Things that can be made into themselves

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Abstract

We investigate which classes of sets have numberings in which the index set of the left-r.e. members of the class is itself a member of the class. For example, the Martin-Löf random sets can be made into themselves. Furthermore we characterise the left-r.e sets whose index set under equality can be made equal to itself. We find that minimal and maximal left-r.e. sets with respect to finite differences on inclusion exist and that, while some minimal left-r.e. sets can be made into themselves, maximal left-r.e. sets cannot be made into themselves. Finally, we show that the inclusion problem for left-r.e. sets cannot be r.e., thereby answering a variant of Kummer's question for r.e. sets.

1 Introduction

The roots of recursion theory entwine self-reference. Even before Turing [40] formalised the notion of computation using machines with infinite tapes and finite control structures, mathematicians conceived of primitive recursive and recursive functions [1, 17]. In particular, Kurt Gödel captured many aspects of computation in his formal system of arithmetic and exploited its self-referential properties in the proof of his famous incompleteness theorem [17]. In order to show that the theory of natural numbers does not have an consistent and complete r.e. axiomatization, Gödel created a first-order

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formula which informally states, with respect to an underlying primitive-recursive set of axioms,

> *"This statement is unprovable."*

so that neither the statement nor its negation has a mathematical proof with respect to the given set of axioms. Gödel's ground-breaking construction contained various important concepts. For that reason, the acceptable numberings are also called, after him, *Gödel numberings*. The expressive strength of a general-purpose computer language is precisely what makes Gödel's self-referential statement possible. Self-reference has manifested itself in computer science and mathematics in the form of fixed point theorems, in particular Kleene's Recursion Theorem [21], Roger's Fixed-Point Theorem [35, Theorem 11-I], the Arslanov Fixed Point Theorem [4] and its generalizations [2, 3, 19], as well as other diagonalization methods [30, 37]. Today research continues in the area of machine self-reference and self-knowledge [8].

Numberings are effective listings of objects; for example, in recursion theory one often studies numberings of r.e. sets and partial-recursive functions. On one hand there are the acceptable numberings introduced by Gödel [17]; on the other hand Friedberg [16] showed that there are also one-to-one numberings of the above named objects where each item occurs exactly once. In this paper, we look at self-reference in terms of numberings of left-r.e. sets. In particular we ask: For which classes $\mathcal{C}$ of sets is it possible to find a universal numbering $A_0, A_1, \ldots$ such that $\{e : A_e \in \mathcal{C}\}$ is again a member of $\mathcal{C}$? Numberings for left-r.e. sets were first studied by Brodhead and Kjos-Hanssen [6, 20] and provide more expressive possibilities than the traditional class of numberings for r.e. sets.

In this paper, we are especially interested in classes of sets having numberings in which the index set of the left-r.e. members of the class is itself a member of the class and sets whose index sets are equal to themselves. For some reason such things exist, and we call this phenomenon "things that can be made into themselves." Here the phrase "can be made into themselves" implicitly refers to the fact that we permit ourselves the flexibility to choose the underlying numbering of all left-r.e. sets for a desired purpose. If we would follow the usual default and use only acceptable left-r.e. numberings, this would prohibit many things from being made into themselves. Indeed Rice's Theorem [33] holds for acceptable left-r.e. numberings and therefore any non-trivial index set is many-one hard either for the halting problem or its complement.

In 1958, Friedberg [16] constructed a maximal r.e. set, that is, an r.e. set maximal under inclusion up to finite differences, thereby bringing Post's program [32, 37, 30] to an abrupt halt. Post [32] had wished to prove the existence of Turing incomplete r.e. sets by building sets with sparse complements. Maximal sets can be Turing complete and have the thinnest possible complements for r.e. sets, so Friedberg's result shows that "thinness" alone cannot achieve Turing incompleteness. As independently discovered by Friedberg and Muchnik [37], Turing incomplete r.e. sets do exist by an alternate methods. In Section 6 we introduce the concept of maximal and minimal left-r.e. sets. Unlike the class of r.e. sets, which has only maximal sets, both minimal and maximal left-r.e. sets exist (Theorem 6.2). Among these two classes, however, only maximal left-r.e. sets can be made into themselves (Theorem 6.5).

We shall show that the Martin-Löf random sets and 1-generic sets can be made into themselves (Corollary 2.5 and Corollary 2.7), though not at the same time (Proposition 5.2), whereas the r.e., co-r.e. and recursive sets each cannot be (Corollary 3.2). We characterise the left-r.e. sets whose index sets can be made equal to the set itself (Theorem 4.1) and describe the complexity of the inclusion problem for left-r.e. numberings (Theorem 7.1).

Notation. A *numbering* φ is a partial-recursive function $\langle e, x\rangle \mapsto \varphi_e(x)$. W_e^φ denotes the domain of φ_e and we may omit the superscript if it is clear from context. There exists a numbering φ such that the plain *Kolmogorov complexity* C of a binary string σ, given by $C(\sigma) = \min\{\log(e) : \varphi_e(0) = \sigma\}$, is optimal up to an additive constant factor [24]. A numbering ψ is called *prefix-free* if $\psi_0(i)$ is not a prefix of $\psi_0(j)$ for all distinct i and j. There is, up to an additive constant, an optimal prefix-free numbering ν for *prefix-free Kolmogorov complexity*, defined analogously to the plain case as $H(\sigma) = \min\{\log(e) : \nu_e(0) = \sigma\}$ [24]. We shall use $|e|$ interchangeably with $\log e$ and the length of the string e. Let $A \triangle B$ denote the symmetric difference of A and B, that is, $A \cup B - A \cap B$. Furthermore, $\overline{A} = \mathbb{N} - A$ is the complement of the set A. Furthermore, $A \subseteq^* B$ means that almost all elements of A are also in B and $A \subset^* B$ means that in addition to the previous, there are infinitely many elements in $B - A$. For finite strings σ and τ, $\sigma \cdot \tau$ denotes concatenation of σ and τ, $\sigma \sqsupseteq \tau$ means σ extends τ and $\sigma \sqsubseteq \tau$ means σ is a prefix of τ. Similarly for sets, $\sigma \sqsubseteq A$ means that σ is a prefix of A. A set is *co-r.e.* if it is the complement of an r.e., or recursively enumerable set, $'$ is the jump operator and $\equiv_{\mathrm{T}}$ is Turing equivalence. We say A is *B-recursive* if $A \leq_{\mathrm{T}} B$. $A \leq_{\mathrm{btt}} B$ if membership in A can be decided by uniformly constructing a Boolean formula over finitely many variables and evaluating it using membership values from B. For a set A, we use $A \upharpoonright n$ to denote the prefix of A's characteristic sequence $A(0)A(1)\ldots A(n)$. A subset of natural numbers is Π^0_n if it can be described by a formula consisting of n alternating quantifiers, starting with a universal quantifier, and ending with a recursive predicate. Furthermore, a set is Σ^0_n iff its complement is Π^0_n.

Background. We will appeal to the following notions throughout our discussion. A set A is called *autoreducible* if for all x, whether x is a member of A can be effectively determined by querying A at positions other than x. For any numbering α, the *α-index set* of a class $\mathcal{C}$ is the set $\{e : \alpha_e \in \mathcal{C}\}$. For sets of nonnegative integers A and B, $A \leq_{\mathrm{lex}} B$ means that either $A = B$ or the least element x of the symmetric difference satisfies $x \in B$. A set A is *left-r.e.* iff there is a uniformly recursive approximation $A_0, A_1, \ldots$ of A such that $A_s \leq_{\mathrm{lex}} A_{s+1}$ for all s. The symbol $'$ denotes the jump operator and $\oplus$ denotes join. For further background on recursion theory and left-r.e. sets, see the textbooks of Downey and Hirschfeldt [13], Li and Vitányi [24], Nies [29], Odifreddi [30, 31], Rogers [35] and Soare [37].

The reader may already be familiar with left-r.e. reals, which admit an increasing, recursive sequence of rationals from below, however in the context of effective enumerations it makes more sense to consider left-r.e. *sets*, see [5, Section 2].

Definition 1.1. A *left-r.e. numbering* α is a partial-recursive function from natural numbers to left-r.e. sets given by

$$e \mapsto \lim_{s \to \infty} \alpha_{e,s} = \alpha_e$$

where:

(I) $\alpha_{e,s}$ is a recursive set, uniform in e and s, and

(II) $\alpha_{e,s} \leq_{\mathrm{lex}} \alpha_{e,s+1}$ for all s.

A left-r.e. numbering is called *universal* if its range includes all left-r.e. sets, and a left-r.e. numbering α is called an (K-)*acceptable* left-r.e. numbering if for every left-r.e. numbering β there exists a (K-)recursive function f

such that $\alpha_{f(e)} = \beta_e$ for all e. Here K denotes the halting set. Acceptable numberings permit an effective means for coding any algorithm, so the class of acceptable numberings is also the class of general purpose programming languages.

Definition 1.2. We say that a left-r.e. set A *can be made into itself* if there exists a universal left-r.e. numbering α such that

$$\{e : \alpha_e = A\} = A.$$

A class of sets $\mathcal{C}$ *can be made into itself* if there exists a universal left-r.e. numbering β such that

$$\{e : \beta_e \in \mathcal{C}\} \in \mathcal{C}.$$

If a set or class can be made into itself via an acceptable left-r.e. numbering, then we say the class can be *strongly* made into itself.

Note that in this context there are classes $\mathcal{C}$ which can be made into themselves and which do not entirely consist of left-r.e. sets. This will be essential for various results; for example the Martin-Löf random sets can be made into themselves while the Martin-Löf random left-r.e. sets cannot be made into themselves. Hence permitting $\mathcal{C}$ to have members which are not left-r.e. is often necessary and is also natural in the case for many classes; see Corollary 2.5 and Proposition 3.3 below.

Our primary tool for making things into themselves will be indifferent sets. An indifferent set is a list of indices where membership in a given set can change without affecting membership in some class.

Definition 1.3 (Figueria, Miller and Nies [14])**.** An infinite set I is called *indifferent for a set A with respect to $\mathcal{C}$* if for any set X,

$$X \triangle A \subseteq I \implies X \in \mathcal{C}.$$

When the class $\mathcal{C}$ is clear from context, we may omit it.

2 Classes that can be made into themselves

We show that any class of nonrecursive sets which either contains the Martin-Löf random sets or contains the weakly 1-generic sets can be made into itself.

Our proof relies crucially on co-r.e. indifferent sets which are retraceable by recursive functions.

A set A is called *Martin-Löf random* [25, 36] if there exists a constant c such that for all n, $H(A \upharpoonright n) \geq n - c$. Intuitively, A is random if every prefix of A is incompressible and therefore lacks a definable pattern. Zvonkin and Levin [42] and later Chaitin [10] gave an example of a left-r.e. Martin-Löf random real called Ω.

Figueira, Miller and Nies [14] constructed indifferent sets for the class of Martin-Löf random sets. One of their approaches is to build indifferent sets for non-autoreducible sets. While this works for Martin-Löf random sets, the technique does not generalise to weaker forms of randomness because recursively random sets may be autoreducible [26]. On the other hand, Franklin and Stephan [15] showed that every complement of a dense simple set is indifferent with respect to Schnorr randomness for all Schnorr random sets. The arguments in Lemma 2.2 and Theorem 2.4 are also essentially due to Figueira, Miller and Nies [14], however we find it useful to make explicit the property of retraceability.

Definition 2.1. A set $A = \{a_0, a_1, a_2 \ldots\}$ is *retraceable* if there exists a partial-recursive function f satisfying $f(a_{n+1}) = a_n$ for all n and $f(x) < x$ whenever $f(x)$ is defined. A set S is *approximable* if there exists an n and a recursive function f such that for any $x_1, \ldots, x_n$ with $x_1 < \ldots < x_n$, the n-bit vector $f(x_1, \ldots, x_n)$ agrees with the characteristic vector $(S(x_1), \ldots, S(x_n))$ in at least one place. In particular, we say S is $(1, n)$-*recursive.*

Lemma 2.2. *For every K-recursive function f, there exists a co-r.e. set $I = \{i_0, i_1, i_2, \ldots\}$ which is retraceable by a recursive function and satisfies $f(n) < i_n < i_{n+1}$ for all n.*

Proof. Let $\{f_s\}$ be a recursive approximation to f satisfying $\max f_s < s$. We construct I by a movable marker argument. The set

$$I_s = \{i_{0,s}, i_{1,s}, i_{2,s}, \ldots\}$$

will be a recursive approximation to I at stage s. Set $I_0 = \omega$. At stage $s+1$, choose the least n satisfying $f_s(n) \neq f_{s+1}(n)$ and enumerate sufficiently many elements into $\overline{I}_{s+1}$ such that

- For all $k \geq n$, $i_{k,s+1} \geq s + 1$, and

- For all $k < n$, $i_{k,s+1} = i_{k,s}$.

For each n, $\{f_t(n)\}$ settles in some stage $s_n + 1$ and so

$$i_n = i_{n,s_n} \geq s_n + 1 > f(n).$$

Furthermore, the recursive function

$$g(x) = \begin{cases} i_0 & \text{if } x \leq i_1 \text{, and} \\ \max I_{x+1} \cap \{0, 1, 2, \ldots x - 1\} & \text{otherwise.} \end{cases}$$

witnesses that I is retraceable. □

The set in Lemma 2.2 is retraced by a total recursive function. Hence there is a recursive function h which maps I surjectively to the set of natural numbers. In the above case, one can also see directly that such a h exists, as one can choose h as

$$h(x) = |I_{x+1} \cap \{0, 1, \ldots, x\}|$$

and then h has the desired property $h(i_n) = n$. A set which is retraceable by a recursive function is $(1, 2)$-recursive [38], and therefore the set I above is also approximable.

Lemma 2.3. *Let $\mathcal{C}$ be a class of nonrecursive sets containing:*

(I) *a K-recursive member A with a co-r.e. and retraceable set I which is indifferent for A with respect to $\mathcal{C}$ and*

(II) *a left-r.e. set $X = \sup X_s$ such that all the recursive approximations X_s to X are not in $\mathcal{C}$.*

Let $\mathcal{D}$ be a superclass of $\mathcal{C}$ not containing any recursive set. Then there exists a K-acceptable universal left-r.e. numbering which makes $\mathcal{D}$ into itself.

Proof. Let $i_0, i_1, i_2, \ldots$ be the elements of I in ascending order and let the numbering $\alpha_0, \alpha_1, \alpha_2, \ldots$ be an acceptable universal left-r.e. numbering. Recall that there is a recursive function h with $h(i_n) = n$ for all n. Let A_s be an approximation of A in the limit. Now define

$$\beta_e = \begin{cases} \alpha_{h(e)} & \text{if } e \in I, \\ X_s & \text{if } e \notin I \text{ and } s \text{ is the largest stage with } A_s(e) = 0 \text{ and} \\ X & \text{if } e \notin I \text{ and } e \in A. \end{cases}$$

Each β_e is left-r.e. because h is recursive, the complement of I is r.e. and $\gamma = \sup_s \gamma_s$. Furthermore, β is a K-acceptable numbering as the mapping $n \mapsto i_n$ is K-recursive. For $e \notin I$, $\beta_e \in \mathcal{D}$ iff $\beta_e \in \mathcal{C}$ iff $e \in A$. The reason is that for $e \in \overline{I} \cap A$ it holds that $\beta_e = X \in \mathcal{C} \subseteq \mathcal{D}$; for $e \in \overline{I} \cap \overline{A}$ it holds that $\beta_e = X_s$ for some s and X_s is recursive, hence outside $\mathcal{C}$ and outside $\mathcal{D}$. As I is indifferent for A with respect to $\mathcal{C}$, it follows that $\{e : \beta_e \in \mathcal{D}\}$ is in $\mathcal{C}$ and therefore also in $\mathcal{D}$. So $\mathcal{D}$ is made into itself by the universal left-r.e. numbering β. □

A set A is called *low* if $A' \equiv_{\mathrm{T}} K$ and A is called *high* if $A' \geq_{\mathrm{T}} K'$.

Theorem 2.4. *For every low Martin-Löf random set A, there exists a co-r.e. set which is indifferent for A with respect to Martin-Löf-randomness and retraceable by a recursive function.*

Proof. Let A be a low Martin-Löf random set, for example

$$A = \{x : 2x \in \Omega\} \tag{2.1}$$

is Martin-Löf random and low by van Lambalgen's Theorem [41] and [12, Theorem 3.4], see also [29, Theorem 3.4.11]. Then

$$f(n) = \max\{m : H(A \upharpoonright m) \leq m + 3n\}$$

is partial-recursive in A and hence K-recursive. By Lemma 2.2, there exists a co-r.e. set I which is retraceable by a recursive function and satisfies

$$f(n) < i_n < i_{n+1} \tag{2.2}$$

for all n. Let $k(m)$ be the number such that

$$i_{k(m)} < m \leq i_{k(m)+1},$$

and let $r(m)$ be the number such that

$$f[r(m)] < m \leq f[r(m) + 1],$$

which is positive for all but finitely many m by Miller and Yu's Ample Excess Lemma [28], see [13, Corollary 6.6.2]. By (2.2) we have $k(m) \leq r(m)$ for all sufficiently large m; otherwise

$$f[r(m) + 1] < i_{r(m)+1} \leq i_{k(m)} < m,$$

which is impossible.

Suppose that there were some Martin-Löf non-random set N such that $N \triangle A \subseteq I$. We can code a prefix of the set A given sufficiently long prefixes for N and I, and so for infinitely many m

$$\begin{aligned} H(A \upharpoonright m) &\leq H(N \upharpoonright m) + H[A(i_0)A(i_1)\ldots A(i_{k(m)})] + 2\log m + O(1) \\ &< m + 2k(m) + 2\log m + O(1) \\ &\leq m + 2r(m) + 2\log m + O(1). \end{aligned}$$

Here the additive log factor is used for coding two implicit programs into a single string. On the other hand, by the definition of f,

$$H(A \upharpoonright m) > m + 3r(m)$$

for all m, a contradiction. Therefore I is indifferent for A. □

We are now ready to prove that several classes can be made into themselves. Since left-r.e. Martin-Löf random sets exist [10, 13], the following result is immediate from Theorem 2.4 and Lemma 2.3.

Corollary 2.5. *If a class $\mathcal{C}$ contains all Martin-Löf random sets and no recursive sets then $\mathcal{C}$ can be made into itself. In particular, the classes of Martin-Löf random sets, recursively random set, Schnorr random sets, Kurtz random sets, bi-immune sets, immune sets and nonrecursive sets can be made into themselves.*

See the usual textbooks on recursion theory and algorithmic randomness for the definition of these notions [13, 24, 29, 30, 35, 37]. It is also straightforward to make non-random sets strongly into themselves: Just enumerate the left-r.e. sets on the even indices and one fixed member of the class on the odd indices. This numbering makes each class containing all non-immune sets (plus perhaps some others) have a non-immune index set.

We now investigate self reference for the class of 1-generic sets, a class of sets orthogonal to Martin-Löf random sets with respect to Baire category and measure. A set of binary strings A is called *dense* if for every string σ there exists $\tau \in A$ extending σ. A set is *weakly 1-generic* if it has a prefix in every dense r.e. sets of binary strings. Furthermore X is *1-generic* if for every (not necessarily dense) r.e. set of strings W, either X has a prefix in W or some prefix of X has no extension in W. Every 1-generic set is weakly 1-generic [29]. The following result isolates and generalises the main idea of [18, Theorem 23].

Theorem 2.6. *Every K-recursive 1-generic set A has a co-r.e. indifferent set which is retraceable by a recursive function.*

Proof. Let $W_0, W_1, \ldots$ be any enumeration of the r.e. sets, and let R_e denote the e^{th} genericity requirement: ρ *satisfies* R_e if either some prefix of ρ belongs to W_e or no proper extension of ρ belongs to W_e. First we show that there exists a K-recursive function f such that

$$(\forall n)\,(\forall e \leq f(n))\,(\forall \sigma \in \{0,1\}^{f(n)})$$
$$[\sigma \cdot A[f(n)]A[f(n)+1]\ldots A[f(n+1)] \text{ satisfies } R_e]\,.$$

For any given σ and e, there must be some sufficiently long segment of A, say $A(|\sigma|)A(|\sigma|+1)\ldots A(c_{\sigma,e})$, satisfying R_e since W_e is an r.e. set and A is 1-generic. Now let $f(0) = 0$ and

$$f(n+1) = \max\{c_{\sigma,e} : |\sigma|, e \leq f(n)\}.$$

f can be computed using an A and a halting set oracle, hence f is K-recursive. Now using Lemma 2.2, obtain a co-r.e. set I which is retraceable by a recursive function and satifies $i_n > f(2n)$ for all n. By the pigeonhole principle, for every n there exist at least n intervals below $f(2n)$ of the form

$$J_k = \{f(k)+1, f(k)+2, \ldots, f(k+1)\} \quad (k \leq 2n)$$

which do not contain a member of I. Hence $J_n \cap I = \emptyset$ for infinitely many n. For any $B \subseteq I$, each such n witnesses that some initial segment of $A \triangle B$ satisfies R_e for all $e \leq f(n)$, hence I is indifferent for A. $\square$

While a left-r.e. set cannot be 1-generic [29], it can be weakly 1-generic [37]. This follows from the fact that a 1-generic set cannot compute a nonrecursive r.e. set [37]. Thus by Theorem 2.6 and Lemma 2.3, we obtain the following result.

Corollary 2.7. *Any class of non-recursive sets containing the weakly 1-generic sets can be made into itself.*

Day has thoroughly investigated indifferent sets for 1-generic sets [11]. He showed that every 1-generic set has an indifferent set which is itself 1-generic and also points out, as follows from Theorem 2.6, that every K-recursive 1-generic set has a co-r.e. indifferent set.

3 Things which cannot be made into themselves

In this section we show that there are many classes which cannot be made into themselves. The easiest example is the class of all finite sets as this class cannot have a finite index set.

Theorem 3.1. *There is no left-r.e. numbering for the non-r.e. left-r.e. sets. Similarly, there is no left-r.e. numbering for the non-recursive left-r.e. sets.*

Proof. Assume $\alpha_0, \alpha_1, \ldots$ is a recursive enumeration containing no cofinite set. It is now shown that there is also a non-r.e. left-r.e. set B which differs from all α_e. For this, let F be the K-recursive function such that $F(e)$ is the maximum of the e-th non-elements in each of the sets $\alpha_0, \alpha_1, \ldots, \alpha_e$. One builds B such that the complement of B consists of elements $x_e = 2^e \cdot 3^{d(e)}$ where $d(e)$ is the supremum of all $F_s(e)$ for a recursive approximation F_s to F; furthermore, whenever $x_e \notin W_{e,s} \wedge 3x_e \in W_{e,s}$ then $d(e)$ is incremented by 1. Note that the latter is done only once after $F(e)$ has converged and that the latter enforces that $W_e(x_e) \neq B(x_e) \vee W_e(3x_e) \neq B(3x_e)$ so that B is not an r.e. set. It is easy to see that B is a left-r.e. set; the reason is that the definition of $d(e)$ permits to make an approximation $x_{e,s}$ to x_e monotonically from below and that therefore the approximation $B_s = \{y : \forall e\, [y \neq x_{e,s}]\}$ is a left-r.e. approximation to B. Hence $\alpha_0, \alpha_1, \ldots$ can neither be the numbering of all nonrecursive left-r.e. sets nor the numbering of all non-r.e. left-r.e. sets. $\square$

Although somewhat disappointing, the next fact follows as a consequence.

Corollary 3.2. *The r.e. sets, co-r.e. sets and recursive sets cannot be made into themselves.*

Proof. Suppose that α is a universal left-r.e. numbering which makes the r.e. sets into themselves, and say the α-index set of the r.e. sets is R. Let X be any set which is left-r.e. but not r.e., for example a left-r.e. Martin-Löf random. Now define a left-r.e. numbering β by

$$\beta_e = \begin{cases} \alpha_e & \text{if } e \notin R, \\ \sigma \cdot X \text{ for some finite } \sigma & \text{otherwise.} \end{cases}$$

In detail, β_e follows the enumeration of α_e until e gets enumerated into R (if this ever happens), at which point β switches to enumerating X. Thus β is an enumeration of the non-r.e. left-r.e. sets, contrary to Theorem 3.1.

Now, suppose that some universal left-r.e. numbering γ makes the co-r.e. sets into themselves. Let Q be the γ-index set of the co-r.e. sets, and note that the class of left-r.e. co-r.e. sets is the class of left-r.e. recursive sets. By a construction analogous to the one for β above, there exists a left-r.e. numbering consisting of the left-r.e. sets with γ-indices in $\overline{Q}$. This is an enumeration of all left-r.e. sets which are non-recursive, contradicting Theorem 3.1. Since Q is also the index set of recursive sets, the recursive sets cannot be made into themselves either. $\square$

Another example of what cannot be done is the following; the class of left-r.e. Martin-Löf random sets is quite natural and also known as the class of Ω-numbers [7, 22, 27]. The reason is that they can be represented as the halting probability of some universal prefix-free Turing machine.

Proposition 3.3. *The left-r.e. Martin-Löf random sets cannot be made into themselves.*

Proof. If the left-r.e. Martin-Löf random reals could be made into themselves, then the set of indices for Martin-Löf non-random reals would be Δ_2 inside this numbering. This contradicts a theorem of Kjos-Hanssen, Stephan, and Teutsch [20] which says that the Martin-Löf non-randoms are never Π^0_3 in any universal left-r.e. numbering. $\square$

We remark that any set that can be strongly made into itself contains an infinite recursive subset by the Padding Lemma [30, 34]. This means that the Martin-Löf randoms, the recursively random sets, the Schnorr randoms, the Kurtz randoms, the bi-immune sets, and immune sets cannot be strongly made into themselves. Figueira, Miller and Nies [14] asked whether Chaitin's Ω can have an infinite co-r.e. indifferent set. A partial solution to this problem follows immediately from the Lemma 2.3 and Proposition 3.3: if such a co-r.e. indifferent set exists, it cannot be retraceable by a recursive function.

In contrast to Proposition 3.3, every acceptable numbering of the left-r.e. reals makes the autoreducible reals strongly into themselves, as the resulting index set is a cylinder and thus autoreducible. Note that not every set is autoreducible, for example Martin-Löf random reals fail to be autoreducible [14, 39]. By Corollary 2.5, the non-autoreducible reals can also be made into themselves, but by the above comment they cannot be made strongly into themselves.

4 Singleton classes

In the case of sets, we can exactly characterise which things can be made into themselves.

Theorem 4.1. *A left-r.e. set A can be made into itself iff $A \neq \emptyset$ and there exists an infinite, r.e. set B such that $A \cap B = \emptyset$.*

Proof. Assume A can be made into itself via a universal left-r.e. numbering α. Then $A \notin \{\emptyset, \omega\}$, so there exists a rational number r with $.A < r < 1$ where ".A" is the set A interpreted as a real number between 0 and 1. Let

$$B = \{e : (\exists s)[\alpha_{e,s} > r]\}.$$

Then $A \cap B = \emptyset$, B is r.e., and B is infinite.

Conversely, assume $A \neq \emptyset$, and $B = \{b_0, b_1, \dots\}$ is an infinite r.e. set satisfying $A \cap B = \emptyset$. Brodhead and Kjos-Hanssen [6] showed that there exists a *Friedberg numbering*, or enumeration without repetition, of the left-r.e. reals. Let α be a Friedberg numbering of the left-r.e. reals with the real A deleted from the enumeration.

First consider the case where A is infinite, and let $A_0, A_1, A_2, \dots$ be a recursive approximation of A from below. We then build a further numbering γ such that

$$\gamma_e = \begin{cases} \alpha_d & \text{if } e = b_d, \\ A_s & \text{if } e \in \overline{A} \cap \overline{B} \text{ and } s = \max\{t : e \in A_t\}, \\ A & \text{if } e \in A \cap \overline{B}. \end{cases}$$

This γ witnesses that A can be made into itself. Note that for $e \in B$ it holds that $\gamma_e = \sup\{\gamma_s : e \in A_s\}$ and hence the γ_e are left-r.e. sets.

If A is finite then in the definition of γ_e, A_s can be replaced with "some finite set not equal to A" and then γ makes A into itself. □

In canonical universal left-r.e. numberings, no set gets made into itself.

Proposition 4.2. *Let α be an acceptable universal left-r.e. numbering. Then for every set B, $\{e : \alpha_e = B\} \neq B$.*

Proof. Every finite set has an infinite index set and is thus not made into itself. For every infinite set consider the left-r.e. numbering β given by

$$\beta_e = B \cap \{x : (\exists y \in W_e)\, [x < y]\}.$$

Note that $\beta_e = B$ iff W_e is infinite and that there is a recursive function f with $\alpha_{f(e)} = \beta_e$ for all e. It follows that W_e is infinite iff $\alpha_{f(e)} = B$. Hence $\{e : \alpha_e = B\}$ is not left-r.e. but rather Π^0_2-complete like the index set for the infinite sets [37]. $\square$

5 Making things into themselves simultaneously

Having made certain classes into themselves and others not, we now investigate which collections of classes can be simultaneously made into themselves using a single numbering.

Definition 5.1. We say that $\mathcal{A}$ and $\mathcal{B}$ can be *simultaneously* made into themselves if there is a numbering which makes both $\mathcal{A}$ into itself and $\mathcal{B}$ into itself.

One thing one we do not get at the same time is Martin-Löf random sets and weakly 1-generic sets. We showed in Corollary 2.5 and Corollary 2.7 that each of these classes can be made into themselves (by themselves), however their combination results in calamity.

Proposition 5.2. *The Martin-Löf random sets and weakly 1-generic sets cannot simultaneously be made into themselves.*

Proof. Assume that α makes the weakly 1-generic sets into themselves. Then the characteristic sequence for the α-index set of the weakly 1-generic sets is itself weakly 1-generic and hence must contain very long runs of 1's [29, Theorem 3.5.5]. On the other hand, no Martin-Löf random sets is weakly 1-generic [13, Proposition 8.11.9], and therefore the α-index set for the Martin-Löf random sets must contain very long runs of 0's. Thus it follows from [29, Theorem 3.5.21], which says that long runs of 0's prevent a set from being Martin-Löf random, that the Martin-Löf random sets do not get made into themselves using α. $\square$

We note that for many classes which can be made into themselves and which have complementary classes which can also be made into themselves, the class and its complementary class cannot be simultaneously made into themselves.

Proposition 5.3. *Any class closed under complements cannot be simultaneously made into itself with its complement.*

Proof. Suppose that some class which is closed under complements can be made into itself. Then the indices for the complement in any universal left-r.e. numbering are also a member of the original class and hence do not belong to its complement. □

Examples of important classes for which Proposition 5.3 applies include the Martin-Löf random sets and the autoreducible sets. Corollary 2.5 established that the Martin-Löf random sets can be made into themselves, and any acceptable universal left-r.e. numbering will make the non-Martin-Löf random sets into themselves via the Padding Lemma [37]. We established in the discussion following Proposition 3.3 that any acceptable universal left-r.e. numbering also makes the autoreducible sets into themselves. Hence the following corollary holds.

Corollary 5.4. *The class of all sets which are not Martin-Löf random and the class of all autoreducible sets are simultaneously made into themselves by any acceptable universal left-r.e. numbering.*

6 Minimal and maximal left-r.e. sets

A coinfinite r.e. set A is called *maximal* [16] iff there is no coinfinite r.e. superset $E \supset A$ with $E - A$ being infinite; in other words, an r.e. set A is maximal iff $A \subset^* \mathbb{N}$ and there is no r.e. set E with $A \subset^* E \subset^* \mathbb{N}$. The corresponding notion of minimal r.e. sets does not exist. Indeed, every infinite r.e. set A contains an infinite recursive subset, and one can recursively remove every other element from this infinite recursive set to obtain an infinite r.e. subset of A with infinitely fewer elements.

To what extent does the inclusion structure for the left-r.e. sets resemble that of the r.e. sets? One difference between these two structures is immediate. Unlike the situation for r.e. sets, intersections and unions of left-r.e. sets need not be left-r.e.; only the join

$$E \oplus F = \{2x : x \in E\} \cup \{2y + 1 : y \in F\}$$

of left-r.e. sets E and F is always left-r.e. For example, Ω intersected with the set of even numbers, call this set A, is not a left-r.e. set. If it were, then one could use this set to build a left-r.e. approximation for the set $B = \{x : 2x \in \Omega\}$ by updating at each stage those B-indices e for which every odd A-index below $2e$ shows a zero. But, as established in (2.1), B is low and Martin-Löf random, contradicting that every left-r.e. Martin-Löf random is an Ω-number [7, 22, 27] and that every Ω-number is weak-truth-table equivalent to, and hence Turing equivalent to, the halting problem [9]. An analogous construction shows that left-r.e. sets are not closed under inclusion.

Definition 6.1. A left-r.e. set A is called a *minimal left-r.e. set* iff $\emptyset \subset^* A$ and there is no left-r.e. set E with $\emptyset \subset^* E \subset^* A$. A left-r.e. set B is called a *maximal left-r.e. set* iff $B \subset^* \mathbb{N}$ and there is no left-r.e. set E with $B \subset^* E \subset^* \mathbb{N}$.

The next result shows that both types of sets exist, in contrast to the r.e. case where only maximal sets exist. Neither maximal left-r.e. sets, nor minimal left-r.e. sets, nor their respective complements need be hyperimmune (in contrast to the complements of maximal r.e. sets [30, Proposition III.4.14]).

Theorem 6.2. *There are a minimal set A and a maximal set B in the partially ordered structure of all left-r.e. sets and $\subset^*$.*

Proof. Let Ω be Chaitin's Martin-Löf random set and let Ω_s be a left-r.e. approximation to it. Furthermore, let

$$c_{n,s} = \sum_{m<2^n} 2^{2^n - m}\Omega_s(m)$$

and $c_n = \lim_{s\to\infty} c_{n,s}$. Let $d_n = c_n - 2^{2^{n-1}} c_{n-1}$ so that d_n is the sum of all $2^{2^n-m}\Omega(m)$ with $m = 2^{n-1}, 2^{n-1+1}, \ldots, 2^n - 1$. Note that $c_n \leq 2^{2^n}$ for all n. Let $I_1, I_2, \ldots$ be a recursive partition of $\mathbb{N}$ into intervals such that each interval I_n contains all numbers $\langle n, x, y\rangle = \min(I_n) + x \cdot 2^{2^n} + y$ with $x, y \in \{0, 1, \ldots, 2^{2^n} - 1\}$. Now let

$$\begin{aligned} a_n &= \langle n, c_{n-1}, 2^{2^n} - 1 - d_n\rangle \text{ for } n > 0, \\ b_n &= g(a_n) \text{ where} \\ g(u) &= \max(I_n) + \min(I_n) - u \text{ for all } n \text{ and all } u \in I_n, \\ A &= \{a_1, a_2, \ldots\} \text{ and } B \;=\; \mathbb{N} - \{b_1, b_2, \ldots\} \end{aligned}$$

So g is defined such that if u is the r^{th} smallest element of I_n then $g(u)$ is the r^{th} largest element of I_n. Note that A and B are btt-equivalent: $u \in A \Leftrightarrow g(u) \notin B$. Now it is shown that A is a minimal left-r.e. set and B is a maximal left-r.e. set.

The set A is left-r.e. as one can start the enumeration at s_0 with $c_{0,s} = c_0$ and letting, for $s \geq s_0$, $A_s = \{a_{1,s}, a_{2,s}, \ldots, a_{s,s}\}$. Then one has for each $s \geq s_0$ that whenever there is an n with $a_{n,s+1} > a_{n,s}$ then there is also a least $m \leq n$ where $a_{m,s+1} \neq a_{m,s}$ and it follows that for this number the change is in the d-part of $a_{m,s} = \langle m, c_{m-1,s}, 2^{2^m} - 1 - d_{m,s}\rangle$ so that $a_{m,s+1} < a_{m,s}$. Hence it holds that $A_s \leq_{\text{lex}} A_{s+1}$ and the approximation of the A_s is an left-r.e. approximation. Furthermore, let $B_s = (I_1 - \{b_{1,s}\}) \cup (I_2 - \{b_{2,s}\}) \cup \ldots \cup (I_s - \{b_{s,s}\})$. Note that g inverts the direction of the approximation in the intervals. Hence, if $s \geq s_0$ and $b_{n,s+1} \neq b_{n,s}$ then the least $m \leq n$ with $b_{m,s+1} \neq b_{m,s}$ satisfies that $b_{m,s+1} > b_{m,s}$. Hence one can see that for $s \geq s_0$ it holds that $B_s \leq_{\text{lex}} B_{s+1}$.

Assume now that E is an infinite left-r.e. subset of A and let E_s be a left-r.e. approximation of E. For any n where $a_{n+1} \notin E$ and $a_{n+2} \in E$, let σ be an n-bit binary string telling which of the first n elements $a_0, a_1, \ldots, a_n$ is in E and let $\psi(\sigma, c_n)$ be a partial-recursive function identifying the first stage $s \geq s_0$ such that $a_{0,s} = a_0$, $a_{1,s} = a_1$, …, $a_{n,s} = a_n$ and

$$E_s \cap J_{n+2} = \{a_{m,s} : m \in \{1, 2, \ldots, n\} \wedge \sigma(m-1) = 1\} \cup \{a_{n+2,s}\};$$

where $J_n = I_1 \cup I_2 \cup \ldots \cup I_n$. Note that $n, a_0, a_1, \ldots, a_n$ can all be computed from c_n. Now, due to $E_s \leq_{\text{lex}} E$, the final value of a_{n+2} must be below $a_{n+2,s}$ for the s computed this way, hence $c_{n+1,s} = c_{n+1}$. This implies that for all the n where $a_{n+1} \notin E \wedge a_{n+2} \in E$ it holds that the Kolmogorov complexity of c_{n+1} given c_n is at most n bits plus a constant; however, the prefix-free Kolmogorov complexity of each c_n is approximately 2^n and therefore there can only be finitely many such n. It follows that almost all a_n are in E. This shows that A is a minimal left-r.e. set.

To see that B is maximal, consider any coinfinite left-r.e. set E containing B. As before one computes for each n with $b_{n+1} \in E \wedge b_{n+2} \notin E$ and σ being an n-bit string telling which of $b_1, b_2, \ldots, b_n$ are in E the stage $\psi(c_n, \sigma)$ as the first stage $s \geq s_0$ such that $b_{1,s} = b_1$, $b_{2,s} = b_2$, …, $b_{n,s} = b_n$ and

$$E_s \cap J_{n+2} = J_{n+2} - \{b_{m,s} : m \in \{1, 2, \ldots, n\} \wedge \sigma(m-1) = 0\} - \{b_{n+2,s}\}.$$

Note again that $n, b_1, b_2, \ldots, b_n$ can be computed from c_n. Now the $s = \psi(c_n, \sigma)$ satisfies that $b_{n+2,s} \leq b_{n+2}$ and hence $c_{n+1,s} = c_{n+1}$. This permits again to conclude by the same Kolmogorov complexity arguments as in the case of the set A that E is the union of B and a finite set; hence B is a maximal left-r.e. set. $\square$

One might ask why we construct a maximal left-r.e. set instead of checking whether some maximal r.e. set is also maximal as a left-r.e. set. Unfortunately this approach does not work, as the following result shows.

Theorem 6.3. *No r.e. set can be a maximal left-r.e. set.*

Proof. Let A be an infinite r.e. set. Without loss of generality assume that exactly one new element gets enumerated into A at each stage of its recursive approximation $A_0, A_1, A_2, \dots$ and for each s, let $x_0, x_1, x_2, \dots$ denote the complement of A_s in ascending order and define

$$E_s = A_s \cup \{x_1, x_3, x_5, \dots\}.$$

Now assume that there is a stage s and $x_n \in A_{s+1} - A_s$ being the unique element enumerated into A at stage s. If n is even, then

$$E_{s+1} = E_s \cup \{x_n, x_{n+2}, x_{n+4}, \dots\} - \{x_{n+1}, x_{n+3}, \dots\},$$

and if n is odd, then

$$E_{s+1} = E_s \cup \{x_{n+1}, x_{n+3}, \dots\} - \{x_{n+2}, x_{n+4}, \dots\}.$$

In either case the minimum of the symmetric difference of E_s and E_{s+1}, which is x_n when n is even and x_{n+1} when n is odd, belongs to E_{s+1}. Hence $E_s \leq_{\text{lex}} E_{s+1}$. The left-r.e. set $E = \lim E_s$ contains all elements of A and every second element of the complement of A, hence A is not maximal in the structure of the left-r.e. sets under inclusion. $\square$

A further interesting question is the following: For maximal r.e. sets C one has the property that there is no r.e. set E with $E - C$ and $\overline{E} - C$ being infinite [37, p. 187]. Do the corresponding properties also hold for minimal and maximal left-r.e. sets? That is, can one make sure that no left-r.e. set splits a minimal left-r.e. set A into two infinite parts or the complement of a maximal left-r.e. set B into two infinite parts? The answer is "no".

Theorem 6.4. *Let A be an infinite left-r.e. set and B be a coinfinite left-r.e. set. Then there is an infinite left-r.e. set E such that $A \cap E$ and $A \cap \overline{E}$ are both infinite. Furthermore there is an infinite left-r.e. set F such that $\overline{B} \cap F$ and $\overline{B} \cap \overline{F}$ are both infinite.*

Proof. Assume by way of contradiction that A and B exist. Then the set of even number neither splits A nor the complement of B into two infinite halves; therefore without loss of generality, all members of A are odd and all non-members of B are odd.

Let $A = \{a_0, a_1, a_2, \ldots\}$ and $\overline{B} = \{b_0, b_1, b_2, \ldots\}$ be denoted such that $a_k < a_{k+1}$ and $b_k < b_{k+1}$ for all k. Now choose E and F such that

$$\begin{aligned} E &= \{a_{2k}, a_{2k+1} - 1 : k \in \mathbb{N}\} \text{ and} \\ \overline{F} &= \{b_{2k}, b_{2k+1} - 1 : k \in \mathbb{N}\}. \end{aligned}$$

One can obtain corresponding approximations E_s and F_s for E and F, respectively, by using analogous formulas to define E_s from A_s and $\overline{F_s}$ from $\overline{B_s}$. Fix left-r.e. approximations A_s to A with $A_s(2x) = 0$ for all x and B_s to B with $B_s(2x) = 1$ for all x. Then $A_s \leq_{\text{lex}} A_{s+1} \Rightarrow E_s \leq_{\text{lex}} E_{s+1}$ and $B_s \leq_{\text{lex}} B_{s+1} \Rightarrow F_s \leq_{\text{lex}} F_{s+1}$. Hence both sets E and F are left-r.e. sets. Furthermore, $A \cap E = \{a_0, a_2, a_4, \ldots\}$, $A \cap \overline{E} = \{a_1, a_3, a_5, \ldots\}$, $\overline{B} \cap F = \{b_1, b_3, b_5, \ldots\}$ and $\overline{B} \cap \overline{F} = \{b_0, b_2, b_4, \ldots\}$. Hence E and F meet the requirements. □

Having established the fundamentals on minimal and maximal left-r.e. sets, the time is ready for the question which of them can be made into themselves.

Theorem 6.5. *There is a minimal left-r.e. set A such that $\{A\}$ can be made into itself. There is no maximal left-r.e. set B such that $\{B\}$ can be made into itself.*

Proof. One can easily see that the intervals I_n in Theorem 6.2 can be chosen large enough so that $a_n \neq \max(I_n)$ for all n; hence $A = \{a_0, a_1, \ldots\}$ is disjoint from an infinite recursive set and so $\{A\}$ can be made into itself by Theorem 4.1.

Assume now that B is a maximal left-r.e. set; one has to show that there is no infinite recursive set R disjoint from B. Assume the contrary and without loss of generality $R \cup B$ is coinfinite (otherwise B is the complement of a recursive set and not maximal). Let $B_0, B_1, \ldots$ be a left-r.e. approximation of B. Now one can select a sequence $s_0, s_1, \ldots$ of stages such that $B_{s_t} \cap \{0, 1, \ldots, t\}$ is disjoint from R. Hence $E_t = (B_{s_t} \cap \{0, 1, \ldots, t\}) \cup R$ is a recursive left-r.e. approximation of $B \cup R$ which then witnesses that B was not, as assumed, a maximal left-r.e. set. Hence there is no infinite recursive set disjoint to B and, by Theorem 4.1, $\{B\}$ cannot be made into itself. □

The next result shows that each of the classes of minimal left-r.e. sets and maximal left-r.e. sets cannot be made into itself; the proof method is to show that the corresponding index-sets cannot be K'-recursive and therefore cannot be left-r.e., let alone minimal or maximal.

Theorem 6.6. *Neither the class of minimal left-r.e. sets nor the class of maximal left-r.e. sets can be made into itself.*

Proof. Let A be the minimal and B be the maximal left-r.e. set from Theorem 6.2. Recall that $I_1, I_2, \ldots$ is a recursive partition of the natural numbers such that A has exactly one element in I_n for each n. Let $\mathrm{ind}(x) = n$ for the unique n with $x \in I_n$; the function ind is recursive. We show that with respect to any universal left-r.e. numbering α, neither the minimal nor the maximal left-r.e. sets can be made into itself.

Let P be the index set of the minimal left-r.e. sets in α. Now consider for any r.e. set W_e the set $\tilde{A}_e$ given as

$$\{3x : x \in A \wedge \mathrm{ind}(x) \in W_e\} \cup \{3x+1, 3x+2 : x \in A \wedge \mathrm{ind}(x) \notin W_e\}.$$

One can easily see that $\tilde{A}_e$ has a left-r.e. approximation; starting with a left-r.e. approximation A_s for A and an enumeration $W_{e,s}$ for W_e, the approximation $\tilde{A}_{e,s}$ is the same as for $\tilde{A}_e$ except A is replaced with A_s and W_e is replaced with $W_{e,s}$.

If W_e is cofinite then the set $\tilde{A}_e$ is a finite variant of $\{3x : x \in A\}$ and thus minimal; if W_e is coinfinite then the set $\tilde{A}_e$ has an infinite left-r.e. subset which has infinitely many less elements than $\tilde{A}_e$, namely

$$\{3x : x \in A \wedge \mathrm{ind}(x) \in W_e\} \cup \{3x+1 : x \in A \wedge \mathrm{ind}(x) \notin W_e\}.$$

There is a K'-recursive mapping which determines for every e the least index d with $\alpha_d = \tilde{A}_e$; now $d \in P$ iff W_e is cofinite. As the set $\{e : W_e \text{ is cofinite}\}$ is not K'-recursive in any acceptable numbering of the r.e. sets [37, Corollary IV.3.5], P cannot be K'-recursive and therefore is not a minimal left-r.e. set.

Now let Q be the index set of the maximal left-r.e. sets in the given enumeration α. Recall that B is a fixed maximal left-r.e. set. Now each join $B \oplus W_e$ is left-r.e. and is a maximal left-r.e. set iff W_e is cofinite. Again there is a K'-recursive mapping which finds for each e an index d with $B \oplus W_e = \alpha_d$; hence one can, relative to K', many-one reduce the index set of the cofinite sets to Q. As the index set of the cofinite sets is not K'-recursive, Q also cannot be K'-recursive; hence Q cannot be left-r.e. and in particular is not a maximal left-r.e. set. $\square$

7 Inclusion

We now turn our attention to the question of which things can be stuck inside other things. Kummer [23] showed that there exists a numbering φ of the partial recursive sets such that the r.e. *inclusion problem*,

$$\mathrm{INC}_\varphi = \{\langle i, j\rangle : W_i^\varphi \subseteq W_j^\varphi\},$$

is recursive in the halting set and asked whether there exists a numbering φ of the partial recursive sets such that INC_φ is r.e. Kummer's question remains open, however in the context of left-r.e. sets we show the answer is negative. Below we use INC_α to denote the left-r.e. inclusion problem.

Theorem 7.1. *For every universal left-r.e. numbering α,*

(I) INC_α *is not r.e. and*

(II) $\mathrm{INC}_\alpha \geq_{\mathrm{T}} K$.

Proof. For part (I), define the following two sets:

$$\begin{aligned} A &= \text{ the set of odd numbers,} \\ B &= \{2x : x \in K\} \cup \{2x+1 : x \notin K\}. \end{aligned}$$

Note that $A \cap B = \{2x+1 : x \notin K\}$ and that A and B are both left-r.e.: the characteristic function of B on $2x, 2x+1$ changes from 01 to 10 whenever x goes into K, hence this is a left-r.e. process.

Let α be a universal left-r.e. numbering and suppose that INC_α were r.e. For each number x, we show how to decide membership in the set $\{y \in K : y < x\}$. We search for a left-r.e. set E and a number s such that the following has happened up to stage s:

- The indices for $E \subseteq A$ and $E \subseteq B$ have both been enumerated into the inclusion problem;
- for all $y < x$, either $y \in K$ or $2y+1 \in E_s$ but not both.

Note that E cannot acquire any further element $2z+1 < 2x$ after stage s as then $2z+1 \in B$ which implies $z \notin K$, contrary to the second item above. Hence E does not change below $2x$ after stage s and therefore one knows for all $y < x$ that $y \in K$ iff $y \in E_s$. An α-index for such a set E exists as every finite set has an index in α, and therefore our search terminates. The recursive algorithm just described thus decides the halting problem, which is impossible.

For part (II), note that instead of searching for enumerations of the inclusion problem, one can run the above algorithm relative to the inclusion problem and so show that K is Turing reducible to the inclusion problem with that algorithm. $\square$

We leave the following open questions for the left-r.e. inclusion problem:

Question 7.2. *Does there exist a numbering α for the left-r.e. sets such that $\mathrm{INC}_\alpha \equiv_\mathrm{T} K$? In particular, can we make INC_α to be left-r.e.?*

Consider the related relation

$$\mathrm{LEX}_\alpha = \{\langle i, j\rangle : \alpha_i \leq_{\mathrm{lex}} \alpha_j\}.$$

Any Friedberg numbering α makes LEX_α recursive in the halting set. The reason is that no two distinct indices in a Friedberg numbering represent the same left-r.e. set, so a halting set oracle suffices to find a sufficiently long prefix which reveals the lexicographical order of the strings. We can improve this result to a numbering such that the left-r.e. relation itself becomes left-r.e.

Theorem 7.3. *There exists a universal left-r.e. numbering α such that LEX_α is an r.e. relation.*

Proof. Let β be a Friedberg left-r.e. numbering which includes indices for all the left-r.e. sets except for $\mathbb{N}$. We define a universal left-r.e. numbering α based on β as follows. Informally, during the first s stages, α follows the first s indices of β for s computation steps, and some finitely many other α-indices e have been defined to be $\alpha_e = \mathbb{N}$. If $\alpha_e = \mathbb{N}$, we say that the index e has been *obliterated*. We describe stage $s+1$. For each pair $\langle i, j\rangle$ with $i < j$ where β_i becomes lexicographically larger than β_j at stage $s+1$, that is, $\beta_{i,s} \leq_{\mathrm{lex}} \beta_{j,s}$ but $\beta_{i,s+1} >_{\mathrm{lex}} \beta_{j,s+1}$, the index for the α-follower of β_j and all larger defined α-indices are obliterated and a new α-follower for β_j and each of the other newly obliterated indices is established. Also in stage $s+1$, an α-follower for β_{s+1} is established so that in the end each β-index will have a unique α-index following it. Note that only finitely many α-indices are defined in any given stage.

For every e, the α-index following β_e eventually converges once sufficiently much time has passed to allow the approximation of β_e's prefix to differ from the approximation of every lesser β-index's prefix and also enough time that these prefixes never again change. Furthermore, obliterating indices can only ever increase membership of the respective set, so α is a universal left-r.e. numbering. Finally, α is r.e. because whenever β's enumeration tries to push $\langle i, j \rangle$ out of LEX_α, the index j gets obliterated and hence $\langle i, j \rangle$ stays inside LEX_α. $\square$

Summaries

自我循环的"家伙". 我们研究什么样的递归枚举类有编号使得它的左递归枚举元素所组成的指标集在这种编号中是这个类本身的一个元素。 例如， Martin-Löf 随机集可以成为自己的一个元素。 而且,我们刻划在等号下的指标集等于自身的左递归枚举集。 我们发现，存在相对于包含的有限差的极大和极小的左递归枚举集； 尽管有些极小左递归枚举集可以成为自身的一员， 但是极大左递归枚举集却不能。 另外,我们证明左递归枚举集的包含问题不可能是递归的,这样就解决了库默尔关于递归集的问题的一种形式.

Aĵoj kiojn oni povas meti en si mem. Aro A estas rekursive enumerabla se A estas la limo de uniforme rekursivaj aroj $A_0, A_1, \ldots$ je kiuj $A_n \subseteq A_{n+1}$ por ĉiu n; A estas maldekstre rekursive enumerabla se A estas la limo de uniforme rekursivaj aroj $A_0, A_1, \ldots$ je kiuj $A_n \leq_{lex} A_{n+1}$ por ĉiu n. La publikaĵo temas pri la sekvanta afero: Se $\alpha_0, \alpha_1, \ldots$ estas numerado da maldekstre rekursive enunmerablaj aroj kaj se P estas abstrakta eco de aroj (kiel esti Martin-Löf hazarda), tiam oni konsideru la indeksa aro $\{e : \alpha_e$ havas econ $P\}$. Oni diras ke oni povas meti la P en si mem se ekzistas numerado $\alpha_0, \alpha_1, \ldots$ de ĉiuj maldekstre rekursive enumerablaj aroj tiel ke la indeksa aro por P je tiu numerado ankaŭ havas la econ P. En tiu-ĉi publikaĵo estas diversaj teoremoj kiuj diras je multaj famaj ecoj el teorioj pri rekursivaj funkcioj kaj algoritmika hazardo se oni povas meti tiujn ecojn en si mem. Ekzemple, oni povas meti la Martin-Löf hazarda arojn en si mem. Plue, se la aro A havas minimume unu membron kaj estas maldekstre rekursive enumerabla, tiam oni povas meti la econ $P(X)$ dirante $X = A$ en si mem ekzakte se ekzistas malfinia rekursive enumerebla aro B kiu havas malplenan komunaĵon kun A. Oni ankaŭ esploras pri minimumaj kaj maksimumaj aroj en la strukturo de maldekstre rekursive enumerablaj aroj je la ordo $\subseteq^*$. Kvankam en la

mondo de rekursive enumerablaj aroj la minimuma aroj ne ekzistas, ambaŭ ekzistas en la mondo de maldekstre rekursive enumerablaj aroj kaj la pruvo malsimilas al tiu de Friedberg por la mondo de rekursive enumerablaj aroj.

Dinge die in sich selbst gemacht werden können. Eine Menge A natürlicher Zahlen heisst rekursiv aufzählbar (r.a.) genau dann wenn es eine uniform-rekursive Folge $A_0, A_1, \ldots$ gibt welche punktweise gegen A konvergiert und $A_n \subseteq A_{n+1}$ für alle n erfüllt; A heisst links-r.a. genau dann wenn es eine uniform-rekursive Folge $A_0, A_1, \ldots$ gibt welche punktweise gegen A konvergiert und $A_n \leq_{lex} A_{n+1}$ für alle n erfüllt. Das Thema der Arbeit ist der folgende Selbstbezug: Man sagt dass eine Eigenschaft P von Mengen natürlicher Zahlen in sich selbst gemacht werden kann wenn es eine Numerierung $\alpha_0, \alpha_1, \ldots$ aller links-r.a. Mengen gibt so dass die Index-Menge $\{e : \alpha_e$ hat die Eingenschaft $P\}$ ebenfalls die Eigenschaft P hat. Es wird untersucht, welche bekannten rekursions-theoretischen Eigenschaften diese Art von Selbstbezug haben, zum Beispiel hat die Eigenschaft "Martin-Löf zufällig" einen solchen Selbstbezug. Man kann auch die Eigenschaft P betrachten wo $P(X)$ bedeutet dass $X = A$ ist für eine feste gegebene nichtleere links-r.a. Menge A. Nun hat P die obenerwähnte Art von Selbstbezug genau dann wenn A zu einer unendlichen rekursiv aufzählbaren Menge B disjunkt ist. Desweiteren wurde die Struktur der links-r.a. Mengen mit der partiellen Ordnung $\subseteq^*$ untersucht. Es wird gezeigt dass es in dieser Struktur, anders als im Fall der r.a. Mengen, nicht nur maximale sondern auch minimale links-r.a. Mengen gibt; die Konstruktion ist recht unterschiedlich von der Konstruktion welche Friedberg im r.a. Fall benutzte. Desweiteren werden die Selbstbezugseigenschaften von minimalen und maximalen links-r.a. Mengen untersucht.

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