ANTI-COMPLEX SETS AND REDUCIBILITIES WITH TINY USE

JOHANNA N. Y. FRANKLIN, NOAM GREENBERG, FRANK STEPHAN, AND GUOHUA WU

ABSTRACT. In contrast with the notion of complexity, a set A is called anti-complex if the Kolmogorov complexity of the initial segments of A chosen by a recursive function is always bounded by the identity function. We show that, as for complexity, the natural arena for examining anti-complexity is the weak-truth table degrees. In this context, we show the equivalence of anti-complexity and other lowness notions such as r.e. traceability or being weak truth-table reducible to a Schnorr trivial set. A set A is anti-complex if and only if it is reducible to another set B with *tiny use*, whereby we mean that the use function for reducing A to B can be made to grow arbitrarily slowly, as gauged by unbounded nondecreasing recursive functions. This notion of reducibility is then studied in its own right, and we also investigate its range and the range of its uniform counterpart.

1. Introduction

In a recent talk [25], Nies gave a general framework for relating lowness notions and their dual highness notions with what he names "weak reducibilities" (with a prominent example being $\leqslant_{LR}$, the low-for-randomness partial ordering). Even before their extensive investigation in the context of effective randomness, in classical recursion theory, both strong and weak reducibilities gave rise to lowness and highness classes. For example, truth-table (or weak truth-table) reducibility induced the classes of superlow and superhigh Turing degrees; in the other extreme, hyperarithmetic reducibility (and the hyperjump) allowed the definition of the useful class of hyperlow hyperdegrees (see [30]). In this paper we give a new notion of relative strength which, surprisingly, leads to a lowness notion which is analogous to familiar ones in the context of the weak truth-table degrees.

The motivation for our notion, which we call "Turing reducibility with tiny use", comes from recent investigations into strengthenings of weak truth-table reducibility in a direction which is incomparable with truth-table reducibility, namely *computable Lipshitz* reducibility $\leqslant_{cl}$ (also known as $\leqslant_{sw}$, *strong weak truth-table* reducibility) and *identity-bounded* Turing reducibility $\leqslant_{ibT}$, and also, to a smaller extent, related reducibilities such as $\leqslant_C$ and $\leqslant_H$. These reducibilities were introduced in order to combine the traditional Turing reduction, that is, computation of the membership relation using an oracle, and calibration of relative randomness, usually on the domain of left-r.e. reals (see [2]).

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Recall that computable Lipschitz reductions are weak truth-table reductions in which the use function is bounded by $n+c$ for some constant c. The idea of a Turing reduction with tiny use is to further restrict the use function of the reduction to recursive functions which grow more and more slowly. A set A is reducible to a set B with tiny use if one can use arbitrarily little of the oracle B to compute arbitrarily much of A, so not only does B contain all the information that A has, it compresses that information arbitrarily well. To make the definition precise, we invoke the following definition first made by Schnorr [31]: an *order function* (or simply an *order*) is a recursive function which is nondecreasing and unbounded.

Definition 1.1. Let $A, B \in \{0,1\}^\omega$. We say that A is *reducible to B with tiny use* and write $A \leqslant_{T(tu)} B$ if for every order function h, there is a Turing reduction of A to B whose use function is bounded by h.

Let us agree on some notation and terminology. If Φ is a Turing reduction (a Turing machine with an oracle) and $\Phi^B = A$, then we let, for every $n < \omega$, the *use* of this reduction, $\varphi(n) = \varphi^B(n)$, be $m+1$, where m is the largest number which is queried by Φ while computing $A \upharpoonright n$. Here $A \upharpoonright n$ is the string $A(0)A(1)\dots A(n-1)$, and we assume that before computing $A(n) = \Phi^B(n)$, Φ first computes $\Phi^B(m)$ for all $m < n$. Thus $B \upharpoonright \varphi(n)$ is the shortest initial segment of B which, serving as an oracle for Φ, outputs $A \upharpoonright n$.

The motivation for considering reducibility with tiny use comes from a result of Greenberg and Nies [12], who showed that if A is a recursively enumerable, strongly jump-traceable set and B is an ω-r.e. random set, then A is reducible to B with tiny use. In fact, in [13] it is shown that this is a characterisation of the strongly jump-traceable r.e. sets.

We note here that this reducibility is unlike the more standard Turing reducibility (and, in fact, all reducibilities mentioned thus far) in that its domain and range are nontrivial. For instance, we will see that it is not the case that every set A is reducible to some set B with tiny use. Therefore, we will often speak of the domain and range of $\leqslant_{T(tu)}$. Furthermore, this reducibility is not reflexive, which suggests that the notation $<_{T(tu)}$ would be more appropriate. However, since the reducibility is not irreflexive, we will keep the notation $\leqslant_{T(tu)}$ and simply note for the reader that the only sets reducible to themselves with tiny use are the recursive sets (Proposition 2.8). When the context excludes the possibility of A or B being equal, we will write $A <_{T(tu)} B$.

The relation $\leqslant_{T(tu)}$ yields a lowness notion in a very simple way: we consider the domain of the relation, i.e. the collection of sets A for which there is some B such that $A \leqslant_{T(tu)} B$. An immediate analysis of $\leqslant_{T(tu)}$ shows that this collection is invariant in the weak truth-table degrees and induces an ideal in these degrees. This ideal can be characterised in three other ways, for which we make a sequence of definitions.

Recall that for their work characterising lowness for Schnorr randomness as recursive traceability, extending a fundamental result of Terwijn and Zambella [33], Kjos-Hanssen, Merkle and Stephan [16] defined a set A to be *complex* if there is an order function f such that $C(A \upharpoonright f(n)) \geqslant n$ for all n (here C denotes plain Kolmogorov complexity[1]). They showed that a set A is complex if and only if there

[1]Recall that a *machine* is a partial recursive function $M\colon \{0,1\}^* \to \{0,1\}^*$. If M is a machine, then the M-complexity of a string σ in the range of M, denoted by $C_M(\sigma)$, is the length of the

is some diagonally nonrecursive function $f \leqslant_{wtt} A$. As an analogue, we make the following definition:

Definition 1.2. A set $A \in \{0,1\}^\omega$ is *anti-complex* if for every order function f, for almost all n, $C(A \upharpoonright f(n)) \leqslant n$.

Thus anti-complexity is a mirror image of complexity: complexity indicates incompressibility in that one can effectively find locations of high complexity, whereas anti-complexity denotes a high level of compressibility and hence low information content.

Traceability, in both its recursive and r.e. versions, is a notion which has turned out to be extremely useful in algorithmic randomness and classical recursion theory. Recent work of Franklin and Stephan [11] has indicated that it is also useful in the context of strong reducibilities. They have shown that the class of Schnorr trivial sets is invariant in the truth table degrees and that a set is Schnorr trivial if and only if its truth-table degree is recursively traceable (this means that only the functions which are truth-table reducible to the set receive a recursive trace, all with a uniform bound of some order). Since the natural environment for $\leqslant_{T(tu)}$ is the weak truth-table degrees, we find that traceability in these degrees plays a role here. The characterisation theorem is as follows.

Theorem 1.3. *The following are equivalent for a set A.*

(1) *There is a set B such that $A \leqslant_{T(tu)} B$.*
(2) *A is anti-complex.*
(3) $\deg_{wtt}(A)$ *is r.e. traceable.*
(4) *A is weak truth-table reducible to a Schnorr trivial set.*

We note that the equivalence of (3) and (4), together with Franklin and Stephan's result, yields a theorem which has no explicit connection to effective randomness, and yet we currently do not know of any direct proof that does not involve $\leqslant_{T(tu)}$ and Kolmogorov complexity: a weak truth-table degree $\mathbf{a}$ is r.e. traceable if and only if there is some weak truth-table degree $\mathbf{b} \geqslant \mathbf{a}$ which contains a set B whose truth-table degree is recursively traceable.

As the existence of an order function witnessing r.e. traceability implies that every order function is such a witness (see Lemma 3.3), it follows that a Turing degree $\mathbf{a}$ is r.e. traceable if and only if every weak truth-table degree contained in $\mathbf{a}$ is r.e. traceable. Theorem 1.3 then implies the following characterisation of r.e. traceability in the Turing degrees.

Theorem 1.4. *The following are equivalent for a Turing degree* $\mathbf{a}$.

(1) $\mathbf{a}$ *is r.e. traceable.*
(2) *Every set $A \in \mathbf{a}$ is anti-complex.*
(3) *Every set $A \in \mathbf{a}$ is weak truth-table reducible to a Schnorr trivial set.*

Among r.e. degrees, we note that the equivalence between array recursiveness and r.e. traceability holds in the weak truth-table degrees. Recall that a *very strong array* $\bar{F} = \langle F_n \rangle_{n<\omega}$ consists of a recursive sequence of pairwise disjoint finite sets such that for all n, $|F_{n+1}| > |F_n|$, and that an r.e. set A is $\bar{F}$-ANR if for every r.e.

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shortest string $\tau \in M^{-1}\{\sigma\}$. If σ is not in the range of M, then we write $C_M(\sigma) = \infty$. A machine U is *optimal* if for every machine M there is some constant c such that for all $\sigma \in \{0,1\}^*$, $C_U(\sigma) \leqslant C_M(\sigma) + c$. Then C denotes C_U for some fixed optimal machine U.

set B there are infinitely many n such that A and B coincide on F_n. Most of the equivalences in the following theorem are known, but we prove the equivalence of (4) and (5) in Section 4.

Theorem 1.5. *The following are equivalent for a weak truth-table degree* $\mathbf{a}$ *containing an r.e. set.*

1. *For no very strong array* $\bar{F}$ *does* $\mathbf{a}$ *contains an* $\bar{F}$*-ANR set.*
2. *For some very strong array* $\bar{F}$*,* $\mathbf{a}$ *contains no* $\bar{F}$*-ANR set.*
3. *There is an* ω*-r.e. function that dominates all functions recursive in* $\mathbf{a}$.
4. $\mathbf{a}$ *is r.e. traceable.*
5. *For all* $A \in \mathbf{a}$*,* $A <_{T(tu)} K$ *(here* $K = \{e : \varphi_e(e)\downarrow\}$ *is the halting set).*

This result implies the result from [7] that the array recursive r.e. wtt-degrees form an ideal.

Together with $\leqslant_{T(tu)}$, we also investigate a uniform version $\leqslant_{uT(tu)}$, where a single reduction witnesses the relation $\leqslant_{T(tu)}$. This relation is, in general, much stronger than $\leqslant_{T(tu)}$ (for example, if A is nonrecursive and $A \leqslant_{uT(tu)} B$, then B is high, which we show does not hold for $\leqslant_{T(tu)}$), but their domains are the same, and so the condition "there is a set B such that $A \leqslant_{uT(tu)} B$" can be added as a fifth equivalent condition in Theorem 1.3. An even stronger version of this theorem which bounds the complexity of such B is Theorem 3.8. We prove Theorem 1.3 in Section 3. In Section 4 we investigate the distribution of the anti-complex sets in the Turing degrees, discuss high and random degrees, prove Theorems 1.4 and 1.5, and investigate anti-complexity and tiny use in the r.e. degrees. One corollary of our investigations is an answer to Question 7.5.13 from Nies's book [23].

Theorem 1.6. *There is a high Turing degree which does not contain a partial-recursively random set.*

The motivation behind this question is to find an exact boundary between weaker notions of randomness, such as Schnorr randomness and recursive randomness, which occur in every high Turing degree, and stronger notions of randomness, such as Martin-Löf randomness, which do not. We provide a proof of Theorem 1.6 in Section 4.

In Section 5, we investigate the dual highness notions: the sets B for which there is a nonrecursive set A such that $A <_{T(tu)} B$ (or the more stringent $A <_{uT(tu)} B$). We investigate the situation in both the hyperimmune-free ($\mathbf{0}$-dominated) degrees and in the r.e. and Δ^0_2 degrees. For example, we show that every high Turing degree contains sets A and B such that $A <_{uT(tu)} B$ and that for every nonrecursive r.e. set B there is some nonrecursive r.e. set A such that $A <_{T(tu)} B$.

Throughout the paper, we also mention strong reducibilities (such as truth-table and many-one) with tiny use. In particular, in Theorem 3.11 we use truth-table reducibility with tiny use to obtain a new characterisation of Schnorr triviality: a set A is Schnorr trivial if and only if it is truth-table reducible to some set B with tiny use. This result strengthens the intuition, arising from Franklin and Stephan's characterisation of Schnorr triviality in terms of recursive traceability in the truth-table degrees, that strong reducibilities have deep connections with weak randomness notions. Along this vein, Day [1] has recently given characterisations of both Schnorr randomness and computable randomness as the complements of the domains of relations weaker than truth-table reducibility with tiny use. For example, he showed that a set A is not Schnorr random if and only if there is some set B such that $A \leqslant_{tt} B$ with use function which does not dominate $n - h(n)$ for some order function h.

In the following section we supply the rest of the basic definitions and make some basic observations.

2. Basics

We first define the uniform reducibility.

Definition 2.1. Let $A, B \in \{0,1\}^\omega$. We say that A is *uniformly reducible to B with tiny use* (and write $A \leqslant_{uT(tu)} B$) if there is a Turing reduction $\Phi^B = A$ whose use function is dominated by every order function.

Observation 2.2.

(1) *If $A \leqslant_{uT(tu)} B$, then $A \leqslant_{T(tu)} B$.*
(2) *If $A \leqslant_{T(tu)} B$, then $A \leqslant_{wtt} B$.*

Remark 2.3. Despite the fact that our reductions imply weak truth-table reductions, we prefer the notation $\leqslant_{T(tu)}$ to $\leqslant_{wtt(tu)}$. This is because a weak truth-table reduction first marks the use, then queries the oracle and finally computes the value, whereas Turing reductions with tiny use would — at least in the uniform case — not do the operations in this order, as otherwise the use is automatically bounded from below by an order function.

Next, we see that our relations are invariant in the wtt-degrees.

Observation 2.4. *If $A \leqslant_{wtt} E$ and $E \leqslant_{T(tu)} B$, then $A \leqslant_{T(tu)} B$; if $A \leqslant_{T(tu)} E$ and $E \leqslant_{wtt} B$, then $A \leqslant_{T(tu)} B$. Thus the relation $\leqslant_{T(tu)}$ is invariant on weak truth-table degrees and is preserved by increasing the degree on the range and decreasing the degree on the domain. The same holds for $\leqslant_{uT(tu)}$.*

Observation 2.5. *For a fixed $B \in \{0,1\}^\omega$, the classes $\{A : A \leqslant_{T(tu)} B\}$ and $\{A : A \leqslant_{uT(tu)} B\}$ are wtt-ideals.*

Another formulation for our notions uses not the use functions but their discrete inverses. If $\Phi^B = A$ is a Turing reduction, then for every $n < \omega$ we let $\Phi(B \upharpoonright n)$ be the longest initial segment of A which is calculated by Φ querying the oracle B only on numbers smaller than n.

In general, if $f\colon \omega \to \omega$ is a nondecreasing and unbounded function but not necessarily recursive, we let f^{-1}, the *discrete inverse* of f, be defined by letting $f^{-1}(k)$ be the greatest n such that $f(n) \leqslant k$ (let us assume that $f(0) = 0$, as it is for every use function, so f^{-1} is total; otherwise f^{-1} is defined for almost all numbers). That is, if $f(n+1) > f(n)$, then the interval $[f(n), f(n+1))$ gets mapped by f^{-1} to n. We note that if f is recursive (and is thus an order function), then so is f^{-1}.

According to this definition, if $\Phi^B = A$ with use φ, then for all n, $\Phi(B \upharpoonright n) = A \upharpoonright \varphi^{-1}(n)$.

Observation 2.6. *Let f and g be nondecreasing and unbounded.*

(1) *If f bounds g, then g^{-1} bounds f^{-1}.*
(2) *$\left(f^{-1}\right)^{-1}$ bounds f. If f grows more slowly than the identity function, that is, if for all n, $f(n+1) \leqslant f(n) + 1$, then $\left(f^{-1}\right)^{-1} = f$.*

This observation suffices for the following corollary, noting that when investigating slow-growing recursive orders, we may assume that the orders grow slower than the identity function.

Corollary 2.7. *Let $A, B \in \{0,1\}^\omega$.*

(1) *$A \leqslant_{T(tu)} B$ if and only if for every order function g, there is a Turing reduction $\Phi^B = A$ such that the map $n \mapsto |\Phi(B \upharpoonright n)|$ bounds g.*
(2) *$A \leqslant_{uT(tu)} B$ if and only if there is a Turing reduction $\Phi^B = A$ such that the map $n \mapsto |\Phi(B \upharpoonright n)|$ dominates every recursive function. (A function which dominates every recursive function is called* dominant*.)*

Some other basic results follow.

Proposition 2.8. *Let $A, B \in \{0,1\}^\omega$.*

(1) *If $A \leqslant_{T(tu)} A$, then A is recursive.*
(2) *If A is recursive, then $A \leqslant_{uT(tu)} B$.*

Proof. Let $f(n) = n+1$. If $\Phi^A = A$ and for all n we have $\Phi(A \upharpoonright n) \supseteq A \upharpoonright n+1$, then we can recursively compute $A(n)$ by applying Φ to $A \upharpoonright n$, which we already computed. For (2), use a reduction $\Phi^B = A$ whose use function is a constant 0. $\square$

Corollary 2.9. *If $A \leqslant_{T(tu)} B$ and A is nonrecursive, then $\deg_{wtt}(A) < \deg_{wtt}(B)$.*

As a result, if $\deg_{wtt}(B)$ is minimal, then every $A \leqslant_{T(tu)} B$ is recursive.

Proposition 2.10. *Let $B \in \{0,1\}^\omega$. If there is some nonrecursive A such that $A \leqslant_{uT(tu)} B$, then B is high.*

Recall that a set B is high if $B' \geqslant_T \emptyset''$.

Proof. For any Turing reduction, if Φ^B is total, then the map $n \mapsto |\Phi(B \upharpoonright n)|$ is computable in B (indeed, weak truth-table reducible to B). The map Φ which witnesses $A \leqslant_{uT(tu)} B$ dominates every recursive function. By Martin [21], this map has high Turing degree. $\square$

We will review the situation in Proposition 2.10 in greater detail in Section 5.

3. Sets bounded by other sets with tiny use

In this section we prove Theorem 1.3. It will follow from Theorem 3.8 and Propositions 3.4, 3.9 and 3.10.

3.1. **Anti-complexity and traceability.**

For functions $f, g\colon \omega \to \omega$, we write $f \leqslant^+ g$ if there is some constant c such that $g + c$ bounds f.

Lemma 3.1. *A set A is anti-complex if and only if for every $f \leqslant_{wtt} A$,*

$$C(f(n)) \leqslant^+ n.$$

This lemma shows that the notion of anti-complexity (like its analogue notion, complexity) is wtt-degree invariant.

Proof. We first note that A is anti-complex if and only if for every order function f, $C(A \upharpoonright f(n)) \leqslant^+ n$. One direction is immediate from Definition 1.2. For the other direction, suppose that for every order function f, $C(A \upharpoonright f(n)) \leqslant^+ n$. Let f be an order function. Applying the hypothesis twice to the functions $n \mapsto f(2n)$ and $n \mapsto f(2n+1)$, there is a constant c such that for all n, $C(A \upharpoonright f(2n)) \leqslant n + c$

and $C(A \upharpoonright f(2n+1)) \leqslant n + c$. If $n \geqslant c$, then $C(A \upharpoonright f(2n))$ and $C(A \upharpoonright f(2n+1))$ are less than or equal to $2n$, so Definition 1.2 holds.

Assume that for every $g \leqslant_{wtt} A$, $C(g(n)) \leqslant^+ n$. Let f be an order function and let $g(n)$ be a natural number code for $A \upharpoonright f(n)$. Then $g \leqslant_{wtt} A$, so as we just observed, A is anti-complex.

Now assume that A is anti-complex. Let $f \leqslant_{wtt} A$ and let g be a recursive bound for the use function for the reduction of f to A. Using this reduction, we see that $C(f(n)) \leqslant^+ C(A \upharpoonright g(n))$. Again, as we just observed, $C(A \upharpoonright g(n)) \leqslant^+ n$. $\square$

We show that anti-complexity can also be characterised as a weak truth-table analogue of a very useful concept in the Turing degrees, that of r.e. traceability. Recall that an *r.e. trace* for a function f is a uniformly recursively enumerable sequence $\langle T_n \rangle$ of finite sets such that for all n, $f(n) \in T_n$, and that a trace $\langle T_n \rangle$ is *bounded* by an order function h if the function $n \mapsto |T_n|$ is bounded by h.

Definition 3.2. A weak truth-table degree $\mathbf{a} \in \mathcal{D}_{wtt}$ is *r.e. traceable* if there is an order function h such that every $f \leqslant_{wtt} \mathbf{a}$ has an r.e. trace which is bounded by h.

The standard argument of Terwijn and Zambella [33] shows that the choice of order doesn't matter:

Lemma 3.3. *A weak truth-table degree* $\mathbf{a}$ *is r.e. traceable if and only if for every order function* h*, every* $f \leqslant_{wtt} \mathbf{a}$ *has an r.e. trace which is bounded by* h*.*

Proof. Suppose that h is an order function which witnesses that a weak truth-table degree $\mathbf{a}$ is r.e. traceable. Let $\hat{h}$ be any other order function and let $f \leqslant_{wtt} \mathbf{a}$.

Let $g(n)$ be the least k such that $\hat{h}(k) \geqslant h(n)$. This function is well defined because $\hat{h}$ is unbounded and is recursive. Hence the map $n \mapsto f \upharpoonright (g(n+1))$ is weak truth-table below $\mathbf{a}$, and so it has a trace $\langle T_n \rangle$ which is bounded by h.

The function g is unbounded because h is unbounded. Let g^{-1} be the discrete inverse of g, so $g^{-1}(k)$ is the greatest n such that $h(n) \leqslant \hat{h}(k)$ (note that g^{-1} is defined on almost every number). Then $|T_{g^{-1}(k)}| \leqslant \hat{h}(k)$ and $g(g^{-1}(k)+1) > k$, so $f \upharpoonright l$ is an element of $T_{g^{-1}(k)}$ for some $l > k$. Hence we can let S_k be the collection of all values $\sigma(k)$ for all $\sigma \in T_{g^{-1}(k)}$ of length greater than k. Then $\langle S_n \rangle$ will be an r.e. trace for f which is bounded by $\hat{h}$. $\square$

Proposition 3.4. *A set* A *is anti-complex if and only if* $\deg_{wtt}(A)$ *is r.e. traceable.*

Proof. Suppose that A is anti-complex and let $f \leqslant_{wtt} A$. By Lemma 3.1, there is some constant c such that for all n, $C(f(n)) \leqslant n + c$. Then letting

$$T_n = \{y : C(y) \leqslant n + c\},$$

$\langle T_n \rangle$ is an r.e. trace for f and for all n, $|T_n| \leqslant 2^{n+c+1}$. Hence (by changing finitely many entries for every function), $\deg_{wtt}(A)$ is r.e. traceable, witnessed by the order function $h(n) = 2^{2n}$.

The other direction follows an idea of Kummer's, who showed that every array recursive r.e. Turing degree contains only sets of low complexity [18] (see also [4]). Suppose that $\deg_{wtt}(A)$ is r.e. traceable and let $f \leqslant_{wtt} A$. By Lemma 3.3, let $\langle T_n \rangle$ be an r.e. trace for f which is bounded by the order function $h(n) = n$. We can construct a machine M which on input σ, first computes $U(\sigma)$, interprets the result as a pair (n, m) and, if $m < n$, outputs the m^{th} element enumerated into T_n.

Then for all n, if $f(n)$ is the m^{th} element enumerated into T_n, then M shows that $C(f(n)) \leqslant^+ C(n,m)$.

Now the standard coding of pairs as numbers is polynomial; so there is some constant c such that for all n and all $m \leqslant n$, $\langle n,m\rangle \leqslant n^c$. For all x, the identity machine witnesses that $C(x) \leqslant^+ \log_2 x$. Hence for all n and all $m \leqslant n$,

$$C(n,m) \leqslant^+ \log_2(\langle n,m\rangle) \leqslant \log_2 n^c = c\log_2 n \leqslant^+ n.$$

Thus we see that the condition of Lemma 3.1 holds. □

Porism 3.5. *If A is anti-complex and $c > 1$ any rational constant, then for all $f \leqslant_{wtt} A$ it holds that $C(f(n)) \leqslant^+ c\log_2 n$.*

3.2. **Tiny use.** Given $A \in \{0,1\}^\omega$, the function $n \mapsto C(A \upharpoonright n)$ is far from monotone. Nevertheless, we are interested in some form of inverse, which is possible because $\lim_n C(A \upharpoonright n) = \infty$. We let $g_A(k)$ be the least n such that for all $m \geqslant n$, $C(A \upharpoonright m) > k$.

Observation 3.6. *For all $A \in \{0,1\}^\omega$, $g_A \leqslant_T A \oplus K$. As before, $K = \emptyset'$ is the halting problem.*

For any string x, we let x^* be the least element of $U^{-1}\{x\}$ (where U is the universal machine we use for plain complexity), so $C(x) = |x^*|$. We also let

$$A^* = \left\{\left(A \upharpoonright g^A(k)\right)^* \,:\, k < \omega\right\}.$$

Once again, we get $A^* \leqslant_T A \oplus K$.

Lemma 3.7. *For every $A \in \{0,1\}^\omega$, the map $k \mapsto (A \upharpoonright g_A(k))^*$ is bounded by some recursive function.*

Proof. There is a constant c such that for all $\tau \in \{0,1\}^*$, $C(\tau 0)$ and $C(\tau 1)$ are both less than or equal to $C(\tau) + c$ (consider the machine which on input σi, for $i = 0,1$, outputs $U(\sigma)i$).

For any $k < \omega$, let τ_k be a binary string and $i \in \{0,1\}$ be such that $A \upharpoonright g_a(k) = \tau_k i$. By the definition of $g_A(k)$, $C(\tau_k) \leqslant k$, and so $C(A \upharpoonright g_A(k)) \leqslant k + c$. Hence $(A \upharpoonright g_A(k))^* < 2^{k+c+1}$.

To ensure the last inequality, we need some agreement about the coding of strings by numbers. This coding is obtained by some ω-ordering of all binary strings; we order binary strings by length first. We let $|x|$ denote the length of the string identified with the number x, so for all x, $2^{|x|} \leqslant x < 2^{|x|+1}$. □

Theorem 3.8. *The following are equivalent for $A \in \{0,1\}^\omega$.*

1. *There is some set B such that $A \leqslant_{T(tu)} B$.*
2. *A is anti-complex.*
3. *g_A is dominant.*
4. *$A \leqslant_{uT(tu)} A^*$.*

We remark that we are not aware of a shorter proof of the equivalence of (2) and (3). This suggests that the study of the relation $\leqslant_{T(tu)}$ is important for the seemingly independent study of anti-complexity in the wtt-degrees.

Proof. (1) implies (2): Assume that $A \leqslant_{T(tu)} B$. For any functional Φ such that $\Phi^B = A$, for all n, $C(\Phi(B \upharpoonright n)) \leqslant^+ C(B \upharpoonright n)$. Also, for all x, $C(x) \leqslant^+ |x|$, so for all n, $C(\Phi(B \upharpoonright n)) \leqslant^+ n$. Suppose that $f \leqslant_{wtt} A$, so there is some functional Γ such that $\Gamma^A = f$ and the use of this computation is bounded by a recursive function g. We can find some Φ such that for all n, $\Phi(B \upharpoonright n)$ is longer than $A \upharpoonright g(n)$, so $C(f(n)) \leqslant^+ n$. By Lemma 3.1, A is anti-complex.

(2) implies (3): Suppose that A is anti-complex and let f be an increasing recursive function. By definition, for almost all n, $C(A \upharpoonright f(n)) \leqslant n$. Hence, for almost all n, $g_A(n) > f(n)$. It follows that g_A dominates every recursive function.

(3) implies (4): For every $A \in \{0,1\}^\omega$ we have $A \leqslant_T A^*$ because

$$A = \bigcup \{U(\sigma) \, : \, \sigma \in A^*\}$$

(in other words, $A(x) = U(\sigma)(x)$ for any $\sigma \in A^*$ such that $x < |U(\sigma)|$, and for every x there is indeed some $\sigma \in A^*$ such that $|U(\sigma)| > x$).

If g_A is dominant, then this reduction witnesses that $A \leqslant_{uT(tu)} A^*$. To see this, let Φ code the described reduction and let f be an increasing recursive function; we see that $n \mapsto |\Phi(A^* \upharpoonright n)|$ dominates f.

Let g be a recursive function which dominates $k \mapsto (A \upharpoonright g_A(k))^*$ (Lemma 3.7), and let n be given. Since g_A is dominant, for almost all k, $g_A(k) > f(g(k+1))$. Suppose that k is large enough that $(A \upharpoonright g_A(k))^* < n \leqslant (A \upharpoonright g_A(k+1))^*$. Then $n \leqslant g(k+1)$ and so $g_A(k) > f(n)$. Then $|\Phi(A^* \upharpoonright n)| \geqslant g_A(k)$ so $|\Phi(A^* \upharpoonright n)| \geqslant f(n)$ as required.

(4) implies (1): This is clear from the definitions. □

3.3. **Schnorr triviality.** Franklin and Stephan [11] characterise the Schnorr trivial sets (defined by Downey and Griffiths in [5]) as those sets whose *truth-table* degree is *recursively traceable*, that is, there is some order function h which bounds traces for all functions f truth-table reducible to the degree $\mathbf{a}$, but where the trace $\langle T_n \rangle$ is required to be given recursively (as a sequence of finite sets) rather than merely uniformly recursively enumerably. In other words, there is a recursive function g such that for all n, $g(n)$ is the canonical index for the finite set T_n (in Soare's [32] notation, $T_n = D_{g(n)}$). Again, the Terwijn-Zambella argument shows that any order would do.

Schnorr triviality is not invariant in the weak truth-table degrees [11, Theorem 4.2]. However, the downward closure of the wtt-degrees containing Schnorr trivial sets is familiar.

Proposition 3.9. *Every Schnorr trivial set is anti-complex.*

Proof. Let A be Schnorr trivial. Fix an order function h. Let Φ be a weak truth-table functional with a recursive bound g on the use function of Φ. Since the map $n \mapsto A \upharpoonright g(n)$ is truth-table reducible to A, by the characterisation mentioned above, there is a recursive trace $\langle T_n \rangle$ for this map which is bounded by h. If Φ^A is total, then we can enumerate a trace S_n for f with bound h by outputting $\Phi^\sigma(n)$ for those $\sigma \in T_n$ for which Φ^σ converges with domain greater than n. Hence $\deg_{wtt}(A)$ is r.e. traceable; by Proposition 3.4, A is anti-complex. □

Proposition 3.10. *Let $A \in \{0,1\}^\omega$. If g_A is dominant, then A is weak truth-table reducible to some Schnorr trivial set.*

Proof. Let $f_0, f_1, \ldots$ be a sequence of (total) recursive functions such that

- each f_i is strictly increasing,
- the range of f_{i+1} is contained in the range of f_i, and
- every recursive function is bounded by some f_i.

(Note that the halting problem K can compute such a sequence.)

By Lemma 3.7, let g be a recursive function which bounds the function $k \mapsto (A \upharpoonright g_A(k))^*$.

For each $k > 0$, let $q_k = \left\langle (A \upharpoonright g_A(k))^*, f_i(k) \right\rangle$, where i is the greatest number such that $\langle g(k), f_i(k) \rangle \leqslant g_A(k-1)$. Then for all $k > 0$, $q_k \leqslant g_A(k-1)$.

Let $B = \{q_k : k > 0\}$. We claim that B is Schnorr trivial and that $A \leqslant_{wtt} B$.

To see the latter, let $n < \omega$. Let $k = g_A^{-1}(n)$ (that is, the greatest k such that $g_A(k) \leqslant n$). Then $q_{k+1} \leqslant g_A(k) \leqslant n$ and $A \upharpoonright g_A(k+1)$ can be effectively obtained from q_{k+1}. This procedure allows us to generate $A \upharpoonright n$ effectively from $B \upharpoonright (n+1)$.

To see that B is Schnorr trivial, we appeal to the characterisation mentioned above. Here is where we use the fact that g_A is dominant. The point is that for every i, all but finitely many elements of B are pairs whose second coordinate is contained in the range of f_i. This is because the map $k \mapsto \langle g(k), f_i(k) \rangle$ is recursive and thus dominated by g_A, so for all but finitely many k we will have $q_k = \left\langle (A \upharpoonright g_A(k))^*, f_{i'}(k) \right\rangle$ for some $i' \geqslant i$, and the range of $f_{i'}$ is contained in the range of f_i.

Now let Ψ be a truth-table functional; there is some i such that f_i bounds the use function of Ψ. After specifying a fixed initial segment of B (specifying those $q_{k'}$ whose second coordinate is not in the range of f_i), there are at most $2^{kg(k)}$ many possibilities for $B \upharpoonright f_i(k)$ because, apart from the finitely many fixed elements, there are only $kg(k)$ many numbers below $f_i(k)$ which can be elements of B, as they all have the form $\langle p, f_i(m) \rangle$ for some $p < g(k)$ and $m < k$. After applying Ψ, we get a recursive trace for $\Psi(B)$ whose k^{th} element has size at most $2^{kg(k)}$. Hence $\deg_{tt}(B)$ is recursively traceable (in the tt-degrees), so as quoted above, B is Schnorr trivial. $\square$

3.4. **Truth-table reductions with tiny use.** Another connection between tiny use and Schnorr triviality is obtained by examining truth-table reducibility. Recall that $A \leqslant_{tt} B$ if and only if there is a Turing reduction Φ for which Φ^X is total for all X and $\Phi^B = A$. We say that $A \leqslant_{tt(tu)} B$ if for every order function h there is such a functional whose use function is bounded by h. Equivalently, for every order function h, there is a truth-table reduction of A to B for which the size of the n^{th} truth table is bounded by $h(n)$. This notion is invariant in the truth-table degrees.

Since the use function for a total Turing functional is recursive (equivalently, the size of the n^{th} truth-table of a tt-reduction is recursive), there is no uniform notion in this context.

The class of all A such that there is a B with $A \leqslant_{tt(tu)} B$ gives us a new characterisation of the Schnorr trivial sets.

Theorem 3.11. *Let $A \in \{0,1\}^\omega$. There is a set B such that $A \leqslant_{tt(tu)} B$ if and only if A is Schnorr trivial.*

Proof. We begin by assuming that $A \leqslant_{tt(tu)} B$. Let h be an order function. There is a total reduction Φ such that $\Phi^B = A$ whose use function is bounded by $n \mapsto \log(h(n))$. Then a recursive trace for $n \mapsto A \upharpoonright n$ with bound h can be obtained by applying Φ. Hence $\deg_{tt}(A)$ is recursively traceable.

Now suppose that A is Schnorr trivial. Again the point is that $\deg_{tt}(A)$ is recursively traceable, so for any recursive function f, the function $A \mapsto A \upharpoonright f(n)$ has a recursive trace bounded by the identity function.

Let $\langle f_i \rangle$ be an enumeration of all increasing total recursive functions. For each $i < \omega$, let $\langle D^i_n \rangle_{n<\omega}$ be a recursive trace for the function $A \mapsto A \upharpoonright f_i(n)$ such that for all n, $|D^i_n| = n$.

We let B be the collection of triples (i,n,m) such that $A \upharpoonright f_i(n)$ is the m^{th} element of D^i_n.

Let $i < \omega$. Let Φ_i be the following truth-table functional: given an oracle X and input $x \in [f_i(n-1), f_i(n))$, find the least $m \leqslant n$ such that $(i,n,m) \in X$; if the m^{th} element of D^i_n is a string σ of length $f_i(n)$, output $\sigma(x)$. If not, or if there is no $m \leqslant n$ such that $(i,n,m) \in X$, output 0. It is clear that for all $i < \omega$, $\Phi_i^B = A$.

The standard coding of triples of natural numbers by natural numbers grows polynomially. Hence, if g is, say, an exponentially growing recursive function, then for almost all i, for all n and $m \leqslant n$, $(i,n,m) < g(n)$. Hence for almost all i, $|\Phi_i(B \upharpoonright g(n))| \geqslant f_i(n)$, whence the function $n \mapsto |\Phi_i(B \upharpoonright n)|$ dominates $f_i \circ g^{-1}$. Of course every recursive function is dominated by some $f_i \circ g^{-1}$, so $A \leqslant_{tt(tu)} B$. $\square$

4. The distribution of anti-complex sets

In this section we investigate how the anti-complex sets are distributed in the Turing degrees and among certain classes of sets. Three question are natural:

- Which Turing degrees contain anti-complex sets?
- Which Turing degrees contain only anti-complex sets?
- What kind of sets can be anti-complex?

The answer to the second question was mentioned in the introduction:

Proposition 4.1. *A Turing degree* $\mathbf{a}$ *contains only anti-complex sets if and only if* $\mathbf{a}$ *is r.e. traceable.*

Proof. A Turing degree $\mathbf{a}$ is r.e. traceable if and only if for every order function h, every $f \in \mathbf{a}$ has an r.e. trace bounded by h. Since a Turing degree $\mathbf{a}$ is the union of the weak truth-table degrees contained in $\mathbf{a}$, by Lemma 3.3, a Turing degree $\mathbf{a}$ is r.e. traceable if and only if every weak truth-table contained in $\mathbf{a}$ is r.e. traceable. The result now follows from Theorem 1.3. $\square$

Theorem 1.4 now follows from Theorem 1.3.

The rest of this section will be dedicated to answering the other two questions. We will see that every high degree contains an anti-complex set, which leads us to a discussion of which types of randomness are compatible with anti-complexity. Then, after we show that there is an r.e. Turing degree that contains no anti-complex sets, we study the properties of r.e. and ω-r.e. sets that are anti-complex.

4.1. High and random anti-complex sets.

Franklin [10] shows that every high degree contains a Schnorr trivial set. It follows from Proposition 3.9 that every high degree contains an anti-complex set. We improve this result in Corollary 5.5.

Nies [24] constructed a Δ^0_2 perfect tree, all of whose branches are jump-traceable and thus have r.e. traceable Turing degree. Every perfect Δ^0_2 tree contains a high path, and so there is a high r.e. traceable Turing degree. It follows from Proposition 4.1 that there is a high Turing degree that has only anti-complex elements. Note that such a high degree cannot be Δ^0_2, as every r.e. traceable Turing degree is GL_2.

Now every high degree contains Schnorr random and recursively random sets [26]. Hence there is a recursively random, anti-complex set. On the other hand, sufficient randomness precludes anti-complexity: Kučera [17] has shown that every Martin-Löf random set weak truth-table computes a diagonally nonrecursive function, so every Martin-Löf random set is complex and thus certainly not anti-complex. This result can be strengthened to show that partial-recursively random sets are not anti-complex.

Now we are ready to prove Theorem 1.6 and we repeat the statement of the theorem for the reader's convenience.

Theorem 1.6. There is a high Turing degree which does not contain a partial-recursively random set.

Proof. Let A be an anti-complex set. By Porism 3.5, there is some constant $c < \omega$ such that $C(A \upharpoonright n) \leqslant^+ c \log_2 n$; so for almost all n, $C(A \upharpoonright n) \leqslant (c+1) \log_2 n$. Hence Theorem 7 of [22] shows that no Mises-Wald-Church stochastic set is anti-complex. Every partial-recursively random set is Mises-Wald-Church stochastic (see Section 7.4 of [3]), and so no partial-recursively random set is anti-complex. As we just discussed, there is a high Turing degree all of whose elements are anti-complex, and so such a degree cannot contain a partial-recursively random set. □

4.2. **Anti-complex-free Turing degrees.** As outlined in Subsection 5.3 below, there are many natural examples of Turing degrees which do not contains anti-complex sets; however, these are not r.e. Turing degrees. In the following, we prove that there is also an r.e. degree not containing anti-complex sets (which does not follow directly from known results). Note that this r.e. Turing degree cannot be very low, as all array recursive (and hence superlow) r.e. degrees are r.e. traceable.

This result extends the result of Downey, Griffiths and LaForte [6] that there is an r.e. degree that contains no Schnorr trivial sets and utilizes their techniques.

These techniques involve prefix-free complexity. Recall that a machine M is *prefix-free* if its domain is an antichain of $\{0,1\}^*$, that is, for all distinct $\sigma, \tau \in \operatorname{dom} M$, σ is not an initial segment of τ. There is a prefix-free machine, optimal among all prefix-free machines, and so *prefix-free Kolmogorov complexity*, which is often denoted by K, but which we denote by H (to differentiate from the halting set $K = \emptyset'$), equals C_V for some optimal prefix-free machine V.

Lemma 4.2. *If $A \in \{0,1\}^\omega$ is anti-complex, then for every order function f, $H(A \upharpoonright f(n)) \leqslant^+ n$.*

Proof. We follow the proof of Proposition 3.4. If A is anti-complex and f is an order function, then since $\deg_{wtt}(A)$ is r.e. traceable, there is an r.e. trace $\langle T_n \rangle$, bounded by the identity function, for the function $n \mapsto A \upharpoonright f(n)$. The same argument in the proof of Proposition 3.4 shows that for all n there is some $m \leqslant n$ such that

$$H(A \upharpoonright f(n)) \leqslant^+ H(m, n).$$

It is no longer true that $H(x) \leqslant^+ \log_2 x$, but even a crude bound such as $H(n) \leqslant^+ 2 \log_2 n$ would do to show that for some constant c we have $H(m,n) \leqslant^+ c \log_2 n \leqslant^+ n$ as required. □

Theorem 4.3. *There is an r.e. Turing degree that contains no anti-complex sets.*

Proof. For any prefix-free subset D of $\{0,1\}^*$, we let

$$\mu(D) = \sum_{\tau \in D} 2^{-|\tau|}$$

be the measure of the subset of the Cantor space defined by D by taking all infinite extensions of elements of D.

Theorem 9 of [6] states that there is an r.e. set A such that for all $B \equiv_T A$ there is a prefix-free machine M such that $\mu(\mathrm{dom}(M))$ is a recursive real and such that for infinitely many m, $H(B \upharpoonright m) \geqslant C_M(m)$.

The r.e. degree we seek is the Turing degree of A. Let $B \equiv_T A$; we show that B is not anti-complex. Let M be a machine for B as described in the previous paragraph.

We first note that $\mathrm{dom}(M)$ is a recursive subset of $\{0,1\}^*$: If $\langle M_s \rangle$ is a some recursive enumeration of M, then $\mathrm{dom}(M) \upharpoonright \{0,1\}^{\leqslant n} = \mathrm{dom}(M_s) \upharpoonright \{0,1\}^{\leqslant n}$ for any stage s such that $\mu(\mathrm{dom}(M)) - \mu(\mathrm{dom}(M_s)) < 2^{-n}$; such a stage s can be found effectively from n. Now the range of M may not be recursive, but $C_M \upharpoonright \mathrm{range}(M)$ is a partial recursive function.

We can compute a strictly increasing recursive function f such that for all n,

$$\sum_{\substack{m > f(n) \\ m \in \mathrm{range}\, M}} 2^{-C_M(m)} \leqslant 2^{-3n}$$

by finding some $s(n)$ such that $\mu(\mathrm{dom}(M)) - \mu(\mathrm{dom}(M_{s(n)})) \leqslant 2^{-3n}$ and letting $f(n)$ be greater than any number in the range of $M_{s(n)}$. Let

$$L = \left\{ (C_M(m) - 2f^{-1}(m), m) \,:\, m \in \mathrm{range}\, M \right\}.$$

The set L is recursively enumerable. Recall that for any set $D \subseteq \omega^2$, the *weight* $\mathtt{wt}(D)$ of D is $\sum_{(n,m) \in D} 2^{-n}$. We have

$$\mathtt{wt}(L) = \sum_{m \in \mathrm{range}\, M} 2^{2f^{-1}(m) - C_M(m)} = \sum_n 2^{2n} \sum_{\substack{m \in \mathrm{range}\, M \\ m \in [f(n), f(n+1))}} 2^{-C_M(m)} \leqslant$$

$$\sum_n 2^{2n} \sum_{\substack{m \in \mathrm{range}\, M \\ m \geqslant f(n)}} 2^{-C_M(m)} \leqslant \sum_n 2^{2n} 2^{-3n} = \sum_n 2^{-n} < \infty.$$

The Kraft-Chaitin Theorem (see [3, 19, 23]) now ensures that for all m,

$$H(m) \leqslant^+ C_M(m) - 2f^{-1}(m)$$

(recall that for $m \notin \mathrm{range}\, M$, we let $C_M(m) = \infty$).

Suppose that B is anti-complex. Then by Lemma 4.2, $H(B \upharpoonright f(n)) \leqslant^+ n$. Let $m < \omega$ and let $n = f^{-1}(m)$. We can uniformly compute $B \upharpoonright m$ if we are given both m and $B \upharpoonright f(n+1)$. Since H measures prefix-free complexity, we have $H(B \upharpoonright m) \leqslant^+ H(m) + H(B \upharpoonright f(n+1))$ (a description for $B \upharpoonright m$ is a description for m concatenated with a description for $B \upharpoonright f(n+1)$). Overall we get, for all m,

$$H(B \upharpoonright m) \leqslant^+ H(m) + f^{-1}(m) \leqslant^+ C_M(m) - f^{-1}(m).$$

Since f is increasing, f^{-1} is unbounded, which would make it impossible to have infinitely many $m \in \mathrm{range}\, M$ such that $H(B \upharpoonright m) \geqslant C_M(m)$. Hence B cannot be anti-complex. □

4.3. **Anti-complex r.e. and ω-r.e. sets.** The results so far show that if A is anti-complex, then there is some set $B \leqslant_T A \oplus K$ such that $A \leqslant_{uT(tu)} B$. In general, as we will see shortly, one cannot improve this to $B \leqslant_{wtt} A \oplus K$. However, if A is r.e., then we get an improved bound as follows.

Proposition 4.4. *If A is an anti-complex r.e. set, then $A <_{uT(tu)} K$.*

Proof. We claim that if A is r.e., then $A^* \leqslant_{wtt} K$; the rest follows from Theorem 3.8. Fix a recursive enumeration $\langle A_s \rangle$ of A and let, at stage s, $g_s(k)$ be the least number n such that no initial segment of A_s of length at least n has a U-description of length at most k. Then g_s converges to g_A and is an ω-r.e. approximation of g_A. We can have $g_{s+1}(k) \neq g_s(k)$ only in three cases:

- there is some $\sigma \in \operatorname{dom} U_{s+1} \setminus \operatorname{dom} U_s$ of length at most k and $U(\sigma) \subset A_{s+1}$;
- there is some $\sigma \in \operatorname{dom} U_s$ of length at most k such that $U(\sigma) \not\subset A_s$ but $U(\sigma) \subset A_{s+1}$; or
- there is some $\sigma \in \operatorname{dom} U_s$ of length at most k such that $U(\sigma) \subset A_s$ but $U(\sigma) \not\subset A_{s+1}$.

For each σ, each case can happen at most once, and the first two cannot both happen at different stages. Hence our approximation for $g_s(k)$ changes at most $2 \cdot 2^{k+1}$ many times.

Hence $g_A \leqslant_{wtt} K$, and it is straightforward to see that $A^* \leqslant_{wtt} g_A \oplus K \oplus A$ for any set A because once we know $g_A(k)$, we only need to query K about strings below $g(k)$ (where $g(k) > (A \upharpoonright g_A(k))^*$ is recursive) to find $(A \upharpoonright g_A(k))^*$ and hence A^*. □

Theorem 1.5 now follows from Proposition 4.4 and the techniques of Downey, Jockusch and Stob [7, 8] and Ishmukhametov [14]. The fact that the array recursive r.e. wtt-degrees form an ideal now follows from Observation 2.5.

One would perhaps hope that the previous result could be extended to classes wider than the class of r.e. sets and their weak truth-table degrees. Of course, if $A <_{T(tu)} K$, then $A \leqslant_{wtt} K$ and so A is ω-r.e.; however, we now show that there are ω-r.e. sets A which are anti-complex and yet $A \not\leqslant_{T(tu)} K$. This shows that the condition $B \leqslant_T A \oplus K$ for the bound for A with tiny use cannot in general be improved to $B \leqslant_{wtt} A \oplus K$.

We first need a lemma which again is not new, but which is not found in standard references (an approximation, insufficient for our purposes, is Theorem 9.14.6 in [3]). Let Ω be the halting probability.

Lemma 4.5. *For any r.e. set A, there is a reduction of A to Ω with use bounded below* $2 \log n$.

Indeed, we can even get a bound of $h(n)$ where h is such that $\sum_n 2^{-h(n)}$ is finite, such as $\log n + 2 \log \log n$.

Proof. Let $\langle \Omega_s \rangle$ be an effective, increasing approximation of Ω and, similarly, let $\langle A_s \rangle$ be an effective enumeration of A. Let h be a recursive function such that $\sum_n 2^{-h(n)}$ is finite.

If n is the smallest number which enters A at stage s, we enumerate the interval $[\Omega_s, \Omega_s + 2^{-h(n)}]$ into a Solovay test G which we enumerate. Since n enters A at most once, the total measure of G is at most $\sum_n 2^{-h(n)}$, which is finite by assumption.

Ω is random, so it belongs to only finitely many of the intervals in G. To compute $A(n)$ from $\Omega \upharpoonright h(n)$, find a stage t at which $\Omega_t \upharpoonright h(n) = \Omega \upharpoonright h(n)$; we claim that $A(n) = A_t(n)$. If n enters A at a later stage s, then $[\Omega_s, \Omega_s + 2^{-h(n)}]$ is in G, but $\Omega - \Omega_t \leqslant 2^{-h(n)}$ and $\Omega_t \leqslant \Omega_s \leqslant \Omega$, so we conclude that Ω is in the interval $[\Omega_s, \Omega_s + 2^{-h(n)}]$. Thus we can get a wrong answer for only finitely many numbers n, and we can find a reduction as required. □

Proposition 4.6. *There is an anti-complex ω-r.e. set which is not reducible to K with tiny use.*

Indeed, as the proof shows, there is such a set which is also the difference of two left-r.e. reals. (We cannot get a left-r.e. real, because every left-r.e. real is weak truth-table equivalent to an r.e. set.)

Proof. By [11, Theorem 4.1], there is a coinfinite r.e. set A such that every superset of A is Schnorr trivial (indeed, any dense simple set would do). Let $B = A \cup \Omega$. B is Schnorr trivial and thus anti-complex. B is also ω-r.e., since it is a Boolean combination of two sets which are wtt-reducible to K.

Now assume for a contradiction that $B \leqslant_{T(tu)} K$. Then there is some reduction $\Gamma^{\Omega} = B$ such that for all n, $|\Gamma(\Omega \restriction n)| > n$ because Ω has the same wtt-degree as K. By Lemma 4.5, there is a reduction $\Delta^{\Omega} = A$ with the same property, as there is a reduction from A to Ω with use below $2 \log n$.

We use the functionals Γ and Δ to define a partial recursive martingale which will succeed on Ω, contradicting the fact that Ω is random. The martingale d is defined by induction on the length of the binary strings which form its domain. We start with the value 1. If $d(\sigma)$ is defined, we first calculate $\Delta(\sigma)(n)$ and $\Gamma(\sigma)(n)$, where $n = |\sigma|$ (if either $|\Gamma(\sigma)| \leqslant n$ or $|\Delta(\sigma)| \leqslant n$, then we know that σ cannot be an initial segment of Ω, so we can stop all betting). If $\Delta(\sigma)(n) = 1$, then we hedge our bets, that is, we let $d(\sigma 0) = d(\sigma 1) = d(\sigma)$. Otherwise, we put all of the capital we have on the outcome $\Gamma(\sigma)(n)$, because in this case, if $\sigma = \Omega \restriction n$, then $A(n) = 0$ and so $B(n) = \Omega(n)$. Thus we let $d(\sigma i) = 2d(\sigma)$ and $d(\sigma(1-i)) = 0$, where $i = \Gamma(\sigma)(n)$.

Since A is coinfinite, there are infinitely many n at which we double our money betting along Ω, so $\lim_n d(\Omega \restriction n) = \infty$ as required for the contradiction. ☐

5. Sets bounding nonrecursive sets with tiny use

We now turn to investigate the ranges of the relations $\leqslant_{T(tu)}$ and $\leqslant_{uT(tu)}$ (where the domain is restricted to the class of nonrecursive sets to avoid triviality). Unlike their domains, these ranges are not equal, because as we observed earlier, if $A \leqslant_{uT(tu)} B$ and A is nonrecursive, then B is high, whereas we will shortly see that there are nonhigh sets which bound nonrecursive sets with tiny use. First, we prove some results on the range of $\leqslant_{uT(tu)}$.

5.1. High degrees.

Unlike for Turing reducibility, with weak truth-table reducibility we have to be careful when we deal with functions (elements of the Baire space ω^{ω}) and sets (elements of the Cantor space $\{0,1\}^{\omega}$). For example, a function is always Turing equivalent to its graph, but if it is not bounded by a recursive function, it may not be wtt-equivalent to its graph. Our primary interest is to investigate $\leqslant_{T(tu)}$ on *sets*, and so far we have not treated functions as oracles in computations with recursive or tiny use. However, as a technical tool, we can extend the definitions of $\leqslant_{T(tu)}$ and $\leqslant_{uT(tu)}$ to include functions as oracles in the standard way; weak truth table invariance still holds. In this context we have the following result.

Observation 5.1. *Let $G(f)$ be the graph of f. If f is a dominant function, then $G(f) \leqslant_{uT(tu)} f$.*

This allows us to characterise the range of $\leqslant_{uT(tu)}$.

Lemma 5.2. *Let $B \in \{0,1\}^\omega$. There is some nonrecursive set A such that $A <_{uT(tu)} B$ if and only if there is some dominant function $f \leqslant_{wtt} B$.*

Proof. In the proof of Proposition 2.10 we noticed that if there is some nonrecursive set A such that $A \leqslant_{uT(tu)} B$ witnessed by some reduction Φ, then the map $n \mapsto |\Phi(B \upharpoonright n)|$ is dominant and is weak truth-table reducible to B. In the other direction, suppose that f is dominant and that $f \leqslant_{wtt} B$. Let A be the graph of f. Then $A \leqslant_{uT(tu)} f$; together with $f \leqslant_{wtt} B$ we get $A \leqslant_{uT(tu)} B$ from Observation 2.4. □

We know that every high Turing degree contains a dominating function, but the weak truth-table degree of that function may not contain any set. This lets us see that there is a high set that is not in the range of $\leqslant_{uT(tu)}$.

Lemma 5.3. *Let f be a function such that $n \mapsto C(f(n))$ is bounded by some recursive function. Then f is wtt-equivalent to some set.*

Proof. Let g be a recursive function which bounds $C(f(n))$. Let A be the set of pairs (n,u) where u is the first number below $g(n)$ which is discovered in some effective enumeration of the universal machine U to be mapped by U to $f(n)$. Then $A \equiv_{wtt} f$. □

Proposition 5.4. *Every high Turing degree contains a dominant function $\hat{f}$ such that $C(\hat{f}(n)) \leqslant^+ n$.*

Proof. Let g be a dominant function; we first find an $f \leqslant_T g$ with the desired properties.

Once again, let $\langle \Omega_s \rangle$ be an effective increasing approximation of Ω. Define f by letting $f(n)$ be the least $s \leqslant g(n)$ such that $\Omega_s \upharpoonright n = \Omega_{g(n)} \upharpoonright n$. It is certainly true that $f \leqslant_T g$.

First we show that $C(f(n)) \leqslant^+ n$. Let M be a machine that on an input σ of length n outputs the least stage s such that $\sigma = \Omega_s \upharpoonright n$ if such a stage exists. Then for all n, $M(\Omega_{g(n)} \upharpoonright n) = f(n)$, so $C_M(f(n)) \leqslant n$ as required.

Next, let h be an order function. We first note that $H(\Omega_{h(n)} \upharpoonright n) \leqslant H(n)$ (as before, H denotes *prefix-free* Kolmogorov complexity), and since Ω is random, for almost all n, $\Omega_{h(n)} \upharpoonright n \neq \Omega \upharpoonright n$, as $H(\Omega \upharpoonright n) \geqslant^+ n$. Thus we can let $\hat{h}(n)$ be the least $s > h(n)$ such that $\Omega_s \upharpoonright n \neq \Omega_{h(n)} \upharpoonright n$; this too is a recursive function, defined on almost every input.

Since g is dominant, for almost all n, $g(n) > \hat{h}(n)$, which implies that $\Omega_{g(n)} \upharpoonright n \neq \Omega_{h(n)} \upharpoonright n$ since the approximation $\Omega_s \upharpoonright n$ does not return to old values, and so $f(n) \geqslant \hat{h}(n) > h(n)$ for almost all n. Thus f is dominant.

Next, we code a set A in the Turing degree of g into f to get a function which is Turing equivalent to g. We let $\hat{f}(n) = 2f(n) + A(n)$. Then $A \leqslant_T \hat{f}$ and $\hat{f} \leqslant_T f \oplus A \leqslant_T g$, so $\hat{f} \equiv_T g$. Since $\hat{f}$ bounds f, $\hat{f}$ is dominant, and $C(\hat{f}(n)) \leqslant^+ C(f(n)) \leqslant^+ n$. □

Corollary 5.5. *Every high Turing degree contains sets A and B such that $A <_{uT(tu)} B$.*

Proof. Let $\mathbf{a}$ be a high Turing degree. By Proposition 5.4 and Lemma 5.3, there is some dominant $f \in \mathbf{a}$ which is wtt-equivalent to some set B, so of course $B \in \mathbf{a}$.

By Lemma 5.2, there is some set A such that $A \leqslant_{uT(tu)} B$. Indeed, we can take A to be the graph of f. A is thus Turing equivalent to f, so $A \in \mathbf{a}$. □

We can improve on the corollary in case, for example, the high degree is also generalised low.

Theorem 5.6. *If* $\mathbf{a}$ *is a Turing degree such that* $\mathbf{a} \vee \mathbf{0}' \geqslant_T \mathbf{0}''$*, then for every* $B \in \mathbf{a}$ *there is some* $A \in \mathbf{a}$ *such that* $A <_{uT(tu)} B$*.*

The point is that under the assumption that every $B \in \mathbf{a}$ is wtt-equivalent to some dominant function f, we can let A be the graph of f. This means that we can prove the following equivalent fact instead.

Proposition 5.7. *If* $\mathbf{a}$ *is a Turing degree such that* $\mathbf{a} \vee \mathbf{0}' \geqslant_T \mathbf{0}''$*, then every* $B \in \mathbf{a}$ *is wtt-equivalent to some dominating function.*

Proof. Let $B \in \mathbf{a}$, and let $\langle \varphi_e \rangle$ be an enumeration of all partial recursive functions. Then Tot, the collection of all indices e such that φ_e is total, has Turing degree $\mathbf{0}''$, so there is some Turing reduction $\Phi^{B \oplus K} = \mathrm{Tot}$.

We show that there is a set $E \subseteq \mathrm{Tot}$ which is recursively enumerable in B and such that for every (total) recursive function g there is some $e \in E$ such that $g = \varphi_e$. The set E is enumerated as follows: at stage s, if $\Phi^{B \oplus K_s}(e)\downarrow = 1$, then we enumerate $g(e, \sigma)$ into E with use $B \upharpoonright u$, where u is the use of the computation, $\sigma = K_s \upharpoonright u$ and the instructions for calculating $\varphi_{g(e,\sigma)}$ are as follows. We emulate φ_e as long as σ is an initial segment of K_t for stages $t \geqslant s$, waiting for computations to converge, but if at some stage we observe that σ is no longer an initial segment of K, we make $\varphi_{g(e,\sigma)}$ total by immediately converging on all inputs for which we have not yet given an output and giving the answer 0. The function g is thus recursive.

Now E is used to construct a dominant function $f \leqslant_{wtt} B$: we let $f(n)$ be the maximum of the values $\varphi_e(n)$ for the e that are enumerated into E by stage n with B-use at most n.

Again, we can modify f to be a dominant function $\hat{f} \equiv_{wtt} B$ by coding B into $\hat{f}$, say again by letting $\hat{f}(n) = 2f(n) + B(n)$. □

We do not know much in general about the range of $\leqslant_{T(tu)}$. We give some partial results in the following subsections. First, we show that every nonrecursive r.e. set is in the range of $\leqslant_{T(tu)}$. Then we consider the hyperimmune-free case and show that there is a hyperimmune-free set in the range of $\leqslant_{T(tu)}$, even though not all hyperimmune-free sets are.

5.2. **Recursively enumerable degrees.** The techniques of the previous subsection can be improved to yield the following result.

Theorem 5.8. *For every nonrecursive r.e. set* B *there is a nonrecursive r.e. set* A *such that* $A <_{T(tu)} B$*.*

In order to prove this theorem, we take dominant functions which have decent approximations. Let $\mathbf{h}$ be a high r.e. Turing degree. By standard manipulations, we can get a dominant function $f \leqslant_T \mathbf{h}$ with an approximation with the following "nice" properties.

- The approximation is increasing: for all n and s, $f_s(n) \geqslant f_{s-1}(n)$.
- If $f_s(n) \neq f_{s-1}(n)$, then $f_s(n) = s$.

- For all s there is at most one n such that $f_s(n) = s$ (this is done by delaying changes in the approximation).

We now procede to the proof of the theorem.

Proof of Theorem 5.8. Let B be a nonrecursive r.e. set. By a standard cone-avoiding addition to the Sacks jump inversion theorem, there is a high r.e. degree $\mathbf{h}$ which does not compute B. Let $f \leqslant_T \mathbf{h}$ be a dominant function with an approximation $\langle f_s \rangle$ as described above.

Enumerate a set A as follows: for all $n \in B$, if n enters B at stage s, enumerate $f_s(n)$ into A.

Suppose that we want to compute A from B. To find out if $t \in A$, we first go to stage t and see if $f_t(n) = t$ for some $n \leqslant t$ — if not, then t is certainly not in A. If so, then t is in A if and only if for the unique $n = n(t)$ such that $f_t(n) = t$, n enters B at a later stage s before $f_s(n)$ changes. This gives a reduction of A to B with identity use.

In fact, $A \leqslant_{T(tu)} B$. The idea is the following. Suppose again that we want to find out whether t is in A and that we find that $f_t(n) = t$. If we knew that $f(n) > t$, in other words, that there is a later stage s at which we have $f_s(n) \neq f_t(n)$, then we could wait for that stage and see if n entered B before that stage or not. Of course, we cannot always do this, because it may happen that $t = f_t(n) = f(n)$ (or else A would be recursive, whereas later we show it is not). But now suppose that h is an order function and that we want to reduce A to B with use bounded by h. Then if $n = n(t) < h(t)$, then we can consult $B(n)$ as before to compute $A(t)$. If $h(t) \leqslant n$, then $h^{-1}(n) \geqslant t$; since f is dominant, $f(n) > t$ except for finitely many n, so we can employ the second tactic of waiting for $f_s(n)$ to change in order to compute $A(t)$. In the second case we do not consult B at all, so overall we get a reduction with use bounded by h.

Finally, to show that A is not recursive, we see that $B \leqslant_T A \oplus f$ and recall that $B \not\leqslant_T f$. To find $B(n)$, we calculate the least stage t at which $f_t(n) = f(n)$; if $n \notin B_t$, then $n \in B$ if and only if $t \in A$. $\square$

Just as we did for $\leqslant_{tt}$, we can apply the "tiny use" operator to many-one reducibility and say that $A \leqslant_{m(tu)} B$ if for every order function h there is a recursive function f dominated by h such that $A = f^{-1}B$. The previous proof can be slightly modified to show that for every nonrecursive r.e. set B, there are an r.e. set $\hat{B}$ which is wtt-equivalent to B and a nonrecursive r.e. set A such that $A \leqslant_{m(tu)} \hat{B}$. We simply enumerate (n, m) into $\hat{B}$ if n is enumerated into B at stage s and $f_s(n)$ is the m^{th} value we see for $f_t(n)$ by stage s, that is, $m = |\{f_t(n) : t \leqslant s\}|$. Then, given t, we can find n and m; then $t \in A$ if and only if $(n, m) \in \hat{B}$ and, as described above, this can be done with tiny use. To get $B \equiv_{wtt} \hat{B}$ rather than just Turing equivalence, we use a function f which is ω-r.e., the existence of which is guaranteed by the proof of Proposition 5.4.

Theorem 5.8 cannot be extended to all Δ^0_2 sets, as Downey, Ng and Solomon [9] constructed a Δ^0_2 set which has minimal wtt-degree.

Finally, there is an r.e. set B which has minimal tt-degree [20]. For such B, there can be no nonrecursive $A \leqslant_{tt(tu)} B$. Thus we cannot improve $\leqslant_{T(tu)}$ in the theorem to $\leqslant_{tt(tu)}$, or, in the comments after the proof, get $\hat{B} \equiv_{tt} B$ rather than $\hat{B} \equiv_{wtt} B$.

5.3. **Hyperimmune-freeness.** Note that most hyperimmune-free Turing degrees do not contain any set B for which there is a nonrecursive A such that $A <_{T(tu)} B$. For example, a Turing degree which is both minimal and hyperimmune-free does not contain such sets because if $\deg_T(X)$ is hyperimmune-free and minimal, then $\deg_{wtt}(X)$ is also minimal, as every nonrecursive $Y \leqslant_{wtt} X$ is Turing equivalent to X and thus also wtt-equivalent to X. Similarly, if X is Martin-Löf random and $\deg_T(X)$ is hyperimmune-free, then every nonrecursive $Y \leqslant_{wtt} X$ is truth-table equivalent to a Martin-Löf random set and so cannot be anti-complex, so again we get that every $A \leqslant_{T(tu)} X$ is recursive.

Thus in the realm of the hyperimmune-free degrees, generic sets (in the sense of either recursive Sacks forcing or forcing with sets of positive measure) do not compute nonrecursive sets with tiny use.

On the other hand, there is a hyperimmune-free B with a nonrecursive $A <_{T(tu)} B$. This follows from the hyperimmune-free basis theorem and the following theorem.

For putting the next result into its context, we remark that it is already known that there is a Π^0_1-class with no recursive elements which consists of anti-complex sets; for example, there is one which consists of Schnorr trivial sets [11] and one which consists of sets A such that $\deg_T(A)$ is r.e. traceable.

Theorem 5.9. *There is a Π^0_1-class with no recursive elements consisting of sets B for which there are nonrecursive sets A such that $A <_{m(tu)} B$.*

Proof. We imitate part of the the proof of Theorem 5.8. Again, let $\langle f_s \rangle$ be an ω-r.e. approximation for a dominant function f with the properties discussed above; say g is a recursive function which bounds the number of possible values $m(n) = |\{f_s(n) : s < \omega\}|$.

For $n < \omega$ and $k \leqslant m(n)$, let $\pi(n,k)$ be the k^{th} value of $f_s(n)$. Thus $\pi(n, m(n)) = f(n)$. Now let $D = \{(n,k) : k < m(n)\}$ and $E = \pi[D]$; both are r.e. sets. Furthermore, E is nonrecursive as $f \leqslant_T D \leqslant_T E$: for each n, we find $\pi(n,k)$ recursively in k and so, consulting E, determine if $k < m(n)$ or not.

We can thus split E into a pair E_0 and E_1 of recursively inseparable r.e. sets. Let $\mathcal{P}$ be a Π^0_1-class of sets which separate $D_0 = \pi^{-1}E_0$ from $D_1 = \pi^{-1}E_1$ (sets that contain D_0 and are disjoint from D_1). Let B be any element of $\mathcal{P}$ and let $A = \pi[B]$. Then A separates E_0 and E_1, so A is not recursive. We claim that $A \leqslant_{T(tu)} B$, indeed that $A \leqslant_{m(tu)} B$.

The argument is similar to that of the proof of Theorem 5.8. For any t, if $t \in A$ then $t = \pi(n,k)$ for some n and some $k \leqslant m(n)$. The pair (n,k) is unique and we can effectively decide if such a pair exists, and if so, find it. Suppose such a pair (n,k) is found. We have $t \in A$ if and only if $(n,k) \in B$.

Let h be an order function. For all but finitely many t, either $(n,k) < h(t)$ or $t \leqslant h^{-1}(n,k) \leqslant h^{-1}(n,g(n)) < f(n)$, whence $k < m(n)$. As before, in the first case we consult B about (n,k). In the latter case, $(n,k) \in D$, so we do not need to consult B. For recall that B is a separator of D_0 and D_1, and that D_0 and D_1 form a partition of D; and that both D_0 and D_1 are r.e., so which one of them contains (n,k) is revealed to us with sufficient patience. □

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J. Franklin, Department of Mathematics, 196 Auditorium Road, University of Connecticut, U-3009, Storrs, CT 06269-3009, USA
E-mail address: `johanna.franklin@uconn.edu`

N. Greenberg, Victoria University, PO Box 600, Wellington 6140, New Zealand
E-mail address: `greenberg@mcs.vuw.ac.nz`

F. Stephan, Department of Computer Science and Department of Mathematics, National University of Singapore, Singapore 117543, Republic of Singapore
E-mail address: `fstephan@comp.nus.edu.sg`

G. Wu, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 639798, Republic of Singapore.
E-mail address: `guohua@ntu.edu.sg`