Midterm Examination 2
MA 3205: Set Theory

03.11.2009, 12.00-12.45h

Matriculation Number: _______________

Rules
Each question contains as many marks as it has subquestion. Each correct subquestion gives 1 mark. The maximum score is 15 marks.

Question 1. Let $(A, <)$ be a well-ordered set and $B = \{f : A \to A\}$ be the set of all functions from A to A. Now define the following relation $\sqsubset$ on $f, g \in B$:

$$f \sqsubset g \Leftrightarrow \exists a \in A \forall b \in A \, [f(a) < g(a) \wedge [b < a \Rightarrow f(b) = g(b)]].$$

(a) What can be said about this ordering:

☐ The relation is reflexive and thus not a partial ordering;
☐ The relation is a partial ordering and has incomparable elements;
☒ The relation is a linear ordering.

(b) Give a justification for your answer.

First, the relation is not reflexive as if $f \subset f$ then there must be an a with $f(a) < f(a)$, what contradicts $<$ being a well-order. Second, the relation is transitive, as if $f \sqsubset g$ and $g \sqsubset h$ then there is a minimal a such that $f(a) \neq g(a)$ or $g(a) \neq h(a)$. For all $b < a$ it holds that $f(b) = g(b) = h(b)$ and at a it holds either that $f(a) = g(a) < h(a)$ or $f(a) < g(a) = h(a)$ or $f(a) < g(a) < h(a)$ which, in all three cases, implies $f(a) < h(a)$ and $f \sqsubset h$. Third, one has to show that any two different functions f, g either satisfy $f \sqsubset g$ or $g \sqsubset f$. There is a least a such that $f(a) \neq g(a)$; hence either $f(a) < g(a)$ or $g(a) < f(a)$. As $f(b) = g(b)$ for all $b < a$, it follows that $f(a) < g(a)$ implies $f \sqsubset g$ and $g(a) < f(a)$ implies $g \sqsubset f$.

Question 2. How many well-orderings exist on $A = \{0, 1, 2, 3\}$? Here call two well-orderings $<_1, <_2$ on a set A are different iff there exist distinct $x, y \in A$ such that $x <_1 y$ and $y <_2 x$.

(a) How many different well-orderings exist on $A = \{0, 1, 2, 3\}$? __24__.

(b) Give a justification for your answer.

Every finite linearly ordered set is well-ordered. Now there are 4 possibilities to choose the smallest, 3 possibilities to choose the second smallest, 2 to choose the third smallest and then one remaining possibility to choose the largest element; so there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to order the 4 elements of A.

Question 3. (a) Given two ordinals ω^α and ω^β, which of the following conditions is equivalent to $\omega^\alpha + \omega^\beta = \omega^\beta + \omega^\alpha$?

☐ $\alpha < \beta$; ☐ $\alpha \leq \beta$; ☒ $\alpha = \beta$; ☐ $\alpha \geq \beta$; ☐ $\alpha > \beta$; ☐ $\alpha \neq \beta$.

(b) Give a justification for your answer.

If $\alpha < \beta$ then $\omega^\alpha + \omega^\beta = \omega^\beta$ while $\omega^\beta + \omega^\alpha$ cannot be simplified, so the two terms are different. Similarly if $\alpha > \beta$. But if $\alpha = \beta$, both terms are equal to $\omega^\alpha \cdot 2$.

Question 4. Which of the following statements about the rank are true always (for all sets X), sometimes (for some but not all sets X) and never (for no set X)?

(a) If $\alpha = \rho(X)$ then $\alpha \subseteq X \subseteq V_\alpha$:

☐ always; ☒ sometimes; ☐ never.

(b) $\rho(\mathcal{P}(X)) = \rho(X) + 1$:

☒ always; ☐ sometimes; ☐ never.

(c) $\rho(X \times X) = \rho(X)$ where $X \times X = \{\{a, \{a, b\}\} : a, b \in X\}$:

☐ always; ☒ sometimes; ☐ never.

Question 5. (a) When is an ordinal α a cardinal?

- [ ] α is a cardinal iff α is a limit ordinal;
- [ ] α is a cardinal iff $\alpha \leq \omega$ or α is uncountable;
- [x] α is a cardinal iff $\forall \beta \in \alpha\, [|\beta| < |\alpha|]$;
- [ ] α is a cardinal iff $\exists \beta \in \alpha\, [|\beta| < |\alpha|]$.

(b) Which of the following statements is true?

- [ ] There is exactly one finite cardinal;
- [x] There is exactly one countable cardinal;
- [ ] There is exactly one uncountable cardinal.

(c) For which cardinals κ does $\kappa + \kappa = \kappa$ hold?

- [ ] $\kappa + \kappa = \kappa$ iff $\kappa = 0$;
- [ ] $\kappa + \kappa = \kappa$ iff $\kappa \geq \aleph_0$;
- [x] $\kappa + \kappa = \kappa$ iff $\kappa = 0 \vee \kappa \geq \aleph_0$;
- [ ] $\kappa + \kappa = \kappa$ iff $\kappa < \aleph_0$;
- [ ] $\kappa + \kappa = \kappa$ iff $\kappa = 0 \vee \kappa = \aleph_0$.

Question 6. Consider the partial ordering $\sqsubset$ on three-tuples of integers by

$$(u, v, w) \sqsubset (x, y, z) \Leftrightarrow u \leq x \wedge v \leq y \wedge w \leq z \wedge u + v + w < x + y + z.$$

An element (x, y, z) of a set A is called minimal iff there is no $(u, v, w) \in A$ with $(u, v, w) \sqsubset (x, y, z)$. Determine the number of minimal elements of the following sets; this number can be any value in $\{0, 1, 2, \ldots, \aleph_0\}$.

(a) $A = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$: __0__.

(b) $B = \{(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : x \geq 5 \wedge y \geq 5 \wedge z \geq 5 \wedge x + y + z \geq 16\}$: __3__.

(c) $C = \{(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : x \geq 5 \wedge y \geq 5 \wedge z \geq 5 \wedge x + y + z \geq 17\}$: __6__.

Working Space

You can use this page to do calculations, but you should write the answers into the space provided. Answers found here are not evaluated.