$ for all $n > 0$, since $A_{b_0} = \emptyset$, while A_{b_n} contains an element. Also, since $A_{b_{n+1}} = \{b_n\} \neq \{b_m\} = A_{b_{m+1}}$, if $b_n \neq b_m$, then $b_{n+1} \neq b_{m+1}$. It follows that there is no least n such that for some $m > n$, $b_n = b_m$; hence there is no such n at all. Furthermore, the sequence $b_0, b_1, \ldots$ is recursive, since the set B_1 of all x with $|A_x| = 1$ is recursive, and b_{n+1} is the unique member of B_1 with $b_n \in A_{b_{n+1}}$. One can find the members of the sequence $b_0, b_1, \ldots$ by searching from them in B_1 inductively.

Proposition 22. *Given any Friedberg model, there is a K-recursive function F that is approximable from below such that the set $E[F]$ is not r.e., and thus, not in the model.*

Proof. Let A_0, A_1, A_2, $\ldots$ be the given Friedberg model, and let $B_k = \{x : |A_x| = k\}$ for all k. Using the indices in the model in place of sets, let b_0, b_1, b_2, $\ldots$ be such that $A_{b_0} = \emptyset$ and $A_{b_{n+1}} = \{b_n\}$. This sequence is recursive. Now, define the following function $G(n)$. Given n, search using K until either (a) or (b) or (c) below holds.

(a) There are an index e and $m \geq n$ such that $\{b_n, b_m\} \subseteq A_e \subseteq \{b_n, b_{n+1}, \ldots, b_m\}$, $|A_e| \geq n+2$,and the index of the triple $\{b_n, b_m, e\}$ is not in A_n.
(b) There is a triple $\{b_n, x, y\}$ with index in A_n such that $x \notin \{b_{n+1}, b_{n+2}, \ldots\}$.
(c) There are a $k > n$ and a y such that the triple $\{b_n, b_k, y\}$ has an index in A_n, and either $A_y \not\subseteq \{b_n, b_{n+1}, \ldots, b_k\}$ or $|A_y| \leq n+1$.

Note that the search terminates for each n. Otherwise, the collection of indices of finite subsets of $\{b_n, b_{n+1}, \ldots\}$ would be r.e., in contradiction to the proof of Theorem 7. Indeed, one could, for each $m > n$, enumerate all triples $\{b_n, b_m, e\}$ with indices in A_n giving the indices of finite subsets of $\{b_n, b_{n+1}, \ldots\}$ with at least $n+2$ elements, plus, for each $k \leq n+1$, those indices $e \in B_k$ where all k elements of A_e are in $\{b_n, b_{n+1}, \ldots\}$. Since this collection is not r.e., the search must terminate. Now, if it terminates by case (a), then let $G(n) = m$ for the m found in this case. If it terminates by cases (b) or (c), let $G(n) = 2n+2$. Since $G \leq_T K$, G can be approximated in the limit by a uniformly recursive sequence of functions G_s. Let $F(n) = \max\{G_s(n) : s \in \mathbb{N}\}$. Since the sequence $G_s(n)$ converges for every n, this maximum $F(n)$ exists, and $F \leq_T K$.

Assume now, by way of contradiction, that $E[F]$ is in the model. Then, there is an n with $A_n = E[F]$. Hence, $G(n)$ cannot be defined by cases (b) and (c) in the definition above. So, there is $m \in \{n, n+1, \ldots, G(n)\}$ and an e such that $\{b_n, b_m\} \subseteq A_e \subseteq \{b_n, b_{n+1}, \ldots, b_m\}$, $|A_e| \geq n$, and the index of $\{b_n, b_m, e\}$ is not in A_n. But, since $F(n) \geq G(n)$, the index of $\{b_n, b_m, e\}$ should be in A_n, a contradiction. Hence, $E[F]$ cannot be A_n, and $E[F]$ is not in the model. ∎

Proposition 23. *Let $F \leq_T K$ be approximable from below. Then, there is a Friedberg model which contains $E[F]$.*

Proof. The construction of the Friedberg model is a variation on the construction in Theorems 4 and 6. For this, one first fixes that $b_n = 3n$, $A_0 = \emptyset$, and $A_{b_{n+1}} = \{3n\}$ for all n. Furthermore, as F is approximable from below, there is a recursive enumeration of all triples $\{b_n, b_m, D\}$ in $E[F]$. Given the s^{th} member $\{b_n, b_m, D\}$ of this enumeration, let A_{3s+1} contain all numbers 3ℓ where $b_\ell \in D$. Note that the minimum $3n$ and maximum $3m$ of A_{3s+1} can be computed from s, and the A_{3s+1} are uniformly recursive. Also, note that for every size k, there are only finitely many A_{3s+1} which have cardinality k. Furthermore, let $A_2 = \mathbb{N}$. The sets A_{3x+5} are now constructed by a priority argument, as in the previous theorems. Again, as in Theorem 4, let E_0, E_1, E_2, $\ldots$ be a one-to-one numbering of all r.e. subsets of $\mathbb{N}$ with at least 2 nonelements. In the following, $A_{y,s}$ denotes the elements enumerated into A_y within s steps, and X is the d^{th} finite set, that is, X is the set with $\sum_{a \in X} 2^a = d$. For each x, assign the set A_{3x+5} according to which of the following cases applies and for which the priority is highest, that is, has the lowest numerical value.

- Priority $3d$: This case applies if $A_{y,x} \neq X$ for all $y < 3x+5$, and $X \neq \{3y\}$ for all y, where X is the d^{th} finite set. If this case is selected, then A_{3x+5} follows the set X as long as $E_{e,s} \neq X$ for all $e < |X|-1$, and no set A_{3s+1} equal to X has been found for any s. In case there is $e < |X|-1$ and a stage $s > x$ with $E_{e,s} = X$ at stage s, or X turns out to be equal to some A_{3s+1}, A_{3x+5} starts to follow $\mathbb{N} \setminus \{u\}$ forever, where u is the least number such that $u \notin X$, and no other set is currently following $\mathbb{N} \setminus \{u\}$.
- Priority $3d+1$: This case applies if $|E_{d,x}| > d+1$ and E_d does not currently have a follower. If this case is selected, then A_{3x+5} starts to follow E_d. If for some $t > x$ and s, the equality $E_{d,t} = A_{3s+1}$ holds, then A_{3x+5} stops to follow E_d, and A_{3x+5} starts to follow $\mathbb{N} \setminus \{u\}$ forever, where u is the least number such that u has not yet been enumerated into A_{3x+5}, and no other set is currently following $\mathbb{N} \setminus \{u\}$.
- Priority $3d+2$: This case applies if no A_y with $1 < y < x$ is currently following $\mathbb{N} \setminus \{d\}$. If this case is selected, then A_x follows $\mathbb{N} \setminus \{d\}$ forever.

Note that the priorities of different choices are different. Therefore, it is always clear which choice is taken to initialise A_x. The algorithm maintains an explicit list of which sets of the form $\mathbb{N} \setminus \{u\}$ it has created, together with the stages at which these sets have been created, so that each of these sets is created once (by the last case), and only once. Each set E_d with at least $d+2$ elements which is also different from all sets A_{3s+1} receives in the limit exactly one follower. Note that, in this context, there are only finitely many sets A_{3s+1} with $\min(E_d) = \min(A_{3s+1})$. Hence, it happens only at finitely many stages t that a follower is abandoned because of $E_{d,t} = A_{3s+1}$. For every finite X, it is made sure that exactly one of the following cases arises (in this order of priority): (a) $X = A_{3s}$, (b) $X = A_{3s+1}$, (c) X is obtained by some A_{3x+5} following E_d with $X = E_d \wedge |E_d| > d+1$, (d) X is obtained by an A_{3x+5} directly following X with some priority $3d$. Note that in cases (c) and (d), whenever (b) or (c) turns out to apply, the follower is abandoned and changed to a set of the form $\mathbb{N} \setminus \{u\}$. Hence, the set X exists in the numbering and has a unique index. Furthermore, one can see that for every $k \in \{1, 2, 3, \ldots\}$, there are only finitely many sets of cardinality k in $E[F]$. Hence, there are only finitely many x such that A_x has initially the cardinality k, and receives later more elements. So, each B_k is a finite variant of $\{x : A_x \text{ has at the initialisation } k \text{ elements}\}$, and each B_k is recursive. Hence, the constructed model is a Friedberg model.

In addition to this, one has to show that $E[F]$ exists in the Friedberg model that has been constructed. It is sufficient to show that $E[F]$ is a set, and defined by an r.e. collection S of indices. Given s, one can compute the minimum $3n$ and maximum $3m$ of A_{3s+1}, and search in B_3 for an index k such that $A_k = \{3n, 3s+1, 3m\}$. This k is then enumerated into S. Hence, S is recursively enumerable. Furthermore, S is a set in the constructed Friedberg model. Indeed, every A_{3n} is a set, and each A_{3s+1} contains only indices of sets of the form A_{3n}, and so, is a set as well. As formed, A_k contains two indices of the form $3n$ and $3m$, plus one index of the form $3s+1$, and so, is a set again. Hence, S contains only indices of sets, and so, is also a set in the Friedberg model. ∎

This result shows only that there is no largest Friedberg model. It remains open whether there is a maximal one, that is, whether there is a Friedberg model such that no other Friedberg model contains strictly more sets.

5 The theory of Friedberg models

The present section is devoted to analysing the complexity of the set of sentences satisfied in a Friedberg model (its theory). One distinguishes two cases: first, quantifiers ranging only over sets; second, quantifiers ranging over classes, but with an additional predicate that permits to distinguish sets from proper classes. The underlying logical language is as follows:

- variables x, y, z, ... ranging over indices of sets in the first case, and ranging over all natural numbers in the second case;
- numerical constants denoting an index of a set in the first case, and denoting an index of a class (possibly a set) in the second case;
- a predicate $x \in A_y$ to establish the element-relation between x and y;
- a predicate $Set(x)$ to express whether A_x is a set or a proper class.

The *pure theory* of a Friedberg model A_0, A_1, A_2, ... is the set of all sentences (formulas without free variables) which are true in the model when all quantified variables range over indices of sets, and the predicate *Set* is always true (since all individuals in the domain of a model of the theory are sets). The *combined theory* of a Friedberg model A_0, A_1, A_2, ... is the set of all sentences which are true in the model when all quantified variables range over indices of all classes (that is, over all natural numbers) and the predicate $Set(x)$ is true iff A_x is a set (not a proper class) in the model.

There is a canonical embedding of the pure theory of Friedberg models into the combined theory of Friedberg models.

Theorem 24. *The pure theory P of Friedberg models is many-one reducible to the combined theory S of Friedberg models.*

Proof. In order to see that P is many-one reducible to S, consider any formula quantifying over indices of sets, say

$$\exists x \, \forall y \in A_x \forall z \in A_y \, [z \in A_x]$$

and assume that all constants in this formula are indices of sets. Now, one can translate this formula into a formula where one uses the predicate *Set* in order to enforce the following: the resulting formula only depends on those cases where the corresponding variables take as values indices of sets in the model. In the existential case, one can just require that the variable takes as value the index of a set. In the universal case, one has to introduce an implication in the main part, which says that if the variable takes the value of a set, then the corresponding formula holds:

$$\exists x \left[Set(x) \wedge \forall y \in A_x \big[Set(y) \rightarrow \forall z \in A_y \, [Set(z) \Rightarrow z \in A_x] \big] \right]$$

or, using that elements of sets are sets, and bringing the formula back into prenex normal form, one would obtain the following simpler formula:

$$\exists x\, \forall y \in A_x \forall z \in A_y\, [Set(x) \wedge z \in A_x].$$

This translation is obviously recursive, and shows how to translate a formula from P into S. Furthermore, since P does not contain any formula with constants that are not indices for sets, one could conjunct each formula with a finite formula requiring that all constants are indices of sets. That would make sure that formulas with constants outside P are not translated into members of S. ∎

Note that the theory of a Friedberg model depends on the model, and is not the same for all models. For example, the following formula is true if the ordinal ω is realised in the model:

$$\exists x \left[Set(x) \wedge \exists y \in A_x\, [\neg \exists z \left[z \in A_y\right] \wedge \forall y \in A_x \exists z \in A_x \forall u\, [u \in A_z \Leftrightarrow u \in A_y \vee u = y]]\right].$$

More precisely, this formula is true iff the Friedberg model contains an inductive set. The smallest Friedberg model does not satisfy this formula. There are also sets which are first-order definable, but do not exist in any Friedberg model: for example, the set of all sets. So, there is no set A_x in the model which satisfies $\forall y\, [y \in A_x]$ in the first case (where one quantifies over indices of sets only), and which would be given by

$$\forall y\, [Set(y) \Leftrightarrow y \in A_x]$$

in the second case (where one quantifies over indices of sets and classes). Furthermore, one can use parameters of sets X, Y in the Friedberg model such that $X \setminus Y$ is not in the Friedberg model (as it is not r.e.), although it is first-order definable.

It should also be noted that many sets are first-order definable and are realised in a given Friedberg model, although one cannot compute their index in a Friedberg model from the parameters. Such is the case with unions, intersections, transitive closures, power sets (if realised in the model), and so on.

Proposition 25. *There is an infinite first-order definable set which is realised in all Friedberg models. Furthermore, the transitive closure of any set is definable* (*using that set as a parameter*).

Proof. As in Example 2, let b_0 be the index of the empty set, and let b_{n+1} be the index of $\{b_n\}$. Then, the set $X = \{b_0, b_1, b_2, \ldots\}$ is r.e. Indeed, one can, using B_1, enumerate b_0, b_1, b_2, … One then has that the range of this enumeration is an r.e. set, hence exists in the given Friedberg model. It follows that X can be first-order defined by the following conditions:

- $\exists y \in X\, \forall z\, [z \notin A_y]$,
- $\forall y \in X\, \exists z \in X\, \forall u\, [u \in A_z \Leftrightarrow y = u]$, and
- $\forall y\, [A_y \subset X \Rightarrow A_y$ violates one of the two conditions above].

Given a set Y and an index y for Y, one can enumerate, starting with y, all elements of A_y, all elements of elements of A_y, and so on. Hence, one gets a set Z satisfying the following conditions:

- $y \in Z$,
- $\forall z \in Z \, \forall x \in A_z \, [x \in Z]$, and
- $\forall z \, [A_z \subset Z \Rightarrow A_z$ violates one of the two conditions above].

Recall that $A_v \subset A_w$ just means that $\forall u \in A_v \, [u \in A_w]$ and $\exists u \in A_w \, [u \notin A_v]$ hold. The verification that these conditions properly define X and Z is straightforward. ▮

Theorem 26. *Let a Friedberg model be given. There is an r.e. set C of constants (indices of r.e. sets in the Friedberg model) and a many-one reduction F from the combined theory S to the pure theory P of the Friedberg model such that every formula in the range of F uses only constants from C. In particular, the Friedberg model contains an inner model with the predicate Set of the combined theory.*

Proof. Let $X = \{b_0, b_1, b_2, \ldots\}$ be defined as in Proposition 25, and let $A = \{(b_n, b_m) : n \in A_m\}$, with (x, y) abbreviating the ordered pair $\{\{x\}, \{x, y\}\}$. Then, A is an r.e. set. Therefore, it has an index, so, it exists in the given Friedberg model. Hence, X and A define an inner model as every class A_x is represented by the set $\{b_y : y \in A_x\}$, which is equal to $\{b_y : (b_y, b_x) \in A\}$.

Now, it will be shown that this inner model also can be equipped with a predicate *Set* defining the sets. In other words, there is a formula $\Phi(x)$ that holds iff b_x represents an A_x that is a set. In the following, variables with a tilde, like $\tilde{x}$, are used to indicate when a variable ranges over values from X, that is, over values b_0, b_1, $\ldots$, and not just over all numbers. Now, define $\Phi(\tilde{x})$ as $\tilde{x} \in X \wedge \exists u, v, w$ such that:

- A_v is the transitive closure of A_u,
- A_w is the graph of a function f with domain A_v,
- $f(u) = \tilde{x}$, and f maps A_v to a subset of X,
- for all $y, z \in A_v$, $y \in A_z$ iff $(f(y), f(z)) \in A$, and
- for all $\tilde{y}, \tilde{z} \in X$, if $\tilde{z}$ in the range of f and $(\tilde{y}, \tilde{z}) \in A$, then $\tilde{y}$ is in the range of f.

The purpose of $\Phi(\tilde{x})$ is mainly to witness that $\tilde{x} \in X$ and A_x is isomorphic to A_u. As the quantifiers range only over indices of sets, u would not come up if A_u was a proper class. Hence, $\Phi(\tilde{x})$ brings the information that A_x is a set over to the inner model. One has to use the transitive closure A_v instead of A_u in order to get hold of the elements, elements of elements, and so on. Then, f is an isomorphism from A_v with the ordering given by $y \in A_z$, to the fragment of X which is the downward-closure along the relation coded by A from $\tilde{x}$. Here, $f(u) = \tilde{x}$ in order to make the connection between both sets. Note that f exists whenever A_v is r.e., since f takes the domain A_v, and is defined by mapping every $y \in A_v$ to b_y. Thus, the index w exists whenever A_u is isomorphic to the set coded by b_x. In other words, whenever $A_u = A_x$, $u = x$. But, since the index u of $\tilde{x}$ is not given, in order to get hold of these three indices, u, v, w have to be existentially quantified. The r.e. set C of the constants used in this translation consists of the union of X and the set consisting of the index of X. ▮

The above result suggests to be a bit more restrictive when dealing with P. In general, the constants used in P do not form an r.e. set, which might be viewed as undesirable. However, one could fix any sufficiently large r.e. set C of constants that are indices of sets, and then work with a set P_C in place of C, only using constants from C. That would avoid coding into P directly which constants are indices of sets, and which are not.

Due to this equivalence, the Turing degrees of the pure theory, and the combined theory of a Friedberg model, are the same. So one can just talk about the complexity of the theory of a Friedberg model, without specifying which of the pure theory or combined theory is meant. The next result determines the minimum complexity of the theory of a Friedberg model.

Theorem 27. *The theory of a Friedberg model is at least as complicated as $\emptyset^\omega$, which recall, is $\emptyset \oplus K \oplus K' \oplus K'' \oplus \ldots$ Moreover, for some Friedberg models, the theory has the same many-one degree.*

Proof. Fix an acceptable ordering $(W_e)_{e\in\mathbb{N}}$ of the r.e. sets of natural numbers. As above, let $X = \{b_0, b_1, b_2, \ldots\}$ with $A_{b_0} = \emptyset$ and for all n, $A_{b_{n+1}} = \{b_n\}$. Set $Y = \{((b_i, b_j), b_k) : k = \langle i, j\rangle\}$ and $Z = \{b_{\langle x,e\rangle} : x \in W_e\}$. The sets X, Y, Z exist in every Friedberg model, and can be used as parameters. Now, let Φ be any arithmetic formula in the natural numbers, say

$$\Phi(v, w) = \exists x\, \forall y\, \exists z\, [\langle v, w, x, y, z\rangle \in W_e]$$

where e is a constant identifying the set W_e, and where tuples of more than two components are defined inductively starting with $\langle i, j, k\rangle = \langle\langle i, j\rangle, k\rangle$. One can translate Φ into a formula in the theory of the Friedberg model with two input variables $\tilde{v}$, $\tilde{w}$ having the values b_v, b_w in place of v, w, and with b_e used to code the parameter e:

$$\begin{aligned}\Phi(\tilde{v}, \tilde{w}) \Leftrightarrow \tilde{v}, \tilde{w} \in X \text{ and } \exists\tilde{x} \in X\, \forall\tilde{y} \in X\, \exists\tilde{z} \in X\, \forall\tilde{m}, \tilde{n}, \tilde{o}, \tilde{p}, \tilde{q} \in X \\ \Big[\big((\tilde{v}, \tilde{w}), \tilde{m}\big) \in Y \wedge \big((\tilde{m}, \tilde{x}), \tilde{n}\big) \in Y \wedge \big((\tilde{n}, \tilde{y}), \tilde{o}\big) \in Y \wedge \\ \big((\tilde{o}, \tilde{z}), \tilde{p}\big) \in Y \wedge \big((\tilde{p}, b_e), \tilde{q}\big) \in Y \Rightarrow \tilde{q} \in Z\Big].\end{aligned}$$

For each $\tilde{v}$, $\tilde{w}$, $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, there is exactly one sequence $\tilde{m}$, $\tilde{n}$, $\tilde{o}$, $\tilde{p}$, $\tilde{q}$ for which the formula

$$((\tilde{v}, \tilde{w}), \tilde{m}) \in Y \wedge ((\tilde{m}, \tilde{x}), \tilde{n}) \in Y \wedge ((\tilde{n}, \tilde{y}), \tilde{o}) \in Y \wedge ((\tilde{o}, \tilde{z}), \tilde{p}) \in Y \wedge ((\tilde{p}, b_e), \tilde{q}) \in Y$$

holds. Then the value q with $b_q = \tilde{q}$ satisfies $b_q = b_{\langle v,w,x,y,z,e\rangle}$. Now $\langle v, w, x, y, z\rangle \in W_e$ iff $b_q \in Z$. Hence, this example shows how to reduce every arithmetic formula into a formula in the theory of the given Friedberg model. It follow that $\emptyset^\omega$ is many-one reducible to this theory.

Furthermore, there is some Friedberg model where the theory is many-one reducible to $\emptyset^\omega$. To see this, consider the Friedberg model constructed in Theorem 12. In this model, the predicate $Set(x)$, which says that A_x is a set, is Π^0_1, hence one can write $x \notin A_u$ for a fixed index u instead of $Set(x)$. Let us transform any given formula into a quantified formula that has Boolean connectives over atomic parts of the form $x \in A_y$ and $x \notin A_y$. Such formulas are exactly those which occur in the arithmetic hierarchy, and their truth-values can be established by querying $\emptyset^\omega$ at the correct place. This reduction from a formula to $\emptyset^\omega$ can be realised by a many-one reduction. ∎

From Theorem 6 and Proposition 14, it follows that there is a Friedberg model whose first-order theory is not hyperarithmetic. On one hand, Theorem 15 gives a model where the predicate *Set* is Π^1_1-complete. To see this, first recall the definitions of b_0, b_1, b_2, ... from the proof of Proposition 25 and r_0, r_1, r_2, ... from the proof of Theorem 15, and then note that the class $\{(b_0, r_0), (b_1, r_1), (b_2, r_2), \ldots\}$ exists in the model and can be used as a parameter in order to get the mapping $b_n \mapsto Set(r_n)$ that realises a Π^1_1-complete predicate in the combined theory. This predicate can then be used to show that the theory of this model is actually many-one equivalent to the ω-jump of the Π^1_1-complete problem $\{n : Set(r_n)\}$. Note that this is the highest possible complexity of the theory of a Friedberg model. The predicate *Set* itself is always Π^1_1 or less, since $Set(x)$ is equivalent to $\forall f\, \exists n\, [f(0) \neq x \vee f(n+1) \notin A_{f(n)}]$. Hence, one has the following corollary to the previous result.

Corollary 28. *The theory of a Friedberg model is many-one reducible to the ω-jump of a Π^1_1-complete set. Moreover, this bound is also obtained for some Friedberg model.*

6 The axiom of power set

The axiom of power set states that given a set in a Friedberg model, there is a set which contains all subsets of that set. Of course, only sets in the model are involved, so the axiom reads more precisely as: for all A_x, there is A_z with $A_z = \{y : A_y \subseteq A_x\}$. The next results show that this is true only for few models, but Theorem 32 shows that such models nevertheless exist.

Proposition 29. *If every set has a power set, then there is no set consisting of infinitely many indices of pairwise disjoint infinite sets.*

Proof. Fix an acceptable ordering $(W_e)_{e \in \mathbb{N}}$ of the r.e. sets of natural numbers. Assume that x_0, x_1, x_2, ... is a recursive one-to-one enumeration of indices of pairwise disjoint r.e. sets. Consider the r.e. set

$$\{z : \exists e, u\, [z \in A_{x_{e,u}} \wedge z < |W_{e,u}|]\}$$

which is equal to some A_y. The power set of A_y cannot be recursively enumerable, since x_e is in the power set of A_y iff W_e is infinite. Hence, the power set of A_y is not in the given Friedberg model. ∎

Another example of a model failing to satisfy the axiom of power set is the smallest Friedberg model. Let A_x be an infinite set in this model. Then, for every set A_z consisting of only indices of subsets of A_x, there is a constant k such that A_z contains only finitely many indices of sets with more than k elements. There are, however, infinitely many indices of subsets of A_x with $k+1$ elements. Hence, one has the following proposition.

Proposition 30. *In the smallest Friedberg model, a set has a power set in the model iff it is finite.*

Though one cannot find uniformly the indices for power sets (in the case that they exist) for all sets, one can, in the smallest Friedberg model, have the following.

Proposition 31. *For every set A_x in a smallest Friedberg model, there is a function f realised in this model such that for all $y \in A_x$, if A_y has a power set, then $A_{f(y)}$ is this power set; else, $A_{f(y)} = \emptyset$.*

Proof. Since A_x is hereditarily k-bounded for some k, one can choose the k such that any A_y with $y \in A_x$ either has at most k elements, or is infinite. Now, if $y \in A_x$ and $y \in B_h$ for some $h \leq k$, then $f(y)$ is the unique $z \in B_{2^h}$ such that all 2^h elements v of A_z and all $w \in A_v$ satisfy $v \in B_0 \cup B_1 \cup \ldots \cup B_h$ and $w \in A_y$. This z can be found by exhaustive search: having $v \in B_0 \cup B_1 \cup \ldots \cup B_h$, one knows how many elements A_v has, and one can enumerate A_v until they are known. If $y \in A_x$ and $y \notin B_0 \cup B_1 \cup \ldots \cup B_h$, then $f(y)$ is the index of the empty set, since A_y is infinite and does not have a power set. Thus, f is a partial-recursive function with domain A_x and is realised by the Friedberg model. ∎

Above examples were provided where the axiom of power set fails. The next result will show, in contrast, that there are Friedberg models satisfying the axiom.

Theorem 32. *There is a Friedberg model $A_0, A_1, A_2, \ldots$ such that for every x, there is a z with $A_z = \{y : A_y \subseteq A_x\}$, that is, every set has a power set in this model.*

Proof. Let $E_0, E_1, E_2, \ldots$ be a Friedberg numbering containing all r.e. subsets of $\mathbb{N}$. Furthermore, let f be a recursive one-to-one mapping from the set of all triples (d, σ, t) to the odd numbers, where σ itself is a mapping from $\{0, 1, 2, \ldots, d-1\}$ to $\{0, 1\}$, intended to indicate which of the sets $E_0, E_1, \ldots, E_{d-1}$ are supersets of E_d, and where $f(d, \sigma, t+1) > f(d, \sigma, t)$ for all d, σ, t. The idea is to create indices for two types of sets.

- Following a set E_d with a parameter σ that is a finite function from $\{0, 1, 2, \ldots, d-1\}$ to $\{0, 1\}$:
  - I1. let $s = 0$;
  - I2. wait until $f(d, \sigma, s)$ elements have been enumerated into E_d, and let Q_s be the set of these elements;
  - I3. wait until the elements of Q_s have been enumerated into all sets $E_{d'}$ such that $d' < d \wedge \sigma(d') = 1$;
  - I4. choose a not yet used index x, initialise A_x as Q_s, and let $t = s + 1$;
  - I5. wait until $f(d, \sigma, t)$ elements have been enumerated into E_d, and let Q_t be the set of these elements;
  - I6. wait until the elements of Q_t have been enumerated into all sets $E_{d'}$ such that $d' < d \wedge \sigma(d') = 1$;
  - I7. update A_x to Q_t;

I8. if there is no $E_{d'}$ with $d' < d \wedge \sigma(d') = 0$ such that all elements of Q_s are already enumerated into $E_{d'}$, then let $t = t + 1$, and go to step 5;

I9. freeze the set A_x and stop considering x to be an index of a follower of E_d; let $s = t + 1$, and go to step 2.

– Candidates for a finite set X. Here s is a stage counter which always goes to infinity. At every stage s, only finitely many sets are initialised. There is also a variable x which is undefined before stage 0. Now, do the following for every stage s:

F1. if x is undefined and there is no $y < s$ with $A_{y,s} = X$, then define x as the least index of a set that is not yet initialised, and let $A_{x,s} = X$;

F2. determine, for each set $Z \subseteq \{0, 1, 2, \ldots, s + 2|X|\}$, the state

$$st_{X,Z,s} = \sum_{d \leq s : X \cup Z \subseteq E_{d,s}} 2^{-d};$$

F3. among all those $Z \subseteq \{0, 1, 2, \ldots, s+2|X|\}$ with $|Z| = X$ and $Z \cap X = \emptyset$, let Y_s be that set Z for which $st_{X,Z,s}$ is as large as possible;

F4. if x is defined, then test whether there is a $y \neq x$ such that A_y is already initialised and currently, $A_{y,s} = X$;

F5. if so, then update $A_{x,s+1} = X \cup Y_s$, freeze A_x, and make x undefined.

First, it will be shown that all infinite r.e. sets occur exactly once in the numbering. So, consider any d and any function σ from $\{0, 1, 2, \ldots, d - 1\}$ to $\{0, 1\}$. Now, consider the following cases.

– The algorithm for (d, σ) remains waiting in step I2 or step I5 forever. Then, the algorithm might have created finitely many sets A_x, all but one of which have eventually been frozen, and there is at most one further set A_x that remains as Q_t for some t forever. This happens only if E_d is finite.
– The algorithm for (d, σ) remains waiting in step I3 or step I6 forever. Then, the algorithm is waiting for some elements of E_d to show up in some set $E_{d'}$ with $d' < d \wedge \sigma(d') = 1$, but they do not show up there. Hence, $E_d \not\subseteq E_{d'}$, although σ says so.
– The algorithm for (d, σ) goes through steps I2 to I7 and step I9 infinitely often. Then, there is a $d' < d$ with $\sigma(d') = 0$ and $E_d \subseteq E_{d'}$, although σ said that there is a noninclusion. Hence, σ is again false.
– The algorithm goes through steps I5 to I8 infinitely often, but goes through all other steps only finitely often. Then, E_d is infinite and $\sigma(d') = 1$ iff $E_d \subseteq E_{d'}$ for all $d' < d$. Furthermore, the last x which is assigned to be a follower is the index of a set A_x with $A_x = E_d$.

Note that all sets enumerated by candidates for a finite set X are finite. Hence, the last set A_x produced by the last entry above is the only infinite set generated in this case-distinction, and it is equal to E_d. Furthermore, the unique σ which predicts the inclusions $E_d \subseteq E_{d'}$ correctly for all $d' < d$ eventually also produces the index x with $A_x = E_d$, but no other σ does. Hence, every infinite set E_d has exactly one index, and that one is given by the last x which is allocated for the correct σ when searching the candidate with (σ, d).

Second, it will be shown that every finite set X is produced exactly once. Note that for each odd u, there is exactly one triple (d, σ, t) such that $f(d, \sigma, t) = u$, and only when the algorithm for (d, σ) reaches the state t, can a candidate of the first type with u elements be generated, which might later be withdrawn in case neither the enumeration algorithm for this candidate becomes stuck, nor the candidate is frozen with this number of indices. If some candidate for E_d enumerates a set of u elements, there might be a collision for sets having $u \cdot 2^r$ elements as step F4 in the enumeration of a set A_x that collided with E_d, and that explicitly followed a finite set X equal to the current value of E_d. Only in that case can there be an update from X to $X \cup Y_s$, and — in a chain-reaction — the same might happen for sets $A_{x'}$ of $2|X|$ elements, and for sets $A_{x''}$ of $4|X|$ elements, and so on. Each such chain-reaction is triggered originally by a candidate of the first type getting u elements, and this happens at most once. Hence, for each odd u, there is only one stage where sets of cardinality $u \cdot 2^r$ are updated. Therefore, it is enough to provide one candidate set Y_s for extending, and there is no need to consider other ones. The test in step F1 of each stage s makes sure that the algorithm always provides an index x with $A_x = X$, whenever currently, no other set has the range X. Furthermore, step F4 explicitly resolves collisions, so that for each finite set X, there is exactly one index x such that A_x is eventually equal to X.

Third, for a given A_x, it will be shown that $\{y : A_y \subseteq A_x\}$ is recursively enumerable, hence equal to some A_z. If A_x is finite, then the existence of A_z is clear. So, assume that A_x is infinite. Now, it is necessary to analyse a bit more closely the conditions by which the Y_s are selected. Let d be the unique index with $A_x = E_d$. Let $g(n)$ be the first stage $s > d$ such that for every $C \subseteq \{0, 1, 2, \ldots, d\}$, the set $\cap_{c \in C} E_{c,s}$ has at least $\min\{2n, \cap_{c \in C} E_c\}$ elements below s. Note that this function g is recursive, since there are only finitely many sets C, and for each of them, one knows whether one can find n elements or only $| \cap_{c \in C} E_c|$ many. Let R be the union of all finite sets of the form $\cap_{c \in C} E_c$ with $C \subseteq \{0, 1, 2, \ldots, d\}$. The set $\{y : A_y \subseteq A_x\}$ consists of the following indices.

1. All indices y where A_y was initialised by a process belonging to some (e, σ) with $e > d$ and $\sigma(d) = 1$.
2. All indices y where A_y was initialised by a process belonging to some (e, σ), and later A_y became frozen as a finite set Q_t, and $Q_t \subseteq E_d$.
3. All indices y where A_y was initialised as X with $X \not\subseteq R$, A_y is updated to $X \cup Y_s$ for some $s \leq g(2|X|)$, and $X \cup Y_s \subseteq E_d$.
4. All indices y where A_y was initialised as a set X with $X \not\subseteq R$, $A_{y,s} = X$ at stage $s = g(2|X|)$, and $X \subseteq E_d$.
5. All indices y where $A_y \subseteq E_d$, and either A_y was initialised by a process belonging to (e, σ) with $e \leq d$ without ever reaching the state of being frozen, or A_y was initialised as some set $X \subseteq R$.

Note that the set in this construction is recursively enumerable, since the fifth case deals only with finitely many y, and the other four cases directly tell how the enumeration procedure works.

But this construction does not provide an effective way to get the enumeration procedure, it only says that this procedure exists.

Now, it will be shown that the y enumerated are exactly those with $A_y \subseteq A_x$. If y is an index of a candidate of the first type generated by the procedure associated with some pair (e, σ) with $\sigma(d)\downarrow = 1$, then the construction guarantees that $A_y \subseteq E_d$. Furthermore, y is enumerated. If y is an index of a candidate of the first type generated by the procedure associated with some pair (e, σ) with $\sigma(d)\downarrow = 0$, then the construction guarantees that $A_y \subseteq E_d$ only if A_y is eventually frozen. Indeed, y is enumerated according to the second case iff A_y is eventually frozen and all of its elements belong to A_x, so, again, the enumeration procedure is correct.

If y is an index of a candidate of the first type generated by some process belonging to (e, σ) with $e \leq d$ and A_y is eventually frozen, then, again, y is enumerated iff $A_y \subseteq A_x$. If y is an index of a candidate of the first type generated by some process belonging to (e, σ) with $e \leq d$ and A_y is never frozen, then the fifth case applies, and, again, y is enumerated iff $A_y \subseteq A_x$.

If y is an index belonging to a candidate of the second type and initialised as X with $X \subseteq R$, then, again, the fifth case applies, and y is enumerated iff $A_y \subseteq A_x$. If y is an index belonging to a candidate of the second type and initialised as X with $X \not\subseteq R$ and $A_{y,g(2|X|)} = X \cup Y_s$ for some $s \leq g(2|X|)$, then the third case applies, and y is enumerated iff $A_y \subseteq A_x$. If y is an index belonging to a candidate of the second type and initialised as X with $X \not\subseteq R$ and $A_y = X$ (that is, A_y is not later modified), then the fourth case of the enumeration procedure applies, and y is enumerated iff A_y is a subset of A_x.

The remaining case is where y belongs to a candidate of the second type, and is initialised as X such that $X \not\subseteq R$ and $A_{y,g(2|X|)} = X$, but there is some $s \geq g(2|X|)$ with $A_y = X \cup Y_s$. Let $C = \{c \leq d : X \subseteq E_{c,s}\}$. Since $X \not\subseteq R$, it follows that $\cap_{c \in C} E_c$ is infinite. Hence, there are, below s, at least $2|X|$ numbers enumerated into $\cap_{c \in C} E_{c,s}$ by stage s. One can choose Y_s such that $X \cup Y_s$ is also a subset of $\cap_{c \in C} E_{c,s}$, and such a Y_s is also taken, for otherwise, the state $st_{X,Y_s,s}$ would not be as large as it could be. Thus, if $X \subseteq E_{d,s}$, then also $Y_s \subseteq E_{d,s}$. It follows that the index y is taken into the constructed r.e. set iff $A_y \subseteq E_d$. This completes the verification that exactly the y with $\{y : A_y \subseteq E_d\}$ are enumerated by the procedure above, so, there is a z such that $A_z = \{y : A_y \subseteq A_x\}$.

Fourth, it is noted that all $B_k = \{x : |A_x| = k\}$ are recursive. This stems from the fact that for all k,

- there is at most one set starting as a candidate of the first type with exactly k elements, and
- among those sets starting as candidates of the second type, there is at most one set with k elements that was initialised with $k/2$ elements, and later had $k/2$ elements enumerated into it, and
- among all sets initialised with k elements, at most one later receives k further elements.

Hence, B_k is a finite variant of $\{x : A_x$ has exactly k elements when initialised$\}$. Therefore, the constructed model is a Friedberg model. ▮

So, one can show that the axiom of power set can be satisfied, when formulated adequately for Friedberg models. An interesting question would be to determine the minimum complexity of the function that maps $x \in \mathbb{N}$ to the index z with $A_z = \{y : A_y \subseteq A_x\}$. This function is not recursive: otherwise, V_ω would be representable in some Friedberg model. Furthermore, it is certainly K'-recursive. Still, it would be interesting to know whether it is K-recursive.

7 Conclusion

This paper was dedicated to investigating the properties of Friedberg models of fragments of set theory, a Friedberg model being defined as a Friedberg numbering $A_0, A_1, A_2, \ldots$ of all r.e. subsets of $\mathbb{N}$, with the additional constraint that every set $B_k = \{x : |A_x| = k\}$ is recursive.

It was investigated the extent to which the axioms of Zermelo Fraenkel set theory hold when the element-relation "x is in y" is defined in a Friedberg model as: $x \in A_y$. The "true sets" are then defined as those which are well-founded with respect to the transitive closure of the membership relation. Full comprehension cannot be satisfied, since the r.e. subsets of $\mathbb{N}$ are not closed under complement, set difference, and universal quantification. Furthermore, the axiom of power set holds only in some but not all Friedberg models. The other axioms of Zermelo Fraenkel set theory have their counterpart in this framework, and a choice function can be implemented easily.

One major question is how comprehensive the models are, that is, how many sets from the von Neumann universe are represented in a Friedberg model. The answer depends on the choice of model. It has been shown that there is a smallest Friedberg model that contains exactly those sets from the von Neumann universe that are in all Friedberg models. But there is no largest Friedberg model. It has been shown that for every Friedberg model, there is some set $E[F]$, defined after Theorem 20 with a suitable K-recursive parameter function F, that does not exist in this model, although it exists in another Friedberg model. It remains open whether there is a maximal Friedberg model such that no other Friedberg model contains strictly more sets from the von Neumann universe than this model. Furthermore, there are some concrete examples of sets which exist or do not exist in Friedberg models. For example, the set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots\}$ is realised by all Friedberg models; the set V_ω of all hereditarily finite sets is not realised by any Friedberg model. Using the ordering of Harrison [7] — see Proposition 5 —, one can show that there is a Friedberg model containing all recursive ordinals (that is, this Friedberg model is of ordinal height ω_1^{CK}) while the smallest Friedberg model contains only the finite ordinals (that is, the smallest Friedberg model is of ordinal height ω) and all sets in this model have rank strictly below ω^2. Another open problem is to characterise those sets in the von Neumann universe which exist in some Friedberg model.

The complexity of the predicate $Set(x)$ stating that "A_x is a set in the model" (and not a proper class) can range from Π^0_1 up to Π^1_1. The complexity of the theory of a Friedberg model depends heavily on the complexity of the predicate, and ranges from $\emptyset^\omega$ to the ω-jump of a Π^1_1-complete set.

One might ask whether Friedberg models are a natural object of study. While Friedberg numberings arises quite naturally from the axiom of extensionality, the question of how effective the B_k should be, is harder to answer. It necessitates some considerations on the results that have been derived. Certainly, one cannot postulate that the B_k are uniformly recursive, since then V_ω would exist, which has been shown to be impossible. Furthermore, B_∞, that is, $\{x : |A_x| = \infty\}$, cannot be recursively enumerable, since otherwise, there would be a recursive enumeration of all infinite r.e. subsets of $\mathbb{N}$, which has also been shown to be impossible. On the other hand, if one drops all effectiveness constraints on the B_k, one could then take a Friedberg numbering in which the set of indices x with A_x being of the form $\mathbb{N} \setminus \{u\}$, is a simple subset of $\mathbb{N}$. Since these A_x are all proper classes, it would follow that no set in the model is infinite. Hence, the model would just contain the sets in V_ω and nothing else. That is of course undesirable. So, the recursiveness of B_1 and B_2 is quite useful to guarantee the existence of infinite sets, as well as functions with infinite domain. It is then natural to postulate that the B_k are all recursive.

If the focus is on r.e. models, then, since Rabin [14] showed that there is no r.e. model of NBG — which is a conservative extension of ZFC —, and this proof easily adjusts to a proof that there is no r.e. model of ZFC, only fragments of ZFC can be considered. There is a large body of work on fragments of ZFC, most notably Kripke-Platek set theory. The work presented in this paper necessitates to consider even weaker fragments. Still, the structures involved have proved to be quite rich, and their study interesting, so the authors hope that this work provides some motivation towards further studies of Friedberg models.

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