Unifying Logic, Topology and Learning in Parametric Logic

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Abstract. Many connections have been established between learning and logic, or learning and topology, or logic and topology. Still, the connections are not at the heart of these fields. Each of them is fairly independent of the others when attention is restricted to basic notions and main results. We show that connections can actually be made at a fundamental level, and result in a logic with parameters that needs topological notions for its early developments, and notions from learning theory for interpretation and applicability.
One of the key properties of first-order logic is that the classical notion of logical consequence is compact. We generalize the notion of logical consequence, and we generalize compactness to β-weak compactness where β is an ordinal. The effect is to stratify the set of generalized logical consequences of a theory into levels, and levels into layers. Deduction corresponds to the lower layer of the first level above the underlying theory, learning with less than β mind changes to layer β of the first level, and learning in the limit to the first layer of the second level. Refinements of Borel-like hierarchies provide the topological tools needed to develop the framework.

1 Introduction

There is an immediate similarity between the compactness theorem of first-order logic and finite learning. Indeed, let T be a (possibly infinite) first-order theory and let φ be a first-order formula. If φ is a logical consequence of T, then φ is a logical consequence of a finite subset of T, which means that if all members of T are enumerated then after some finite time, a learner can discover that $T \models \varphi$ and will not have to change its mind. This simple observation could have initiated a common destiny for logic and learning, but the story of their developments is different. Finite learning is hardly considered as learning, and learning theory thrived over more complex criteria of learnability. On the other hand, classical first-order logic promoted the view that the compactness theorem significantly accounts for its successfulness. So the analogy between the compactness theorem and finite learning seems more like a case for an early divorce than for a natural union of logic and learning.

The idea of a logic of induction has had advocates before formal deductive logic was invented, even if the term 'logic' applied to induction could either mean a framework that closely follows the line of deductive logic—define a syntax, a semantics and a proof theory—or a framework based on probability theory, hence quite different in nature from Frege's creation. In any case, a 'logic of induction' is generally opposed to deductive logic: for instance, the former should be based on nonmonotonic forms of reasoning [14] contrary to the latter. But is there a natural formal system that would encompass both deduction and induction? Such a system or *paradigm* $\mathcal{P}$ would be based on a notion of generalized logical consequence that would not be compact. We would identify the 'deductive consequences' (w.r.t. $\mathcal{P}$) of a theory T with the generalized logical consequences of T that enjoy the compactness property. And we could hope that a weaker form of compactness would be the hallmark of the 'inductive consequences' (w.r.t. $\mathcal{P}$) of T [17]. As will be seen, this idea can be refined and results in a notion of 'β-weak compactness' that has a natural interpretation by the notion of learning with less than β mind changes [2, 7, 10, 24]. Such a correspondence is appealing, and not just because the notion of β-weak compactness is formally amazingly simple. It can also provide characterizations of the syntax of the hypotheses that can be learned with less than β mind changes, and related to what is known in topology as the difference hierarchy [11]. This kind of characterization has been obtained when β is an integer [6], together with other results that relate various learning paradigms derived from [9] to some topological spaces [13, 20, 21, 23]. For instance, it has been pointed out that learnability in the limit can be related to being Δ_2 in the Borel hierarchy defined over some topological space [8, 23]. We will reestablish and extend these connections in a more systematic and general setting (in particular, for an arbitrary set of possible data), where logic, topology and learning are interrelated.

We proceed as follows. After fixing some notation in Section 2, we describe the fundamental model-theoretic notions of Parametric logic in Section 3: the parameters, the generalized notion of logical consequence, and the logical hierarchies. In Section 4, we define a natural topological space from some of the parameters introduced in Section 3. Considering the Borel and the difference hierarchies over this topological space, and then the formulas that hereditarily represent sets in these hierarchies, we establish fundamental connections between logical and topological complexity. In Section 5, basic concepts from Inductive inference are redefined in the logical setting, and classification in the limit, with or without a mind change bound, is shown to be related in an essential way to the logical and topological concepts. We conclude in Section 6.

2 General notation

We first introduce some general notation. The class of ordinals is denoted by Ord. Given $\alpha, \beta, \alpha', \beta' \in \mathrm{Ord}$, we write $(\alpha, \beta) \prec (\alpha', \beta')$ iff (α, β) is lexicographically before (α', β'), *i.e.*, either $\alpha < \alpha'$, or $\alpha = \alpha'$ and $\beta < \beta'$.

Let a set X be given. The set of finite sequences of members of X, including the empty sequence $()$, is represented by $X^\star$. Given a finite or an infinite sequence σ of members of X and a natural number i that, in case σ is finite, is at most equal to the length of σ, we represent by $\sigma_{|i}$ the initial segment of σ of length i. Concatenation between sequences is represented by $\star$, and sequences consisting of a unique element are often identified with that element. We write $\subset$ (respect., $\subseteq$) for strict (respect., nonstrict) inclusion between sets, as well as for the notion of a finite sequence being a strict (respect., nonstrict) initial segment of another sequence. Let two sets X, Y and a partial function f from X into Y be given. Given $x \in X$, we write $f(x) =\downarrow$ when $f(x)$ is defined, and $f(x) =\uparrow$ otherwise. Given two members x, x' of X, we write $f(x) = f(x')$ when both $f(x)$ and $f(x')$ are defined and equal; we write $f(x) \neq f(x')$ otherwise.

Let R be a binary relation over a set X. Recall that R is well-founded iff every nonempty subset Y of X contains an element x such that no member y of Y satisfies $R(y, x)$. Suppose that R is well founded. We then denote by ρ_R the unique function from X into Ord such that for all $x \in X$:

$$\rho_R(x) = \sup\{\rho_R(y) + 1 : y \in X,\ R(y, x)\}.$$

The length of R is the least ordinal not in the range of ρ_R. Note that the length of R is equal to 0 iff $X = \emptyset$.

3 Logical paradigms

3.1 The parameters of Parametric logic

Parametric logic is a unified family of logics, also called (*logical*) *paradigms*, defined as particular settings for five parameters.[3] All fundamental notions are defined independently of the values of the parameters. Classical first-order logic is a particular paradigm and some classical learning paradigms from Inductive inference, once cast in the current framework, are other examples of logical paradigms. In this section, we define the five parameters.

The first parameter of Parametric logic is a *vocabulary*, *i.e.*, a set of predicate symbols and function symbols (constants are 0-ary function symbols), possibly containing equality (we depart from the standard practice of including equality in the logical part of the language).

Notation 1 *We denote by $\mathcal{V}$ a countable vocabulary.*

[3] In full generality, we define a paradigm as a set of six parameters. We do not include here one of the parameters, namely, a countable ordinal κ that represents the number of meta-levels above the object level at which reasoning is being formalized. In this paper, κ is set to 0 for the sake of simplicity, and all notions are defined at the object level. Still, most results hold for arbitrary values of κ.

We refer to terms over $\mathcal{V}$ simply as terms. If we say that $\mathcal{V}$ contains $\overline{0}$ and s, then $\overline{0}$ denotes a constant and s a unary function symbol; moreover, given a nonnull $n \in \mathbb{N}$, $\overline{n}$ is used as an abbreviation for the term obtained from $\overline{0}$ by n applications of s to $\overline{0}$. We denote by $\mathcal{L}^{\mathcal{V}}_{\omega\omega}$ the set of finite statements (*over* $\mathcal{V}$), *i.e.*, well formed expressions built from $\mathcal{V}$ and a countable set of (first-order) variables using negation, disjunction and conjunction of (possibly empty) finite sets of expressions, and quantification.[4] So for all finite $D \subseteq \mathcal{L}^{\mathcal{V}}_{\omega\omega}$, the disjunction of all members of D, written $\bigvee D$, and the conjunction of all members of D, written $\bigwedge D$, both belong to $\mathcal{L}^{\mathcal{V}}_{\omega\omega}$. We denote by $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ the extension of $\mathcal{L}^{\mathcal{V}}_{\omega\omega}$ that accepts disjunctions and conjunctions of countable sets of expressions in which finitely many variables only occur free. So for all countable $X \subseteq \mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ in which finitely many variables occur free, the disjunction of all members of X, written $\bigvee X$, and the conjunction of all members of X, written $\bigwedge X$, both belong to $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$.[5] We refer to the members of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ as infinitary statements (*over* $\mathcal{V}$). Note that the occurrence or nonoccurrence of $=$ in $\mathcal{V}$ determines whether $\mathcal{L}^{\mathcal{V}}_{\omega\omega}$ and $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ are languages with or without equality. Given an infinitary statement φ, we identify $\bigvee\{\varphi\}$ and $\bigwedge\{\varphi\}$ with φ. Given two infinitary statements φ_1 and φ_2, $\varphi_1 \vee \varphi_2$ and $\varphi_2 \vee \varphi_1$ are abbreviations for $\bigvee\{\varphi_1, \varphi_2\}$, $\varphi_1 \wedge \varphi_2$ and $\varphi_2 \wedge \varphi_1$ are abbreviations for $\bigwedge\{\varphi_1, \varphi_2\}$, $\varphi_1 \rightarrow \varphi_2$ is an abbreviation for $\neg\varphi_1 \vee \varphi_2$, and $\varphi_1 \leftrightarrow \varphi_2$ is an abbreviation for $(\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$. We denote by $\forall\varphi$ the universal closure of an infinitary statement φ.

The natural classes of languages to be dealt with in this framework are *countable fragments of the infinitary logic* $\mathcal{L}_{\omega_1\omega}$ [15]. More precisely, a fragment of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ is a subset L of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ with the following properties.

- All finite statements are in L.
- For all $\varphi \in L$, variables x and terms t, $\varphi[t/x]$ is in L.
- For all $\varphi \in L$ and variables x, $\exists x\varphi$ and $\forall x\varphi$ are in L.
- For all $\varphi \in L$, $\neg\varphi$ is in L.
- For all finite $D \subseteq L$, $\bigvee D$ and $\bigwedge D$ are in L.
- For all $\varphi \in L$ and substatements ψ of φ, ψ is in L.

For all subsets X of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$, there exists a least fragment of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ that contains X; we call it the fragment of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ generated by X. For all countable subsets X of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$, the fragment of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ generated by X is countable. The second parameter of Parametric logic is a countable fragment of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$.

Notation 2 *We denote by* $\mathcal{L}$ *a countable fragment of* $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$.[6]

[4] We use the word 'statement' here because we reserve the word 'formula' for a more specific use, to be introduced with the second parameter.

[5] $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ is a more natural extension of $\mathcal{L}^{\mathcal{V}}_{\omega\omega}$ when disjunction and conjunction in $\mathcal{L}^{\mathcal{V}}_{\omega\omega}$ are defined as unary operators whose arguments are finite sets, rather than being defined as operators taking two statements as arguments. It also simplifies many definitions and proofs.

[6] When a countable ordinal κ is introduced as a sixth parameter, a notion of infinitary κ-statement is defined, and $\mathcal{L}$ denotes a countable fragment of the set of infinitary κ-statements.

We refer to the members of $\mathcal{L}$ as *formulas*, and to closed formulas as *sentences*.

By structure, we mean structure over $\mathcal{V}$. In Artificial intelligence and in Inductive inference, not every structure is an intended interpretation. For example, the latter investigates the learnability of classes of *languages*, a language being a recursively enumerable subset of $\mathbb{N}$. If we choose for $\mathcal{V}$ a vocabulary that contains $\overline{0}$, s, a unary predicate P, and possibly other predicate symbols, then a language L can be represented as the structure $\mathfrak{M}$ whose domain is $\{\overline{n} : n \in \mathbb{N}\}$, such that for all $n \in \mathbb{N}$, $\mathfrak{M} \models P(\overline{n})$ iff $n \in L$, and such that the interpretation of all other predicate symbols in $\mathcal{V}$ is fixed. Such a structure is *standard*. More generally, given a structure $\mathfrak{M}$, $\mathfrak{M}$ is *Henkin* iff $\mathfrak{M}$'s individuals are the nonempty sets of closed terms that they interpret, $\mathfrak{M}$ is *Herbrand* iff $\mathfrak{M}$ is Henkin and $\mathfrak{M}$'s individuals are singletons, and $\mathfrak{M}$ is *standard* if either $\mathcal{V}$ contains equality and $\mathfrak{M}$ is Henkin, or $\mathcal{V}$ is equality-free and $\mathfrak{M}$ is Herbrand. By the previous remark, the languages considered in Inductive inference are naturally represented as standard structures; in the resulting logical paradigm, nonstandard structures are unintended interpretations. Though the class of intended interpretations is often a class of standard structures, we make no a priori assumption and accept any class of structures as a potential class of intended interpretations. This justifies the introduction of the third parameter, together with a first generalization of the classical notion of logical consequence that will be used in the formal developments, though it is still a particular case of the generalized notion of logical consequence we will define in the next section.

Notation 3 *We denote by $\mathcal{W}$ a class of structures.*[7]

Given $T \subseteq \mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$, we denote by $\mathrm{Mod}_{\mathcal{W}}(T)$ the class of models of T in $\mathcal{W}$.

Definition 4 *Given $T \subseteq \mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$, an infinitary formula φ is said to be a* logical consequence of T in $\mathcal{W}$, *written $T \models_{\mathcal{W}} \varphi$, iff $\mathrm{Mod}_{\mathcal{W}}(T) \subseteq \mathrm{Mod}_{\mathcal{W}}(\varphi)$.*

Given $\varphi, \psi \in \mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$, if $\mathrm{Mod}_{\mathcal{W}}(\varphi) = \mathrm{Mod}_{\mathcal{W}}(\psi)$ then we say that φ is logically equivalent to ψ in $\mathcal{W}$.

Members of $\mathcal{W}$ will be referred to as *possible worlds*. Consider a learning paradigm where the set L of even numbers is one of the languages to be learnt. When learning from *positive data* only, a learner has to converge in the limit to an r.e. index for L, on any enumeration of L (such an enumeration is a *text* for L). Cast in the current framework, an enumeration of L becomes an enumeration of $T = \{P(\overline{2n}) : n \in \mathbb{N}\}$, and an r.e. index for L becomes a formula φ that uniquely characterizes the standard structure $\mathfrak{M}$ that represents the set of even numbers, among the class of intended interpretations. So φ has the property that $\mathrm{Mod}_{\mathcal{W}}(\varphi) = \{\mathfrak{M}\}$, and could be for instance the formula $\varphi = P(\overline{0}) \wedge \forall x(P(x) \leftrightarrow \neg P(s(x)))$. One of our aims is to relate identification in the limit to *some notion* of logical inference, in such a way that learning can be seen as a particular way of drawing logical inferences w.r.t. to the notion that

[7] When a countable ordinal κ is introduced as a sixth parameter, a notion of κ-structure is defined, and $\mathcal{W}$ denotes a class of κ-structures.

has been chosen. Applied to the previous example, this requirement translates to: the formula φ above should be, *in some sense*, a logical consequence of the set T above. Obviously, φ is not a logical consequence of T in the usual sense. It is not even the case that φ is a logical consequence of T in $\mathcal{W}$. Remember that a text for the set L of even numbers is an enumeration of *all* members of L: the nonoccurrence of 3 in any text for L tacitly implies that 3 is not a member of L. This tacit assumption on texts for L is lost when considering T, which can be expanded with $\neg P(\overline{3})$ as well as with $P(\overline{3})$. This is the reason why it is *not* the case that every model of T in $\mathcal{W}$ is a model of φ. Taken alone, the parameter $\mathcal{W}$ does not offer enough flexibility; more generality is needed.To solve this issue, a fourth parameter is needed to represent the formulas that can be used as *data*.

Notation 5 *We denote by $\mathcal{D}$ a set of formulas.*

We refer to a member of $\mathcal{D}$ as a *possible datum*. With the previous example, $\mathcal{D}$ would be set to $\{P(\overline{n}) : n \in \mathbb{N}\}$. If we considered learning from both positive and negative data instead of learning from positive data, hence if we considered *informants* instead of texts, we would instead set $\mathcal{D}$ to $\{P(\overline{n}), \neg P(\overline{n}) : n \in \mathbb{N}\}$ (as done in [16]). In full generality, $\mathcal{D}$ is an arbitrary set of formulas.

Classical first-order logic, being purely hypothetico-deductive, has no notion of data, which amounts to setting $\mathcal{D}$ to $\emptyset$. But every set F of formulas can be the starting point for logical investigation: F can be interpreted as *facts*, *axioms*, or *explicit knowledge* on the basis of which inferences can be drawn. On the other hand, paradigms from Inductive inference, once cast in this framework, are perfectly determined by $\mathcal{W}$ and $\mathcal{D}$, and have no notion of explicit knowledge. Between these extreme cases, it is common in Inductive logic programming to deal with sets of formulas consisting of both data and *background knowledge*, consisting of formulas of a given kind, for instance, definite Horn clauses [19]. A fifth parameter is needed to represent the formulas that can play the role of pieces of information not conveyed by data.

Notation 6 *We denote by $\mathcal{F}$ a set of formulas.*

We refer to a member of $\mathcal{F}$ as a *possible fact*. As pointed out, casting Inductive inference into this framework usually requires to set $\mathcal{F}$ to $\emptyset$, whereas classical first-order logic sets $\mathcal{F}$ to the set of first-order formulas (which, in this case, is also the value of $\mathcal{L}$). It could be assumed that $\mathcal{D}$ and $\mathcal{F}$ are disjoint sets, or that $\mathcal{D}$ is a subset of $\mathcal{F}$, without affecting the formal developments in any significant way. We prefer not to impose any relationship between $\mathcal{D}$ and $\mathcal{F}$.

We have defined all parameters.

Notation 7 *The quintuple $(\mathcal{V}, \mathcal{L}, \mathcal{W}, \mathcal{D}, \mathcal{F})$ is denoted $\mathcal{P}$.*

In the formal developments, $\mathcal{P}$ refers to the class of logical paradigms or to the logical paradigm under investigation, depending on which assumptions are made on its members: some results place no condition on $\mathcal{P}$, while other results impose some conditions on $\mathcal{P}$, the extreme case being when all members of $\mathcal{P}$ have to be set to particular values so that $\mathcal{P}$ refers to a unique paradigm. Classical first-order logic is the particular case where $\mathcal{V}$ is arbitrary, $\mathcal{L} = \mathcal{L}^{\mathcal{V}}_{\omega\omega}$, $\mathcal{W}$ is the class of all structures, $\mathcal{D} = \emptyset$, and $\mathcal{F} = \mathcal{L}$. We have pointed out that learning paradigms can be cast in the current framework and result in other particular cases.

3.2 Generalized logical consequence

The notion of logical consequence we are after is not captured by the notion of logical consequence in $\mathcal{W}$ defined in Definition 4. It cannot be determined by the parameter $\mathcal{W}$ alone; but it can be naturally defined from $\mathcal{W}$ and $\mathcal{D}$. We first introduce a technical concept. Given a structure $\mathfrak{M}$, the $\mathcal{D}$-*diagram of* $\mathfrak{M}$, denoted $\mathrm{Diag}_{\mathcal{D}}(\mathfrak{M})$, is the set of all members of $\mathcal{D}$ that are true in $\mathfrak{M}$ [12]. When Inductive inference is cast in this framework, a learner is basically presented with an enumeration of $\mathrm{Diag}_{\mathcal{D}}(\mathfrak{M})$ for some possible world $\mathfrak{M}$, and $\mathfrak{M}$ is perfectly determined by its $\mathcal{D}$-diagram among the members of $\mathcal{W}$, in the same way that a language L is perfectly determined by a text for L, in the class of languages to be learnt: $\mathfrak{M}$ is the only intended model of $\mathrm{Diag}_{\mathcal{D}}(\mathfrak{M})$. More generally, the intended models of an arbitrary set T of infinitary statements are defined as follows.

Definition 8 *Let a set T of infinitary statements be given.*

A $\mathcal{D}$-minimal model of T in $\mathcal{W}$ *is a structure $\mathfrak{M}$ such that $\mathfrak{M} \in \mathrm{Mod}_{\mathcal{W}}(T)$ and for all $\mathfrak{N} \in \mathrm{Mod}_{\mathcal{W}}(T)$, $\mathrm{Diag}_{\mathcal{D}}(\mathfrak{N}) \not\subset \mathrm{Diag}_{\mathcal{D}}(\mathfrak{M})$.*

We denote by $\mathrm{Mod}^{\mathcal{D}}_{\mathcal{W}}(T)$ the class of $\mathcal{D}$-minimal models of T in $\mathcal{W}$.

Going back to our running example, remember that $T = \{P(\overline{2n}) : n \in \mathbb{N}\}$ is not *by itself* the proper counterpart to a text for the language L consisting of the even natural numbers: in an *intended model* of T, $P(\overline{3})$ should be false, as a counterpart to the fact that 3 belongs to $\mathbb{N} \setminus L$. On the other hand, *together with* $\mathcal{W}$ *and* $\mathcal{D}$, T is a counterpart to L, because all formulas of the form $P(\overline{2n+1})$, $n \in \mathbb{N}$, are false in all $\mathcal{D}$-minimal models of T in $\mathcal{W}$: the class of $\mathcal{D}$-minimal models of T in $\mathcal{W}$ is arguably the class of intended models of T. Note how Definition 8 generalizes the notion of minimal Herbrand model of a set of definite clauses: $\mathcal{W}$ is then assumed to be the set of Herbrand structures, while $\mathcal{D}$ is supposed to be the set of closed atomic formulas [5]. In the general case, since $\mathcal{W}$ and $\mathcal{D}$ are arbitrary, a set of formulas can have a unique $\mathcal{D}$-minimal model in $\mathcal{W}$, many of them, or none at all. We can now define the notion of generalized logical consequence determined by the parameters of a paradigm, on the basis of the claim that the class of $\mathcal{D}$-minimal models of a set T of infinitary statements is the class of intended models of T.

Definition 9 *Let a set T of infinitary statements be given.*

An infinitary statement φ is said to be a $\mathcal{D}$-minimal logical consequence of T in $\mathcal{W}$*, written $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$, iff $\mathrm{Mod}^{\mathcal{D}}_{\mathcal{W}}(T) \subseteq \mathrm{Mod}_{\mathcal{W}}(\varphi)$.*

We set $\mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T) = \{\varphi \in \mathcal{L} : T \models^{\mathcal{D}}_{\mathcal{W}} \varphi\}$.

Definition 9 is not only appropriate to model learning scenarios; it also provides a natural formalization for many of the nonclassical forms of reasoning investigated in Artificial intelligence, and as pointed out, it is also related to a fundamental notion from Logic programming.

A key feature of the framework is to study the relation $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$ for *specific* subsets of $\mathcal{L}$. When $\mathcal{P}$ captures a learning paradigm, the only subsets T of $\mathcal{L}$ we are interested in are the $\mathcal{D}$-diagrams of the members of $\mathcal{W}$. More generally, by our interpretation of $\mathcal{D}$ as the set of possible data, it is natural to assume that all theories of interest contain the $\mathcal{D}$-diagram of a possible world. By our interpretation of $\mathcal{F}$ as the set of possible facts, all theories of interest should be subsets of $\mathcal{D} \cup \mathcal{F}$. Finally, if we restrict investigations to theories that have at least one intended model, *i.e.*, at least one $\mathcal{D}$-minimal model in $\mathcal{W}$, we are led to the following notation.

Notation 10 $\mathcal{B}$ *denotes* $\{\mathrm{Diag}_{\mathcal{D}}(\mathfrak{M}) \cup F : \mathfrak{M} \in \mathrm{Mod}_{\mathcal{W}}(F),\ F \subseteq \mathcal{F}\}$.

We refer to a member of $\mathcal{B}$ as a *possible knowledge base*. The members of $\mathcal{B}$ are the theories that are the legitimate starting points for logical investigation. In case $\mathcal{P}$ is the paradigm of classical first-order logic, $\mathcal{B}$ is nothing but the set of consistent theories.

We close this section with a generalization of a classical logical notion that will be used repeatedly.

Definition 11 *Given a subset T of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ and a member φ of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$, we say that T* $\mathcal{D}$-minimally decides φ in $\mathcal{W}$ *iff either $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$ or $T \models^{\mathcal{D}}_{\mathcal{W}} \neg\forall\varphi$.*

3.3 Logical hierarchies

In this section, we define a model-theoretic measure of complexity for the notion of $\mathcal{D}$-minimal logical consequence in $\mathcal{W}$; such a measure results, for every possible knowledge base T, in a hierarchy of members of $\mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$. A first observation is that the notion $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$, where T varies over $\mathcal{B}$ and where φ varies over $\mathcal{L}$, is usually not compact: for most settings of the parameters in $\mathcal{P}$, it is possible to have $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$ though there does not necessarily exist a finite subset K of T such that $\mathrm{Mod}_{\mathcal{W}}(K) \subseteq \mathrm{Mod}_{\mathcal{W}}(\varphi)$. We claim that compactness is the hallmark of deduction, in the sense that φ can be described as a $\mathcal{D}$-*minimal deductive consequence of T in* $\mathcal{W}$ just in case φ is a member of $\mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ such that the compactness property holds. When $\mathcal{P}$ is the paradigm of classical first-order logic, the compactness theorem implies that '$\mathcal{D}$-minimal logical consequence in $\mathcal{W}$' and '$\mathcal{D}$-minimal deductive consequence in $\mathcal{W}$' are equivalent notions, equivalent to the classical notion of logical consequence. In the general case, given $T \in \mathcal{B}$, the hierarchy of $\mathcal{D}$-minimal logical consequence of T in $\mathcal{W}$ is defined in such a way that its first layer (over T) consists precisely of the $\mathcal{D}$-minimal deductive (compact) consequence of T in $\mathcal{W}$. So in the particular case of classical first-order logic, hierarchies have one layer only.

Given $T \in \mathcal{B}$, the second layer of the hierarchy of $\mathcal{D}$-minimal logical consequence of T in $\mathcal{W}$ can be described as the set of *$\mathcal{D}$-minimal inductive consequences of T in $\mathcal{W}$*. What is meant by *inductive* is a property of *weak compactness*, or compactness weakened with a notion of 'refutability' [22]. Let $T \in \mathcal{B}$ and $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ be given. Suppose that there exists a finite subset K of T such that for all possible knowledge bases T' that contain K, either φ is a $\mathcal{D}$-minimal logical consequence of T' in $\mathcal{W}$, or the negation of the universal closure of φ is a $\mathcal{D}$-minimal *deductive* consequence of T' in $\mathcal{W}$. It is then legitimate, on the basis of K, to *induce* φ since in case φ is not true in some intended model of a possible knowledge base T' that contains K, then a finite subset of T' will eventually enable to *conclusively* refute φ.

The compactness property and the weak compactness property can both be generalized to a property of *β-weak compactness*, where β is an ordinal, the former being the case $\beta = 0$, the latter being the case $\beta = 1$. Given a possible knowledge base T, layer β of the hierarchy built over T will consist of the members of $\mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ that satisfy the β-weak compactness property. The β-weak compactness property essentially states that on the basis of a finite subset K of T, it is legitimate to infer φ because for all possible knowledge bases T', the following holds. Assume that K is a subset of T', but that φ is not a $\mathcal{D}$-minimal logical consequence of T' in $\mathcal{W}$. Then T' can *$\mathcal{D}$-minimally β-refute φ in $\mathcal{W}$*, in the sense that the negation of the universal closure of φ occurs in the hierarchy over T' *below* layer β.

We have roughly described the *layers* that make up the first *level* of the hierarchy built over some possible knowledge base. Reiterating the process will create other levels above the first level, indexed by an ordinal α greater than 1, partitioned into layers indexed by an ordinal β. The hierarchies of $\mathcal{D}$-minimal logical consequences in $\mathcal{W}$ of a possible knowledge base are formally defined next. Note that from level 2 onwards, the property of β-weak compactness is applied to a finite set of formulas for which, in the general case, we have to know a bit more than the mere fact that all of its members have been already put on a lower level of the hierarchy being built: for a given possible knowledge base T, it is necessary to be able to distinguish between members of $\mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ that *do* belong to T, and members of $\mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ that *only possibly* belong to T.

Definition 12 *Let $\alpha, \beta \in \mathrm{Ord}$ with $\alpha \neq 0$ be given, and suppose that $\Lambda^{\mathcal{P}}_{\alpha',\beta'}(T')$ has been defined for all α' $\beta' \in \mathrm{Ord}$ with $\alpha' \neq 0$ and for all $T' \in \mathcal{B}$. Let a possible knowledge base T be given. We define $\Lambda^{\mathcal{P}}_{\alpha,\beta}(T)$ to be the set of all $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ for which there exists a finite $K \subseteq T$ and a finite $H \subseteq \bigcup_{(\alpha',\beta') \prec (\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ such that for all $T' \in \mathcal{B}$:*

($\star$) *if $K \subseteq T'$, $H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$ and $\neg\forall\varphi \notin \bigcup_{\beta' < \beta} \Lambda^{\mathcal{P}}_{\alpha,\beta'}(T')$ then $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$.*

In the previous definition, φ is intuitively put on layer β of level α of the hierarchy over T, on the basis of the sets K and H, thanks to the property of β-weak compactness formalized as ($\star$). For technical purposes, we need to keep track of the sets K and H that enable to place φ in the hierarchy over T. The next notation is introduced for this purpose.

Notation 13 *We denote by* $\mathfrak{d}$ *the set of quintuples of the form* $(K, H, \alpha, \beta, \varphi)$ *where* α, β *are ordinals with* $\alpha \neq 0$*,* K, H *are finite subsets of* $\mathcal{L}$*, and* φ *is a member of* $\mathcal{L}$ *such that for all* $T \in \mathcal{B}$*:*

if $K \subseteq T$*,* $H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ *and* $\neg\forall\varphi \notin \bigcup_{\beta' < \beta} \Lambda^{\mathcal{P}}_{\alpha,\beta'}(T)$ *then* $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$*.*

Note how the logical hierarchies can be defined in terms of $\mathfrak{d}$:

Property 14 *For all* $\alpha, \beta \in \mathrm{Ord}$ *with* $\alpha \neq 0$ *and for all* $T \in \mathcal{B}$*,* $\Lambda^{\mathcal{P}}_{\alpha,\beta}(T)$ *is the set of all* $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ *for which there exists a finite* $K \subseteq T$ *and a finite* $H \subseteq \bigcup_{(\alpha',\beta') \prec (\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ *with* $(K, H, \alpha, \beta, \varphi) \in \mathfrak{d}$*.*

We will use the straightforward property below that compares pairs of hierarchies for two related paradigms that share the same vocabulary, the same language and the same set of possible data, but such that one of the paradigms has more possible worlds and more possible facts than the other.

Property 15 *Let* $\mathcal{W}' \subseteq \mathcal{W}$ *and* $\mathcal{F}' \subseteq \mathcal{F}$ *be given. Set* $\mathcal{P}' = (\mathcal{V}, \mathcal{L}, \mathcal{W}', \mathcal{D}, \mathcal{F}')$*. Let* $\mathfrak{M} \in \mathcal{W}'$ *and* $F \subseteq \mathcal{F}'$ *with* $\mathfrak{M} \models F$ *be given. Let* T *be the theory* $\mathrm{Diag}_{\mathcal{D}}(\mathfrak{M}) \cup K$*. For all* $\alpha, \beta \in \mathrm{Ord}$ *with* $\alpha \neq 0$*,* $\Lambda^{\mathcal{P}}_{\alpha,\beta}(T) \subseteq \Lambda^{\mathcal{P}'}_{\alpha,\beta}(T)$*.*

Note that the compactness property of first-order logic implies that in the particular case where $\mathcal{L} = \mathcal{L}^{S}_{\omega\omega}$, $\mathcal{W}$ is the class of all structures, $\mathcal{D} = \emptyset$, and $\mathcal{F} = \mathcal{L}$ then for all $T \in \mathcal{B}$, $\Lambda^{\mathcal{P}}_{1,0}(T) = \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T) = \{\varphi \in \mathcal{L} : T \models \varphi\}$.

The logical, model-theoretic complexity (in $\mathcal{P}$) of a formula is then naturally defined as follows.

Definition 16 *Let nonnull ordinals* α, β *and a formula* φ *be given.*

We say that φ *is* $\Sigma^{\mathcal{P}}_{\alpha,\beta}$ *just in case for all* $T \in \mathcal{B}$*, if* φ *is a* $\mathcal{D}$*-minimal logical consequence of* T *in* $\mathcal{W}$*, then* $\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\gamma}(T)$ *for some ordinal* $\gamma < \beta$*.*

We say that φ *is* $\Pi^{\mathcal{P}}_{\alpha,\beta}$ *iff* $\neg\forall\varphi$ *is* $\Sigma^{\mathcal{P}}_{\alpha,\beta}$*.*

We say that φ *is* $\Delta^{\mathcal{P}}_{\alpha,\beta}$ *iff* φ *is both* $\Sigma^{\mathcal{P}}_{\alpha,\beta}$ *and* $\Pi^{\mathcal{P}}_{\alpha,\beta}$*.*

The following property can immediately be derived from Definitions 12 and 16. We will use it in the case $\alpha = 1$ when we relate the logical hierarchies to classification with less than β mind changes.

Property 17 *For all nonnull* $\alpha, \beta \in \mathrm{Ord}$ *and* $\varphi \in \mathcal{L}$*,* φ *is* $[\Sigma^{\mathcal{P}}_{\alpha,\beta}, \Pi^{\mathcal{P}}_{\alpha,\beta}]$ *iff there exists a subset* $\mathcal{B}'$ *of* $\mathcal{B}$ *and a sequence* $(\beta_T)_{T \in \mathcal{B}'}$ *of ordinals smaller than* β *with the following property.*

- *Every member of* $\mathcal{B}'$ $\mathcal{D}$*-minimally decides* φ *in* $\mathcal{W}$*.*
- $\mathcal{B}'$ *contains all* $T \in \mathcal{B}$ *such that* $[T \models^{\mathcal{D}}_{\mathcal{W}} \varphi, T \models^{\mathcal{D}}_{\mathcal{W}} \neg\forall\varphi]$*.*

– *For all $T \in \mathcal{B}'$, there exists $K \subseteq T$ and $H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ such that for all $T' \in \mathcal{B}$ with $K \subseteq T'$ and $H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$, $T' \in \mathcal{B}'$ and:*

$$\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T') \text{ iff } \varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T) \text{ whenever } \beta_T \leq \beta_{T'}.$$

Corollary 18 *Let $\mathcal{B}'$ be the set of members of $\mathcal{B}$ that $\mathcal{D}$-minimally decide φ in $\mathcal{W}$. For all nonnull $\alpha, \beta \in \mathrm{Ord}$ and $\varphi \in \mathcal{L}$, φ is $\Delta^{\mathcal{P}}_{\alpha,\beta}$ iff there exists a sequence $(\beta_T)_{T\in\mathcal{B}'}$ of ordinals smaller than β with the following property. For all $T \in \mathcal{B}'$, there exists $K \subseteq T$ and $H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ such that for all $T' \in \mathcal{B}$ with $K \subseteq T'$ and $H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$, $T' \in \mathcal{B}'$ and:*

$$\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T') \text{ iff } \varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T) \text{ whenever } \beta_T \leq \beta_{T'}.$$

3.4 Derivations

This section introduces technical notions and lemmas that will be needed to prove the main results of the paper. In particular, we need to show that it is possible to modify the property of β-weak compactness expressed as $(\star)$ in Definition 12 and replace the condition '$H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$' by '$H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T')$.' The basic idea is *not* to consider all possible knowledge bases T' such that all members of H are $\mathcal{D}$-minimal logical consequences of T' in $\mathcal{W}$, but only those possible knowledge bases T' such that all members of H have already been derived and put in the hierarchy built over T' below level α. The last result in this section shows that if, in Definition 12, '$H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$' is replaced by the stronger condition '$H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T')$,' then the logical hierarchies remain unaffected. It will be derived from a still stronger alternative to the property of β-weak compactness. Basically, we can make sure that it is possible to *trace* in H how each member ψ of H can be derived, on the basis of a finite subset H_ψ of H such that each member of H_ψ either belongs to T or occurs in the hierarchy over T *below* the level on which ψ occurs. This amounts to a stronger property of β-weak compactness, that still defines the same logical hierarchies. We now define the notions needed to state the results that have just been introduced.

Definition 19 *A* trace *is a sequence of the form*

$$(K, (\alpha_1, \beta_1, H_1), \ldots, (\alpha_i, \beta_i, H_i))$$

where $i \in \mathbb{N}$, K is a finite set of formulas, $H_1, \ldots, H_i$ are nonempty, finite sets of formulas, $\alpha_1, \ldots, \alpha_i$ are nonnull ordinals and $\beta_1 \ldots, \beta_i$ are ordinals such that for all nonnull $j < i$, $(\alpha_j, \beta_j) \prec (\alpha_{j+1}, \beta_{j+1})$.

The height bound *of a trace of the form (K) is equal to 0. The* height bound *of a trace that ends with a triple of the form (α, β, H) is equal to (α, β).*

When a trace contains a member of the form $(\alpha, \beta, \{\varphi\})$, this member is usually represented by (α, β, φ).

Definition 20 *A* derivation *is a trace of the form $\sigma \star ((\alpha, \beta, \varphi))$ where σ is a trace whose height bound is smaller than $(\alpha, 0)$—with the convention that* 0 *is smaller than $(\alpha, 0)$.*

A derivation of the form $\sigma \star ((\alpha, \beta, \varphi))$ is said to be a derivation of *φ.*

Definition 21 *Given $T \in \mathcal{B}$, a trace $(K, (\alpha_1, \beta_1, H_1), \dots, (\alpha_i, \beta_i, H_i))$ is said to be* from T (in $\mathcal{P}$) *just in case $K \subseteq T$ and $\bigcup_{0<j\leq i} H_j \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$.*

As a counterpart to $\mathfrak{d}$, introduced in Notation 13 as a means to describe the logical hierarchies, we now define a set $\mathfrak{D}$ as a means to describe hierarchies based on the alternative notion of β-weak compactness discussed at the beginning of the section.

Notation 22 *Let $\mathfrak{D}$ be the* (*unique*) *set of derivations that satisfies the following conditions. Let $\varphi \in \mathcal{L}$, $\alpha, \beta \in \mathrm{Ord}$ with $\alpha \neq 0$, and a trace σ whose height bound is smaller than $(\alpha, 0)$ be given. Assume that for all members of σ of the form (α', β', H) and for all $\psi \in H$, if τ is the longest initial segment of σ whose height bound is smaller than $(\alpha', 0)$, then $\tau\star((\alpha', \beta', \psi)) \in \mathfrak{D}$. Then $\sigma\star((\alpha, \beta, \varphi))$ belongs to $\mathfrak{D}$ iff the following holds. For all possible knowledge bases T, if σ is from T and $\varphi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$, then $\mathfrak{D}$ contains a derivation of $\neg\forall\varphi$ from T whose height bound is at least equal to $(\alpha, 0)$ and smaller than (α, β).*

The following lemma parallels Property 12. It is an interesting result in its own right, but the main reason to include it here is that it is an essential technical tool for the proof of more fundamental results.

Lemma 23 *Given $T \in \mathcal{B}$, $\alpha, \beta \in \mathrm{Ord}$ with $\alpha \neq 0$, and $\varphi \in \mathcal{L}$, $\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\beta}(T)$ iff there exists a derivation of φ from T in $\mathfrak{D}$ whose height bound is (α, β) at most.*

Proof. Given a nonnull $\alpha \in \mathrm{Ord}$, $\beta \in \mathrm{Ord}$, a finite $K \subseteq \mathcal{L}$, a finite $H \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$, and a derivation $\sigma = (E, (\alpha_1, \beta_1, H_1), \dots, (\alpha_i, \beta_i, H_i), (\alpha_{i+1}, \beta_{i+1}, \psi))$, say that $(K, H, \alpha, \beta, \varphi)$ is a *projection of* σ iff $\varphi = \psi$, $\alpha = \alpha_{i+1}$, $\beta = \beta_{i+1}$, $K = E$ and $H = \bigcup_{0<j\leq i} H_i$. Let X be the set of quintuples of the form $(K, H, \alpha, \beta, \varphi)$ that are the projection of some member of $\mathfrak{D}$. Let nonnull $\alpha \in \mathrm{Ord}$ and $\beta \in \mathrm{Ord}$ be given, and suppose that for all $(K, H, \alpha', \beta', \varphi) \in X$ with $(\alpha', \beta') \prec (\alpha, 0)$, $(K, H, \alpha', \beta', \varphi) \in \mathfrak{d}$. Let $\sigma = (K, H, \alpha, \beta, \varphi) \in X$ and $T \in \mathcal{B}$ be such that $K \subseteq T$, $H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ and $\varphi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$. Choose a derivation of the form $\tau \star ((\alpha, \beta, \varphi))$ in X which σ is a projection of. Note that τ is from T. Then $\mathfrak{D}$ contains a derivation of $\neg\forall\varphi$ from T whose height bound is at least equal to $(\alpha, 0)$ and smaller than (α, β), which by assumption implies that $\neg\forall\varphi$ belongs to $\bigcup_{\beta'<\beta} \Lambda^{\mathcal{P}}_{\alpha,\beta'}(T)$. It follows immediately that $\sigma \in \mathfrak{d}$. We conclude that for all $T \in \mathcal{B}$, if there exists a derivation of φ from T in $\mathfrak{D}$ whose height bound is (α, β) at most, then $\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\beta}(T)$.

Given two functions U and V defined on $\mathcal{B} \times (\mathrm{Ord} \setminus \{0\}) \times \mathrm{Ord} \times \mathcal{L}$, we say that (U, V) is a *possible history* iff the following holds. Let $\alpha, \beta \in \mathrm{Ord}$ with $\alpha \neq 0$, $T \in \mathcal{B}$, and a member φ of $\Lambda^{\mathcal{P}}_{\alpha,\beta}(T) \setminus \bigcup_{(\alpha',\beta')\prec(\alpha,\beta)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ be given. Then there exists a finite $K \subseteq T$, an integer k and for all nonnull $i \leq k$, a nonnull ordinal $\alpha_i < \alpha$, an ordinal β_i and a member φ_i of $\Lambda^{\mathcal{P}}_{\alpha_i,\beta_i}(T) \setminus \bigcup_{(\alpha',\beta')\prec(\alpha_i,\beta_i)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ such that:

- $(K, \{\varphi_1, \ldots, \varphi_k\}, \alpha, \beta, \varphi) \in \mathfrak{d}$,
- $U(T, \alpha, \beta, \varphi) = \{(\alpha_i, \beta_i, \varphi_i) : 1 \leq i \leq k\}$, and
- $V(T, \alpha, \beta, \varphi) = K$.

Let a possible history $h = (U, V)$, $\alpha, \beta \in \mathrm{Ord}$ with $\alpha \neq 0$, $T \in \mathcal{B}$, and a finite $H \subseteq \Lambda^{\mathcal{P}}_{\alpha,\beta}(T) \setminus \bigcup_{(\alpha',\beta')\prec(\alpha,\beta)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ be given. We define by induction on p two families of sets, $(X^h_p(T, \alpha, \beta, H))_{p\in\mathbb{N}}$ and $(Y^h_p(T, \alpha, \beta, H))_{p\in\mathbb{N}}$. Set $X^h_0(T, \alpha, \beta, H) = \{(\alpha, \beta, \varphi) : \varphi \in H\}$ and $Y^h_0(T, \alpha, \beta, H) = \emptyset$. Let $p \in \mathbb{N}$ be given, and suppose that $X^h_p(T, \alpha, \beta, H)$ and $Y^h_p(T, \alpha, \beta, H)$ have been defined. Define $X^h_{p+1}(T, \alpha, \beta, H)$ as the union of $X^h_p(T, \alpha, \beta, H)$ with all sets of the form $U(T, \alpha', \beta', \psi)$, and $Y^h_{p+1}(T, \alpha, \beta, H)$ as the union of $Y^h_p(T, \alpha, \beta, \varphi)$ with all sets of the form $V(T, \alpha', \beta', \psi)$, where (α', β', ψ) describes $X_p(T, \alpha, \beta, H)$. Set:

- $X^h(T, \alpha, \beta, H) = \bigcup_{p\in\mathbb{N}} X^h_p(T, \alpha, \beta, H)$;
- $Y^h(T, \alpha, \beta, H) = \bigcup_{p\in\mathbb{N}} Y^h_p(T, \alpha, \beta, H)$.

Clearly, $X^h(T, \alpha, \beta, H)$ and $Y^h(T, \alpha, \beta, H)$ are both finite sets. We denote by $\sigma^h(T, \alpha, \beta, H)$ the (unique) trace with both following properties. The first member of $\sigma^h(T, \alpha, \beta, H)$ is $Y^h(T, \alpha, \beta, H)$. For all $\alpha', \beta' \in \mathrm{Ord}$ and finite $H' \subseteq \mathcal{L}$, (α', β', H') occurs in $\sigma^h(T, \alpha, \beta, H)$ iff there exists a member of $X^h(T, \alpha, \beta, H)$ of the form $(\alpha', \beta', \varphi)$, and H' is the set of all $\varphi \in \mathcal{L}$ such that $(\alpha', \beta', \varphi)$ belongs to $X^h(T, \alpha, \beta, H)$. Note that for all members φ of $\Lambda^{\mathcal{P}}_{\alpha,\beta}(T) \setminus \bigcup_{(\alpha',\beta')\prec(\alpha,\beta)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$, $\sigma^h(T, \alpha, \beta, \{\varphi\})$ is a derivation of φ from T. Define Y as the set of all derivations that are of the form $\sigma^h(T, \alpha, \beta, \{\varphi\})$ where h is a possible history, α a nonnull ordinal, β an ordinal, T a possible knowledge base, and φ a member of $\Lambda^{\mathcal{P}}_{\alpha,\beta}(T) \setminus \bigcup_{(\alpha',\beta')\prec(\alpha,\beta)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$. Clearly, for all α, β with $\alpha > 0$, $T \in \mathcal{B}$ and $\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\beta}(T) \setminus \bigcup_{(\alpha',\beta')\prec(\alpha,\beta)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$, there exists a derivation of φ from T in Y whose height bound is equal to (α, β). Also, for all $\alpha, \beta \in \mathrm{Ord}$ with $\alpha \neq 0$, for all traces σ, for all $\varphi \in \mathcal{L}$, for all members of σ of the form (α', β', H), and for all $\psi \in H$, if $\sigma \star ((\alpha, \beta, \varphi)) \in Y$ and τ is the longest initial segment of σ whose height bound is smaller than $(\alpha', 0)$, then $\tau \star ((\alpha', \beta', \psi)) \in Y$. Let ordinals α, β with $\alpha > 0$ be given, and suppose that for all nonnull $\alpha' \in \mathrm{Ord}$ and $\beta' \in \mathrm{Ord}$ with $(\alpha', \beta') \prec (\alpha, \beta)$, for all traces σ, for all $T \in \mathcal{B}$, and for all $\varphi \in \mathcal{L}$, if σ is from T, $\varphi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ and $\sigma \star ((\alpha', \beta', \varphi))$ belongs to Y, then Y contains a derivation of $\neg\forall\varphi$ from T whose height bound is at least equal to $(\alpha', 0)$ and smaller than (α', β'). Let a trace σ, $T \in \mathcal{B}$, and $\varphi \in \mathcal{L}$ be such that $\sigma \star ((\alpha, \beta, \varphi))$ belongs to Y, σ is from T, $\varphi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$, and $\sigma \star ((\alpha, \beta, \varphi))$ belongs to Y. We can choose a possible history h such that $\sigma \star ((\alpha, \beta, \varphi)) = \sigma^h(T, \alpha, \beta, \{\varphi\})$, and it is easily verified that $\neg\forall\varphi$ belongs to $\bigcup_{\beta'<\beta} \Lambda^{\mathcal{P}}_{\alpha,\beta'}(T)$. Hence Y contains a derivation of $\neg\forall\varphi$ from T whose height bound is at least equal to $(\alpha, 0)$ and smaller than (α, β). It follows immediately that $Y \subseteq \mathfrak{D}$. We conclude that for all $T \in \mathcal{B}$, if $\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\beta}(T)$ then there exists a derivation of φ from T in $\mathfrak{D}$ whose height bound is (α, β) at most. This completes the proof of the lemma.

The following application of Lemma 23 will also be used in the proofs of the main results.

Lemma 24 *Let $\alpha, \beta \in \mathrm{Ord}$ with $\alpha \neq 0$ and $T \in \mathcal{B}$ be given. For all $\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\beta}(T)$, there exists a finite $K \subseteq T$ and a finite $H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ such that $(K, H, \alpha, \beta, \varphi) \in \mathfrak{d}$ and for all $T' \in \mathcal{B}$, the following are equivalent.*

- *$K \subseteq T'$ and $H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$;*
- *$K \subseteq T'$ and $H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T')$.*

Proof. Let a member φ of $\mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ and a member $(K, H, \alpha, \beta, \varphi)$ of $\mathfrak{d}$ be such that $K \subseteq T$ and $H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$. By Lemma 23, given $\xi \in H$, we can choose a derivation σ_ξ of ξ from T in $\mathfrak{D}$, such that the height bound of σ_ξ is smaller than $(\alpha, 0)$. For all members ξ of H, let K_ξ denote the first member of ξ. Set $K' = K \cup \bigcup_{\xi\in H} K_\xi$. Let H' be the set of all $\psi \in \mathcal{L}$ for which there exists $\xi \in H$ such that σ_ξ contains a triple of the form (α', β', H'') with $\psi \in H''$. Note that $K \subseteq K'$ and $H \subseteq H'$, which implies that $(K', H', \alpha, \beta, \varphi) \in \mathfrak{d}$. Also note that $K' \subseteq T$ and $H' \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$. Moreover, it is easy to verify that for all possible knowledge bases T', $K' \subseteq T'$ and $T' \models^{\mathcal{D}}_{\mathcal{W}} H'$ iff $K' \subseteq T'$ and $H' \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T')$. The lemma follows immediately.

4 Connections with topology

4.1 Topology hierarchies

Our aim now is to show that the logical hierarchies are fundamentally related to the Borel and the difference hierarchies over a natural topological space defined from $\mathcal{W}$ and $\mathcal{D}$. We first define the Borel and the difference hierarchies over an arbitrary topological space, in a way that is appropriate for our purposes.

Borel hierarchies are usually defined over topological spaces whose closed sets are countable intersections of open sets. This condition will not be satisfied when $\mathcal{D}$ is not closed under $\neg$ (in particular, when $\mathcal{P}$ represents a learning paradigm where learning is from positive data only). So slight adjustments are needed in order to define Borel hierarchies over more general topological spaces, and can be described as follows. Let a set X and a set B of subsets of X be given. When we consider the topology over X generated by B, yielding a topological space $\mathcal{X}$, we call the sets built from B by finite unions and finite intersections, the Π_0 *Borel* sets of $\mathcal{X}$. Their complements are the Σ_0 *Borel* sets of $\mathcal{X}$. Let $\alpha > 0$ be given. The Σ_α *Borel* sets of $\mathcal{X}$ are built from the Π_β Borel subsets of X, with $\beta < \alpha$, by countable unions. Their complements are the Π_α *Borel* sets of $\mathcal{X}$. Subsets of X that are both $\Sigma_{\alpha,\beta}$ and $\Pi_{\alpha,\beta}$ Borel in $\mathcal{X}$, are said to be Δ_α *Borel in* $\mathcal{X}$.

The difference hierarchy introduces a further granularity in the Borel hierarchy. The natural way to introduce the difference hierarchy in this framework is as follows.

Definition 25 *Let a nonnull ordinal α, a topological space $\mathcal{X}$ over a set X, and a subset Z of X be given.*

We say that Z is $\Sigma_{\alpha,1}$ Borel in $\mathcal{X}$ iff Z is Σ_α Borel in $\mathcal{X}$.

Given an ordinal $\beta > 1$, we say that Z is $\Sigma_{\alpha,\beta}$ Borel in $\mathcal{X}$ iff there exists two families $(A_i)_{i\in\mathbb{N}}$ and $(Z_i)_{i\in\mathbb{N}}$ of subsets of X and a family $(\beta_i)_{i\in\mathbb{N}}$ of nonnull ordinals smaller than β such that the following holds.

- *For all $i \in \mathbb{N}$, A_i is Σ_α Borel in $\mathcal{X}$ and Z_i is Σ_{α,β_i} Borel in $\mathcal{X}$.*
- *For all $i, i' \in \mathbb{N}$ and $x \in A_i \cap A_{i'}$, $x \in Z_i$ iff $x \in Z_{i'}$.*
- *$Z = \bigcup_{i\in\mathbb{N}}(A_i \setminus Z_i)$.*

Given a nonnull ordinal β, we say that Z is $\Pi_{\alpha,\beta}$ Borel in $\mathcal{X}$ iff $X \setminus Z$ is $\Sigma_{\alpha,\beta}$ Borel in $\mathcal{X}$.

Given a nonnull ordinal β, we say that Z is $\Delta_{\alpha,\beta}$ Borel in $\mathcal{X}$ iff Z is both $\Sigma_{\alpha,\beta}$ and $\Pi_{\alpha,\beta}$ Borel in $\mathcal{X}$.

The previous definition is equivalent to the classical definition (see [11]) of the difference hierarchy over the Σ_α Borel sets of $\mathcal{X}$ for a given $\alpha > 0$, as we now show.

Proposition 26 *Let a topological space $\mathcal{X}$ over a set X be given. For all nonnull $\alpha \in \mathrm{Ord}$, nonnull countable $\beta \in \mathrm{Ord}$ and $Z \subseteq X$, Z is $\Sigma_{\alpha,\beta}$ Borel in $\mathcal{X}$ iff there exists a $\subseteq$-increasing sequence $(Y_\gamma)_{\gamma<\beta}$ of Σ_α Borel subsets of X with the following property. For all $x \in X$, $x \in Z$ iff $x \in \bigcup_{\gamma<\beta} Y_\gamma$ and the parity of the least $\gamma < \beta$ such that $x \in Y_\gamma$ is opposite to the parity of β.*

Proof. Proof is by induction on ordinals. Let a nonnull $\alpha \in \mathrm{Ord}$ and a nonnull countable $\beta \in \mathrm{Ord}$ be given. Suppose that for all nonnull $\alpha', \beta' \in \mathrm{Ord}$ with $(\alpha', \beta') \prec (\alpha, \beta)$ and for all $Z \subseteq X$, if Z is $\Sigma_{\alpha',\beta'}$ Borel in $\mathcal{X}$ then there exists a $\subseteq$-increasing sequence $(Y_\gamma)_{\gamma<\beta'}$ of $\Sigma_{\alpha'}$ Borel subsets of X such that for all $x \in X$, $x \in Z$ iff $x \in \bigcup_{\gamma<\beta'} Y_\gamma$ and the parity of the least ordinal $\gamma < \beta'$ such that $x \in Y_\gamma$ is opposite to the parity of β'. Let $Z \subseteq X$ be $\Sigma_{\alpha,\beta}$ Borel in $\mathcal{X}$. Suppose that $\beta = 1$. Set $Y_0 = Z$. Then $(Y_\gamma)_{\gamma<\beta}$ is a $\subseteq$-increasing sequence of Σ_α Borel subsets of X, for all $x \in X$, $x \in Z$ iff $x \in \bigcup_{\gamma<\beta} Y_\gamma$ and the parity of the least $\gamma < \beta$ such that $x \in Y_\gamma$ is opposite to the parity of β. Suppose that $\beta > 1$. Let $(A_i)_{i\in\mathbb{N}}$ be a family of Σ_α Borel subsets of X, let $(\beta_i)_{i\in\mathbb{N}}$ be a family of nonnull ordinals smaller than β, and let $(Z_i)_{i\in\mathbb{N}}$ be a family of subsets of X such that:

- for all $i \in \mathbb{N}$, Z_i is Π_{α,β_i} Borel in $\mathcal{X}$;
- for all $i, j \in \mathbb{N}$ and $x \in A_i \cap A_j$, $x \in Z_i$ iff $x \in Z_j$;
- $Z = \bigcup_{i\in\mathbb{N}}(A_i \cap Z_i)$.

Without loss of generality, we can assume that:

- if $\beta = \gamma + 1$ for some ordinal γ, then $\beta_i = \gamma$ for all $i \in \mathbb{N}$;
- if β is a limit ordinal, then β_i is odd for all $i \in \mathbb{N}$.

By inductive hypothesis, for all $i \in \mathbb{N}$ we can choose a $\subseteq$-increasing sequence $(Y_{i,\gamma})_{\gamma<\beta_i}$ of Σ_α subsets of X such that for all $x \in X$, $x \notin Z_i$ iff $x \in \bigcup_{\gamma<\beta_i} Y_{i,\gamma}$ and the parity of the least ordinal $\gamma < \beta_i$ such that $x \in Y_{i,\gamma}$ is opposite to the parity of β_i. For all ordinals $\gamma < \beta$, set:

$$Y_\gamma = \bigcup\{A_i : i \in \mathbb{N},\ \beta_i \leq \gamma\} \cup \bigcup\{A_i \cap Y_{i,\gamma} : i \in \mathbb{N},\ \gamma < \beta_i\}.$$

Clearly, $(Y_\gamma)_{\gamma<\beta}$ is a $\subseteq$-increasing sequence of Σ_α Borel subsets of X. Let $x \in X$ be given. Assume that $x \in Z$. Choose $p \in \mathbb{N}$ such that $x \in A_p \cap Z_p$. Note that $x \in Y_{\beta_p}$. Let γ be the least ordinal smaller than β such that $x \in Y_\gamma$. We distinguish two cases.

Case 1: there exists $q \in \mathbb{N}$ such that $\gamma < \beta_q$ and $x \in A_q \cap Y_{q,\gamma}$. By the choice of γ, γ is the least ordinal $\gamma' < \beta_q$ such that $x \in Y_{q,\gamma'}$. Moreover, the fact that $x \in (A_p \cap Z_p) \cup A_q$ implies that $x \in Z_q$. We infer that γ and β_q have the same parity. Hence γ and β have opposite parities.

Case 2: for all $i \in \mathbb{N}$ such that $\gamma < \beta_i$, $x \notin A_i \cap Y_{i,\gamma}$. Since $x \in Y_\gamma$, we infer that γ is equal to β_i for some $i \in \mathbb{N}$. Hence γ and β have opposite parities.

Now assume that $x \in \bigcup_{\gamma<\beta} Y_\gamma$, denote by γ the least ordinal smaller than β such that $x \in Y_\gamma$, and assume that γ and β have opposite parities. Clearly, we can choose $p \in \mathbb{N}$ such that $x \in A_p$. We distinguish two cases again.

Case 1: $x \notin \bigcup_{\gamma'<\beta_p} Y_{p,\gamma'}$. Then $x \in Z_p$, and we infer that x belongs to Z.

Case 2: $x \in \bigcup_{\gamma'<\beta_p} Y_{p,\gamma'}$. By the choice of γ, γ is the least ordinal $\gamma' < \beta_p$ such that $x \in Y_{p,\gamma'}$. Since γ and β_p have the same parity, $x \in Z_p$, and we infer that x belongs to Z.

So we have verified that for all $x \in X$, $x \in Z$ iff $x \in \bigcup_{\gamma<\beta} Y_\gamma$ and the parity of the least $\gamma < \beta$ such that $x \in Y_\gamma$ is opposite to the parity of β.

Conversely, suppose that for all nonnull $\alpha', \beta' \in \mathrm{Ord}$ with $(\alpha', \beta') \prec (\alpha, \beta)$ and for all $Z \subseteq X$, if there exists a $\subseteq$-increasing sequence $(Y_\gamma)_{\gamma<\beta'}$ of $\Sigma_{\alpha'}$ Borel subsets of X such that for all $x \in X$, $x \in Z$ iff $x \in \bigcup_{\gamma<\beta'} Y_\gamma$ and the parity of the least $\gamma < \beta'$ such that $x \in Y_\gamma$ is opposite to the parity of β', then Z is $\Sigma_{\alpha',\beta'}$ Borel in $\mathcal{X}$. Let a subset Z of X and let a $\subseteq$-increasing sequence $(Y_\gamma)_{\gamma<\beta}$ of Σ_α Borel subsets of X be such that for all $x \in X$, $x \in Z$ iff $x \in \bigcup_{\gamma<\beta} Y_\gamma$ and the parity of the least $\gamma < \beta$ such that $x \in Y_\gamma$ is opposite to the parity of β. If $\beta = 1$ then it is immediately verified that Z is $\Sigma_{\alpha,\beta}$ Borel in $\mathcal{X}$, so suppose that $\beta > 1$. Let $(A_i)_{i\in\mathbb{N}}$ be an enumeration of $\{Y_\gamma : \gamma < \beta,\ \gamma \text{ and } \beta \text{ have opposite parities}\}$. For all $i \in \mathbb{N}$, set $Z_i = A_i \setminus Z$. Clearly, $Z = \bigcup_{i\in\mathbb{N}} A_i \setminus Z_i$. Obviously, for all $i, j \in \mathbb{N}$ and $x \in A_i \cap A_j$, $x \in Z_i$ iff $x \in Z_j$. Moreover, for all $i \in \mathbb{N}$, A_i is Σ_α Borel in $\mathcal{X}$. So to complete the proof of the proposition, it suffices to show that for all $i \in \mathbb{N}$, Z_i is $\Sigma_{\alpha,\gamma}$ Borel in $\mathcal{X}$ for some nonnull $\gamma < \beta$. Let $i \in \mathbb{N}$ be given. Let $\gamma < \beta$ be such that γ and β have opposite parities and $A_i = Y_\gamma$. Let $x \in X$ be given. Suppose that $x \in Z_i$. So $x \in Y_\gamma$. Since $x \notin Z$, the parity of the least $\gamma' < \beta$ such that $x \in Y_{\gamma'}$ is equal to the parity of β . Together with the fact that β and γ have opposite parities, this implies that $x \in Y_{\gamma'}$ for some $\gamma' < \gamma$ and the parity of the least ordinal $\gamma' < \gamma$ such that $x \in Y_{\gamma'}$ is opposite to the parity of γ. Suppose that $x \notin Z_i$. If $x \in Y_\gamma$ then $x \in A_i$, hence $x \in Z$, and the parity of the least $\gamma' \leq \gamma$ such that $x \in Y_{\gamma'}$ is opposite to the parity of β, hence equal to the parity of γ. So we infer that $x \in Z_i$ iff $x \in Y_{\gamma'}$ for some $\gamma' < \gamma$ and the parity of the least $\gamma' < \gamma$ such that $x \in Y_{\gamma'}$ is opposite to the parity of γ. By inductive hypothesis, this implies that Z_i is $\Sigma_{\alpha,\gamma}$ Borel in $\mathcal{X}$, and we are done.

4.2 Relations between logical and topological complexity

Some topological space is closely related to the logical notions introduced in the first part of the paper. It is defined as follows.

Definition 27 *We denote by* $\mathbb{W}$ *the topological space over* $\mathcal{W}$ *generated by all sets of the form* $\mathrm{Mod}_{\mathcal{W}}(\varphi)$ *where* φ *ranges over* $\mathcal{D}$.

To relate the logical hierarchies to the Borel and to the difference hierarchies over $\mathbb{W}$, we examine how a Σ_α or a $\Sigma_{\alpha,\beta}$ Borel subset of $\mathbb{W}$ can be represented by a formula. Let a nonnull ordinal α, a subset X of $\mathcal{W}$ such that X is Σ_α Borel in $\mathbb{W}$, and a formula φ be given. For φ to represent X, it is not enough that the set of models of φ in $\mathcal{W}$ be precisely equal to X. What is needed is a *hereditary* representation of X. It should be possible to write X as the countable union of a family $(X_i)_{i\in\mathbb{N}}$ of subsets of $\mathcal{W}$, each of which is Π_β Borel in $\mathbb{W}$ for some $\beta < \alpha$, in such a way that each set X_i, $i \in \mathbb{N}$, is itself the set of models in $\mathcal{W}$ of some formula. In other words, it should possible to obtain X from the generators of the topology using countable disjunctions and conjunctions, in such a way that the resulting representation of X can be matched to a formula with members of $\mathcal{D}$ replacing the generators of the topology, countable disjunctions replacing countable unions, and countable conjunctions replacing countable intersections. These considerations lead to the following definition.

Definition 28 *Let a formula* φ *be given.*

We say that φ *is* [$\Sigma^{\mathcal{L}}_0$, $\Pi^{\mathcal{L}}_0$] Borel in $\mathbb{W}$ *iff* $\mathrm{Mod}_{\mathcal{W}}(\varphi)$ *is* [Σ_0, Π_0] *Borel in* $\mathbb{W}$.

Let a nonnull ordinal α *be given. We say that* φ *is* [$\Sigma^{\mathcal{L}}_\alpha$, $\Pi^{\mathcal{L}}_\alpha$] Borel in $\mathbb{W}$ *iff there exists a set* X *of sentences each of which is* [$\Pi^{\mathcal{L}}_\beta$, $\Sigma^{\mathcal{L}}_\beta$] *Borel in* $\mathbb{W}$ *for some* $\beta < \alpha$ *such that* φ *is logically equivalent in* $\mathcal{W}$ *to* [$\bigvee X$, $\bigwedge X$].

Given an ordinal α*, we say that* φ *is* $\Delta^{\mathcal{L}}_\alpha$ Borel in $\mathbb{W}$ *iff* φ *is both* $\Sigma^{\mathcal{L}}_\alpha$ *Borel and* $\Pi^{\mathcal{L}}_\alpha$ *Borel in* $\mathbb{W}$.

The same considerations apply *mutatis mutandis* to the notion of a formula representing a $\Sigma_{\alpha,\beta}$ Borel subset of $\mathbb{W}$.

Definition 29 *Let a nonnull ordinal* α *and a formula* φ *be given.*

We say that φ *is* [$\Sigma^{\mathcal{L}}_{\alpha,1}$, $\Pi^{\mathcal{L}}_{\alpha,1}$] Borel in $\mathbb{W}$ *iff* φ *is* [$\Sigma^{\mathcal{L}}_\alpha$, $\Pi^{\mathcal{L}}_\alpha$] *Borel in* $\mathbb{W}$.

Given $\beta \in \mathrm{Ord}$ *greater than 1, we say that* φ *is* [$\Sigma^{\mathcal{L}}_{\alpha,\beta}$, $\Pi^{\mathcal{L}}_{\alpha,\beta}$] Borel in $\mathbb{W}$ *iff there exists two families* $(\psi_i)_{i\in\mathbb{N}}$ *and* $(\varphi_i)_{i\in\mathbb{N}}$ *of sentences and a family* $(\beta_i)_{i\in\mathbb{N}}$ *of nonnull ordinals smaller than* β *such that the following holds.*

– *For all* $i \in \mathbb{N}$, ψ_i *is* [$\Sigma^{\mathcal{L}}_{\alpha}$, $\Pi^{\mathcal{L}}_{\alpha}$] *Borel in* $\mathbb{W}$.
– *For all* $i \in \mathbb{N}$, φ_i *is* [$\Pi^{\mathcal{L}}_{\alpha,\beta_i}$, $\Sigma^{\mathcal{L}}_{\alpha,\beta_i}$] *Borel in* $\mathbb{W}$.
– *For all* $i, j \in \mathbb{N}$, $\psi_i \wedge \psi_j \models_{\mathcal{W}} \varphi_i \leftrightarrow \varphi_j$.
– φ *is logically equivalent in* $\mathcal{W}$ *to* [$\bigvee_{i\in\mathbb{N}}(\psi_i \wedge \varphi_i)$, $\bigwedge_{i\in\mathbb{N}}(\psi_i \vee \varphi_i)$].

Given a nonnull ordinal β, *we say that* φ *is* $\Delta^{\mathcal{L}}_{\alpha,\beta}$ Borel in $\mathbb{W}$ *iff* φ *is* $\Sigma^{\mathcal{L}}_{\alpha,\beta}$ *Borel and* $\Pi^{\mathcal{L}}_{\alpha,\beta}$ *Borel in* $\mathbb{W}$.

The proof of the property that follows is immediate, by induction on the complexity of Borel sets. The corollary to the property plays an essential role in the proofs of the forthcoming propositions.

Property 30 *Let a formula* φ *be such that* $\mathrm{Mod}_{\mathcal{W}}(\varphi)$ *is a Borel subset of* $\mathbb{W}$. *Then every possible knowledge base* $\mathcal{D}$*-minimally decides* φ *in* $\mathcal{W}$.

Corollary 31 *For all formulas* φ, *if* φ *is* $\mathcal{L}$*-Borel in* $\mathbb{W}$ *then every possible knowledge base* $\mathcal{D}$*-minimally decides* φ *in* $\mathcal{W}$.

The result that follows expresses that topological complexity is a refinement of logical complexity.

Proposition 32 *Let nonnull ordinals* α, β *and a formula* φ *be given. Assume that* φ *is* [$\Sigma^{\mathcal{L}}_{\alpha,\beta}$, $\Pi^{\mathcal{L}}_{\alpha,\beta}$, $\Delta^{\mathcal{L}}_{\alpha,\beta}$] *Borel in* $\mathbb{W}$. *Then* φ *is* [$\Sigma^{\mathcal{P}}_{\alpha,\beta}$, $\Pi^{\mathcal{P}}_{\alpha,\beta}$, $\Delta^{\mathcal{P}}_{\alpha,\beta}$].

Proof. Proof is by double induction on ordinals. Let nonnull ordinals α, β be given, and assume that for all nonnull $\alpha', \beta' \in \mathrm{Ord}$ with $(\alpha', \beta') \prec (\alpha, \beta)$, all formulas that are [$\Sigma^{\mathcal{L}}_{\alpha',\beta'}$, $\Pi^{\mathcal{L}}_{\alpha',\beta'}$, $\Delta^{\mathcal{L}}_{\alpha',\beta'}$] Borel in $\mathbb{W}$ are [$\Sigma^{\mathcal{P}}_{\alpha',\beta'}$, $\Pi^{\mathcal{P}}_{\alpha',\beta'}$, $\Delta^{\mathcal{P}}_{\alpha',\beta'}$]. Let φ be a sentence that is $\Sigma^{\mathcal{L}}_{\alpha,\beta}$ Borel in $\mathbb{W}$. To complete the proof of the proposition, it suffices to show that φ is $\Sigma^{\mathcal{P}}_{\alpha,\beta}$. Suppose that $\beta = 1$. Then $\mathrm{Mod}_{\mathcal{W}}(\varphi)$ is equal to $\mathrm{Mod}_{\mathcal{W}}(\bigvee\{\psi : \psi \in X\})$ for some set X of sentences each of which is $\Pi^{\mathcal{L}}_{0}$ Borel in $\mathbb{W}$ or $\Pi^{\mathcal{L}}_{\alpha',1}$ Borel in $\mathbb{W}$ for some nonnull $\alpha' < \alpha$, and if follows easily from Corollary 31 and the inductive hypothesis that φ is $\Sigma^{\mathcal{P}}_{\alpha,\beta}$. So suppose that $\beta > 1$. For all $i \in \mathbb{N}$, let a sentence ψ_i, a nonnull ordinal $\beta_i < \beta$, and a sentence χ_i be such that the following holds.

– For all $i \in \mathbb{N}$, ψ_i is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$ and χ_i is $\Pi^{\mathcal{L}}_{\alpha,\beta_i}$ Borel in $\mathbb{W}$.
– For all $i, i' \in \mathbb{N}$, $\psi_i \wedge \psi_{i'} \models_{\mathcal{W}} \chi_i \leftrightarrow \chi_{i'}$.
– $\models_{\mathcal{W}} \varphi \leftrightarrow \bigvee_{i\in\mathbb{N}}(\psi_i \wedge \chi_i)$.

Let $T \in \mathcal{B}$ be such that $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. By Corollary 31, we can choose $p \in \mathbb{N}$ such that $T \models^{\mathcal{D}}_{\mathcal{W}} \psi_p \wedge \chi_p$. By inductive hypothesis, ψ_p is $\Sigma^{\mathcal{P}}_{\alpha}$ and χ_p is $\Pi^{\mathcal{P}}_{\alpha,\beta_p}$. Since ψ_p is $\Sigma^{\mathcal{P}}_{\alpha}$, Lemma 24 implies that there exists a finite $K \subseteq T$ and a finite $H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T)$ such that for all $T' \in \mathcal{B}$:

1. $K \subseteq T'$ and $T' \models^{\mathcal{D}}_{\mathcal{W}} H$ iff $K \subseteq T'$ and $H \subseteq \bigcup_{(\alpha',\beta')\prec(\alpha,0)} \Lambda^{\mathcal{P}}_{\alpha',\beta'}(T')$;
2. if $K \subseteq T'$ and $H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$ then $\psi_p \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$.

Set $\mathcal{B}' = \{T' \in \mathcal{B} : K \subseteq T'$ and $H \subseteq \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')\}$. We prove that:

($*$) For all $T' \in \mathcal{B}'$ and for all $\beta' \in \mathrm{Ord}$, $\chi_p \in \Lambda^{\mathcal{P}}_{\alpha,\beta'}(T')$ iff $\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\beta'}(T')$, and $\neg\chi_p \in \Lambda^{\mathcal{P}}_{\alpha,\beta'}(T')$ iff $\neg\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\beta'}(T')$.

Let $T' \in \mathcal{B}'$ be given. So $T' \models^{\mathcal{D}}_{\mathcal{W}} \psi_p$ by 2. Assume that $T' \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. By Corollary 31 again, we can choose $q \in \mathbb{N}$ such that $T' \models^{\mathcal{D}}_{\mathcal{W}} \psi_q \wedge \chi_q$. Since $\psi_q \wedge \psi_p \models_{\mathcal{W}} \chi_q \leftrightarrow \chi_p$, we infer that $T' \models^{\mathcal{D}}_{\mathcal{W}} \chi_p$. Conversely, if $T' \models^{\mathcal{D}}_{\mathcal{W}} \chi_p$ then $T' \models^{\mathcal{D}}_{\mathcal{W}} \bigvee_{i\in\mathbb{N}}(\psi_i \wedge \chi_i)$, hence $T' \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. So we have shown that $T' \models^{\mathcal{D}}_{\mathcal{W}} \varphi$ iff $T' \models^{\mathcal{D}}_{\mathcal{W}} \chi_p$, and equivalently $T' \models^{\mathcal{D}}_{\mathcal{W}} \neg\varphi$ iff $T' \models^{\mathcal{D}}_{\mathcal{W}} \neg\chi_p$. It is then easy to verify ($*$) by induction on β'. Let $T' \in \mathcal{B}$ be such that $T' \not\models^{\mathcal{D}}_{\mathcal{W}} \varphi$. Then $T' \models^{\mathcal{D}}_{\mathcal{W}} \neg\chi_p$, and since $\neg\chi_p$ is $\Sigma^{\mathcal{P}}_{\alpha,\beta_p}$, we infer from 1. and ($*$) that $\neg\varphi \in \Lambda^{\mathcal{P}}_{\alpha,\beta_p}(T)$. We conclude that φ is $\Sigma^{\mathcal{P}}_{\alpha,\beta}$, as wanted.

By Corollary 31, the converse to Proposition 32 can only be established for formulas that are $\mathcal{D}$-minimally decided in $\mathcal{W}$ by every possible knowledge base. It turns out that not further condition is necessary to prove that logical complexity and topological complexity are equivalent notions. This result will easily follow from the next lemma.

Lemma 33 *Suppose that $\mathcal{D} = \emptyset$. Let a nonnull ordinal α, an ordinal β, a trace σ, and a formula φ be such that the height bound of σ is smaller than $(\alpha, 0)$, σ is from a member of $\mathcal{B}$, and $\sigma \star ((\alpha, \beta, \varphi))$ belongs to $\mathfrak{D}$. Let H be the set of formulas that occur in σ.*

- *$\bigwedge H$ is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$.*
- *If $\beta = 0$ then $H \models_{\mathcal{W}} \varphi$. If $\beta > 0$ then $H \models_{\mathcal{W}} \varphi \leftrightarrow \chi$ for some sentence χ that is $\Pi^{\mathcal{L}}_{\alpha,\beta}$ Borel in $\mathbb{W}$.*

Proof. Proof is by double induction on ordinals. Given a trace τ in $\mathfrak{D}$, denote by H_τ the set of formulas occurring in τ. Let a nonnull ordinal α and an ordinal β be given. Assume that for all nonull $\alpha' \in \mathrm{Ord}$, $\beta' \in \mathrm{Ord}$, traces σ' and formulas φ', if $(\alpha', \beta') \prec (\alpha, \beta)$, the height bound of σ' is smaller than $(\alpha', 0)$, $\sigma' \star ((\alpha', \beta', \varphi'))$ belongs to $\mathfrak{D}$, and σ' is from a member of $\mathcal{B}$ then the following holds.

- $\bigwedge H_{\sigma'}$ is $\Sigma^{\mathcal{L}}_{\alpha'}$ Borel in $\mathbb{W}$.
- If $\beta' = 0$ then $H_{\sigma'} \models_{\mathcal{W}} \varphi'$. If $\beta' > 0$ then $H_{\sigma'} \models_{\mathcal{W}} \forall\varphi' \leftrightarrow \chi$ for some sentence χ that is $\Pi^{\mathcal{L}}_{\alpha',\beta'}$ Borel in $\mathbb{W}$.

Let a trace σ and a formula φ be such that the height bound of σ is smaller than $(\alpha, 0)$, σ is from a member of $\mathcal{B}$, and $\sigma \star ((\alpha, \beta, \varphi))$ belongs to $\mathfrak{D}$. Without loss of generality, we can assume that all formulas in $\mathcal{D}$ and all formulas occurring in σ are closed. We first verify that $\bigwedge H_\sigma$ is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$. If the height bound of σ is equal to 0, then $H_\sigma \subseteq \mathcal{D}$, and $\bigwedge H_\sigma$ is trivially $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$. Suppose that the height bound of σ is equal to (α', β') for some nonnull $\alpha' \in \mathrm{Ord}$ and $\beta' \in \mathrm{Ord}$. Let σ' be the largest initial segment of σ whose height bound is smaller than $(\alpha', 0)$. Let trace σ'' be such that $\sigma = \sigma' \star \sigma''$. It follows from the inductive hypothesis that for all sentences ψ that occur in σ'', $\bigwedge H_{\sigma'} \wedge \psi$ is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$. So $\bigwedge H_\sigma$ is logically equivalent to a finite conjunction of sentences that are $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$, hence is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$.

Let X be the set of all traces τ whose height bound is smaller than $(\alpha, 0)$, that are from a member of $\mathcal{B}$, such that all formulas occurring in τ are closed, and such that $\tau\star((\alpha,\gamma,\neg\forall\varphi))$ belongs to $\mathfrak{D}$ for some $\gamma < \beta$ (for later, note that γ can be assumed to be nonnull whenever $\beta > 1$). Set $Y = \{\bigwedge H_\tau \wedge \neg\forall\varphi : \tau \in X\}$. We show that $H_\sigma \models_{\mathcal{W}} \varphi \leftrightarrow \neg \bigvee Y$. (Note that in case $\beta = 0$, this yields $H_\sigma \models_{\mathcal{W}} \varphi$.) For a contradiction, choose $\mathfrak{M} \in \mathcal{W}$ such that $\mathfrak{M} \models H_\sigma \cup \{\neg \bigvee Y, \neg\forall\varphi\}$. Using Corollary 31 together with the hypothesis that $\mathcal{D} = \emptyset$, the derived fact that $\bigwedge H_\sigma$ is $\mathcal{L}$-Borel in $\mathbb{W}$, and the choice of $\mathfrak{M}$ as a model of H_σ, it can be easily derived that σ is from $\mathrm{Diag}_{\mathcal{D}}(\mathfrak{M})$. Hence by Lemma 23, there exists a trace τ and $\gamma < \beta$ such that $\tau\star((\alpha,\gamma,\neg\forall\varphi))$ is a derivation from $\mathrm{Diag}_{\mathcal{D}}(\mathfrak{M})$ in $\mathfrak{D}$. Then $\bigwedge H_\tau \wedge \neg\forall\varphi \in Y$, hence $\mathfrak{M} \models \neg \bigwedge H_\tau \vee \varphi$. But $\mathfrak{M} \not\models \varphi$ by the choice of $\mathfrak{M}$, and $\mathfrak{M} \not\models \neg \bigwedge H_\tau$ since τ is from $\mathrm{Diag}_{\mathcal{D}}(\mathfrak{M})$, contradiction. Conversely, it is clear that if $\mathfrak{M}$ is a model of $H_\sigma \cup \{\varphi\}$ then $\mathfrak{M}$ is a model of $\neg \bigvee Y$. So we have shown that $H_\sigma \models_{\mathcal{W}} \varphi \leftrightarrow \neg \bigvee Y$. Now assume that $\beta > 0$. To complete the proof of the lemma, it suffices to show that $\bigvee Y$ is $\Sigma^{\mathcal{L}}_{\alpha,\beta}$ Borel in $\mathbb{W}$. We distinguish two cases.

Case 1: $\beta = 1$. By inductive hypothesis, for all $\tau \in X$, $\bigwedge H_\tau$ is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$ and $\models_{\mathcal{W}} (\bigwedge H_\tau \wedge \neg\forall\varphi) \leftrightarrow \bigwedge H_\tau$. It follows immediately that $\bigvee Y$ is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$.

Case 2: $\beta > 1$. By inductive hypothesis, for all $\tau \in X$, $\bigwedge H_\tau$ is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$ and there exists a sentence χ_τ and a nonnull $\gamma < \beta$ such that χ_τ is $\Pi^{\mathcal{L}}_{\alpha,\gamma}$ Borel in $\mathbb{W}$ and $H_\tau \models_{\mathcal{W}} \neg\forall\varphi \leftrightarrow \chi_\tau$. Moreover, for all $\tau, \tau' \in X$ and $\mathfrak{M} \in \mathrm{Mod}_{\mathcal{W}}(H_\tau \cup H_{\tau'})$, $\mathfrak{M} \models \chi_\tau$ iff $\mathfrak{M} \models \neg\forall\varphi$ iff $\mathfrak{M} \models \chi_{\tau'}$. Also note that $\models_{\mathcal{W}} \bigvee Y \leftrightarrow \bigvee\{\bigwedge H_\tau \wedge \chi_\tau : \tau \in X\}$. We conclude immediately that $\bigvee Y$ is $\Sigma^{\mathcal{L}}_{\alpha,\beta}$ Borel in $\mathbb{W}$.

We are now in a position to formulate one of the main results of the paper, and prove that logical and topological complexity are equivalent notions, when applied to formulas that are $\mathcal{D}$-minimally decided in $\mathcal{W}$ by every possible knowledge base.

Proposition 34 *Let a formula φ be $\mathcal{D}$-minimally decided in $\mathcal{W}$ by every member of $\mathcal{B}$. For all nonnull ordinals α, β, the following are equivalent.*

- *φ is [$\Sigma^{\mathcal{P}}_{\alpha,\beta}$, $\Pi^{\mathcal{P}}_{\alpha,\beta}$, $\Delta^{\mathcal{P}}_{\alpha,\beta}$].*
- *φ is [$\Sigma^{\mathcal{L}}_{\alpha,\beta}$, $\Pi^{\mathcal{L}}_{\alpha,\beta}$, $\Delta^{\mathcal{L}}_{\alpha,\beta}$] Borel in $\mathbb{W}$.*

Proof. Let nonnull $\alpha, \beta \in \mathrm{Ord}$ be given. By Proposition 32, the second clause implies the first one. Set $\mathcal{P}' = (\mathcal{V}, \mathcal{L}, \mathcal{W}, \mathcal{D}, \emptyset)$. Property 15 implies that if φ is $\Sigma^{\mathcal{P}}_{\alpha,\beta}$ then φ is $\Sigma^{\mathcal{P}'}_{\alpha,\beta}$. Hence in order to prove that the first clause of the proposition implies the second one, we can without loss of generality assume that $\mathcal{D} = \emptyset$. So suppose that $\mathcal{D} = \emptyset$ and φ is $\Sigma^{\mathcal{P}}_{\alpha,\beta}$. Let X be the set of all possible knowledge bases T such that $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. Given $T \in X$, choose a trace σ_T from T and an ordinal $\gamma < \beta$ such that the height bound of σ is smaller than $(\alpha, 0)$ and $\sigma_T \star ((\alpha,\gamma,\varphi)) \in \mathfrak{D}$. For all $T \in X$, let H_T denote the set of universal closures of the formulas that occur in σ_T. Set $\psi = \bigvee\{\bigwedge H_T \wedge \varphi : T \in X\}$. It is immediately verified that $\models_{\mathcal{W}} \varphi \leftrightarrow \psi$. Let a member T of $\mathcal{B}$ be given. By Lemma 33, $\bigwedge H_T$ is $\Sigma^{\mathcal{L}}_{\alpha}$ Borel in $\mathbb{W}$ and there exists a sentence χ_T and a nonnull ordinal $\gamma < \beta$ such that χ_T is $\Pi^{\mathcal{L}}_{\alpha,\gamma}$ Borel in $\mathbb{W}$ and $H_T \models_{\mathcal{W}} \varphi \leftrightarrow \chi_T$. Clearly, for all $T, T' \in X$ and $\mathfrak{M} \in \mathrm{Mod}_{\mathcal{W}}(H_T \cup H_{T'})$, $\mathfrak{M} \models \chi_T$ iff $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \chi_{T'}$. Note that $\models_{\mathcal{W}} \psi \leftrightarrow \bigvee\{\bigwedge H_T \wedge \chi_T : T \in X\}$. It follows that φ is $\Sigma^{\mathcal{L}}_{\alpha,\beta}$ Borel in $\mathbb{W}$.

5 Connections with Inductive inference

5.1 Basic notions

The basic concepts from Inductive inference are immediately adapted to the present framework. The notion of classification which is defined next is, in the absence of computability requirements, more fundamental than the notion of identification in the limit. As usual, it uses a symbol $\sharp$ whose intended meaning is 'no datum presented.'

Definition 35 *A* classifier (for $\mathcal{P}$) *is a partial function* $f : (\mathcal{L} \cup \{\sharp\})^{\star} \rightharpoonup \{0, 1\}$.

It is convenient to talk about *consistent in* $\mathcal{W}$ not only for sets of formulas, but also for members of $(\mathcal{L} \cup \{\sharp\})^{\star}$. Given a member σ of $(\mathcal{L} \cup \{\sharp\})^{\star}$, we denote by $\mathrm{cnt}(\sigma)$ the set of members of $\mathcal{L}$ that occur in σ.

Definition 36 *We say that a member σ of $(\mathcal{L} \cup \{\sharp\})^{\star}$ is* consistent in $\mathcal{W}$ *just in case there exists a possible world $\mathfrak{M}$ such that $\mathfrak{M} \models \mathrm{cnt}(\sigma)$.*

Suppose that $\mathcal{F}$ is empty. Depending on the choice of $\mathcal{D}$, the enumeration of a possible knowledge base (with $\sharp$ possibly occurring in the enumeration), *i.e.*, the enumeration of the $\mathcal{D}$-diagram of a possible world, can correspond to a text, or to an informant, or have no counterpart in Inductive inference. More generally, we want to consider enumerations of possible knowledge bases that, in case $\mathcal{F}$ is nonempty, also go beyond the enumerations of data considered in Inductive inference. We use *environment* as a general term for an enumeration of a possible knowledge base.

Definition 37 *Given a member T of $\mathcal{B}$, an* environment for T *is any member e of $(\mathcal{L} \cup \{\sharp\})^{\mathbb{N}}$ such that for all $\varphi \in \mathcal{L}$, φ occurs in e iff φ belongs to T.*

The concept of classification is defined as follows.

Definition 38 *Let a formula φ and a classifier f be given.*
We say that f positively classifies $\mathcal{B}$ in the limit following φ (in $\mathcal{P}$) *just in case for all $T \in \mathcal{B}$ and environments e for T:*

- *$T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$ iff $\{k \in \mathbb{N} : f(e_{|k}) = 1\}$ is cofinite.*
- *If $\{k \in \mathbb{N} : f(e_{|k}) = 0\}$ is cofinite then $T \models^{\mathcal{D}}_{\mathcal{W}} \neg\forall\varphi$*

We say that f negatively classifies $\mathcal{B}$ in the limit following φ (in $\mathcal{P}$) *just in case for all $T \in \mathcal{B}$ and environments e for T:*

– *$T \models^{\mathcal{D}}_{\mathcal{W}} \neg\forall\varphi$ iff $\{k \in \mathbb{N} : f(e_{|k}) = 0\}$ is cofinite.*
– *If $\{k \in \mathbb{N} : f(e_{|k}) = 1\}$ is cofinite then $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$*

We say that f classifies $\mathcal{B}$ in the limit (in $\mathcal{P}$) *just in case f positively and negatively classifies $\mathcal{B}$ in the limit following φ.*

The concept of classification with a bounded number of mind changes is captured by the following definition. The usual notion of mind change complexity considers the least ordinal β such that *at most*, rather than *less than*, β mind changes are sometimes necessary for the procedure to converge [2, 1, 7]. Note that if β is a limit ordinal, then converging after at most β mind changes can mean either converging after less than β mind changes, or converging after less then $\beta + 1$ mind changes. So the 'less than' formulation is more precise than the 'at most' formulation. But more fundamentally, the 'less than' formulation is justified by the propositions in this part of the paper, which elegantly and simply relate classification with less than β mind changes to the logical and topological notions investigated in the first two parts of the paper.

Definition 39 *Let a nonnull ordinal β, a formula φ, and a classifier f be given. Let X be the set of all $\sigma \in (\mathcal{D} \cup \mathcal{F} \cup \{\sharp\})^\star$ such that σ is consistent in $\mathcal{W}$ and $f(\tau) =\downarrow$ for some initial segment τ of σ. We denote by R_f the binary relation over X such that for all $\sigma, \tau \in X$, $R_f(\sigma, \tau)$ holds iff $\tau \subset \sigma$ and $f(\sigma) \neq f(\tau)$.*

We say that f [positively classifies, negatively classifies, classifies] $\mathcal{B}$ with less than β mind changes following φ (in $\mathcal{P}$) *just in case the length of R_f is defined and smaller than β, and f [positively classifies, negatively classifies, classifies] $\mathcal{B}$ in the limit following φ.*

Finally, we derive some more terminology from Definitions 38 and 39.

Definition 40 *Let a formula φ be given.*

We say that $\mathcal{B}$ is [positively classifiable, negatively classifiable, classifiable] in the limit following φ (in $\mathcal{P}$) *iff some classifier [positively classifies, negatively classifies, classifies] $\mathcal{B}$ in the limit following φ.*

Given a nonnull ordinal β, we say that $\mathcal{B}$ is [positively classifiable, negatively classifiable, classifiable] with less than β mind changes following φ (in $\mathcal{P}$) *iff some classifier [positively classifies, negatively classifies, classifies] $\mathcal{B}$ with less than β mind changes following φ.*

When $\beta = 1$, we also say 'no mind change' *rather than 'less than β mind changes' in the previous expressions.*

Remember that given a classifier f and $\sigma, \tau \in (\mathcal{L} \cup \{\sharp\})^\star$, $f(\sigma) \neq f(\tau)$ whenever $f(\sigma) =\uparrow$ or $f(\tau) =\uparrow$. The next property follows immediately from this observation together with Definition 39.

Property 41 *Let $T \in \mathcal{B}$, $\varphi \in \mathcal{L}$, an environment e for T, and a classifier f be such that f [positively classifies, negatively classifies, classifies] $\mathcal{B}$ in the limit following φ, and $f(\sigma) =\downarrow$ for some $\sigma \subset e$. Then either $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$ and $\{k \in \mathbb{N} : f(e_{|k}) \neq 1\}$ is finite, or $T \models^{\mathcal{D}}_{\mathcal{W}} \neg\forall\varphi$ and $\{k \in \mathbb{N} : f(e_{|k}) \neq 0\}$ is finite.*

The usual notion of *locking sequence* and associated lemma obviously take the following form in this framework.

Definition 42 *Given a possible knowledge base T and a classifier f, a member σ of $(\mathcal{L} \cup \{\sharp\})^\star$ is said to be a* locking sequence for f and T *iff* $\mathrm{cnt}(\sigma) \subseteq T$ *and for all $\tau \in (T \cup \{\sharp\})^\star$, $f(\sigma \star \tau) = f(\sigma)$.*

Lemma 43 *Let a classifier f and a member T of $\mathcal{B}$ be given. Suppose that for every environment e for T, $\{k \in \mathbb{N} : f(e_{|k}) \neq f(e_{|k+1})\}$ is finite. Then for all $\sigma \in (T \cup \{\sharp\})^\star$, there exists a locking sequence for f and T that extends σ.*

5.2 Relations between logical complexity and learnability

In Inductive inference, the class of possible realities is usually countable (for instance, when it is the class of all recursively enumerable languages). This is to be contrasted with the fact that $\mathcal{W}$ or even $\mathcal{B}$, is usually noncountable. When $\mathcal{B}$ is countable, it is easy to obtain a characterization of classification in the limit that generalizes Angluin's finite tell-tail characterization of learnability in the limit [3].

Proposition 44 *Suppose that $\mathcal{B}$ is countable. Let a formula φ be given. Then $\mathcal{B}$ is [positively classifiable, negatively classifiable] in the limit following φ iff for all $T \in \mathcal{B}$ with $[\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T), \neg\forall\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)]$, there exists a finite subset K of T such that:*

($\star$) *for all $T' \in \mathcal{B}$, if $K \subseteq T' \subseteq T$ then $[\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T'), \neg\forall\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')]$.*

Proof. Let a classifier f be such that f positively classifies $\mathcal{B}$ in the limit following φ. Let T be a possible knowledge base such that $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. By Lemma 43, choose a member σ of $(T \cup \{\sharp\})^\star$ such that for all $\tau \in (T \cup \{\sharp\})^\star$ that extend σ, $f(\tau) = 1$. Let $T' \in \mathcal{B}$ with $\mathrm{cnt}(\sigma) \subseteq T' \subseteq T$ be given. Choose an environment e for T' that begins with σ. Since $T' \subseteq T$, $f(e_{|k}) = 1$ for all $k \in \mathbb{N}$ at least equal to the length of σ. Since f positively classifies $\mathcal{B}$ in the limit following φ, we infer that $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$.

Conversely, suppose that condition ($\star$) holds. If $\varphi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ for all $T \in \mathcal{B}$, then $\mathcal{B}$ is trivially positively classifiable in the limit following φ, so suppose otherwise. Fix an enumeration $(T_i)_{i\in\mathbb{N}}$ of all $T \in \mathcal{B}$ such that $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$. Given $i \in \mathbb{N}$, denote by K_i a finite subset of T_i such that for all $T \in \mathcal{B}$, if $K_i \subseteq T \subseteq T_i$ then $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$. Define a classifier f as follows. Set $f(()) =\uparrow$. Let $\sigma \in (\mathcal{L} \cup \{\sharp\})^\star$ be given, and suppose that $f(\sigma)$ has been defined. Let $x \in \mathcal{L} \cup \{\sharp\}$ be given. If there is no $i \in \mathbb{N}$ with $K_i \subseteq \mathrm{cnt}(\sigma \star x) \subseteq T_i$, then $f(\sigma \star x) =\uparrow$. Suppose otherwise, and let i be the least $j \in \mathbb{N}$ such that $K_j \subseteq \mathrm{cnt}(\sigma\star x) \subseteq T_j$. If i is the least $j \in \mathbb{N}$ such that $K_j \subseteq \mathrm{cnt}(\sigma) \subseteq T_j$, then $f(\sigma \star x) = 1$; otherwise $f(\sigma \star x) =\uparrow$. We show that f positively classifies $\mathcal{B}$ in the limit following φ. Let a possible knowledge base T and an environment e for T be given. Suppose that $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$. Choose $i \in \mathbb{N}$ with $T = T_i$. Let $n \in \mathbb{N}$ be such that for all $j \leq i$, if $K_j \subseteq T$ then $K_j \subseteq \mathrm{cnt}(e_{|n})$, and if $T \not\subseteq T_j$ then $\mathrm{cnt}(e_{|n}) \setminus T_j \neq \emptyset$. It is easy to verify that for all $k > n$, $f(e_{|k}) = 1$. Suppose that $\varphi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$. To complete the proof of the proposition it suffices to show that $\{k \in \mathbb{N} : f(e_{|k}) \neq 1\}$ is infinite. Let $m \in \mathbb{N}$ be given, and suppose that $f(e_{|n}) = \downarrow$ for some $n \geq m$ (if there is no such n then we are done). By the definition of f, there is a least $i \in \mathbb{N}$ such that $K_i \subseteq \mathrm{cnt}(e_{|n}) \subseteq T_i$. Since $\varphi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$, it follows from the choice of K_i that $T \not\subseteq T_i$, hence there exists a least $p > n$ such that i is not the least $j \in \mathbb{N}$ with $K_j \subseteq \mathrm{cnt}(e_{|p}) \subseteq T_j$. With the definition of f, we infer that $f(e_{|p}) = \uparrow$. So $\{k \in \mathbb{N} : f(e_{|k}) \neq 1\}$ is infinite, as wanted.

Corollary 45 *Suppose that $\mathcal{B}$ is countable. For all formulas φ, $\mathcal{B}$ is classifiable in the limit following φ iff for all $T \in \mathcal{B}$ that $\mathcal{D}$-minimally decide φ in $\mathcal{W}$, there exists a finite subset K of T such that for all $T' \in \mathcal{B}$:*

$$\text{if } K \subseteq T' \subseteq T, \text{ then } \varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T) \text{ iff } \varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T').$$

The next fundamental result shows that classification with a bounded number of mind changes is perfectly characterized by the property of β-weak compactness on the basis of which the logical hierarchies are defined. Note that no assumption on $\mathcal{P}$ is needed; in particular, $\mathcal{B}$ can be uncountable.

Proposition 46 *For all $\beta \in \mathrm{Ord} \setminus \{0\}$ and $\varphi \in \mathcal{L}$, the following are equivalent.*

- *φ is $[\Sigma^{\mathcal{P}}_{1,\beta}, \Pi^{\mathcal{P}}_{1,\beta}, \Delta^{\mathcal{P}}_{1,\beta}]$.*
- *$\mathcal{B}$ is [positively classifiable, negatively classifiable, classifiable] with less than β mind changes following φ.*

Proof. Let a nonnull ordinal β and a formula φ be given. By Property 17, it suffices to show that $\mathcal{B}$ is positively classifiable with less than β mind changes following φ iff the following holds.

($*$) There exists a subset $\mathcal{B}'$ of $\mathcal{B}$, a sequence $(\beta_T)_{T \in \mathcal{B}'}$ of ordinals smaller than β, and a sequence $(K_T)_{T \in \mathcal{B}'}$ of finite sets of formulas with the following properties.

(i) Every member of $\mathcal{B}'$ $\mathcal{D}$-minimally decides φ in $\mathcal{W}$.
(ii) $\mathcal{B}'$ contains all $T \in \mathcal{B}$ such that $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$.
(iii) For all $T \in \mathcal{B}'$, $K_T \subseteq T$.
(iv) Let $T \in \mathcal{B}'$ and $T' \in \mathcal{B}$ be such that $K_T \subseteq T'$. Then $T' \in \mathcal{B}'$. Moreover, $\beta_T \leq \beta_{T'}$ implies that $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$ iff $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$.

Let f be a classifier that positively classifies $\mathcal{B}$ with less than β mind changes following φ. Let $\mathcal{B}'$ be the set of all $T \in \mathcal{B}$ such that there exists $\sigma \in (T \cup \{\sharp\})^\star$ with $f(\sigma) = \downarrow$. Given $T \in \mathcal{B}'$, let β_T be the least ordinal $\gamma < \beta$ such that there exists $\sigma \in (T \cup \{\sharp\})^\star$ with $\rho_{R_f}(\sigma) = \gamma$, and either $f(\sigma) = 1$ and $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$ or $f(\sigma) = 0$ and $T \models^{\mathcal{D}}_{\mathcal{W}} \neg\forall\varphi$. By Property 41 and the choice of f, conditions (i) and (ii) in ($*$) clearly hold, and $(\beta_T)_{T \in \mathcal{B}'}$ is well defined. Let $T \in \mathcal{B}'$ be given. Suppose without loss of generality that $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. Choose $\sigma_T \in (T \cup \{\sharp\})^\star$ such that $\rho_{R_f}(\sigma_T) = \beta_T$ and $f(\sigma_T) = 1$. Suppose that there exists $T' \in \mathcal{B}$

such that $\mathrm{cnt}(\sigma_T) \subseteq T'$, $\beta_T \leq \beta_{T'}$, and $T' \not\models^{\mathcal{D}}_{\mathcal{W}} \varphi$. Choose an environment e for T' that begins with σ_T. By Property 41 again, $T' \models^{\mathcal{D}}_{\mathcal{W}} \neg\forall\varphi$ and there is a $k \in \mathbb{N}$ greater than the length of σ_T such that $f(e_{|k}) = 0$. But we then infer that $\rho_{R_f}(e_{|k}) < \rho_{R_f}(\sigma_T) \leq \beta_{T'}$, which contradicts the definition of $\beta_{T'}$. Hence conditions (iii) and (iv) in $(*)$ hold if for all $T \in \mathcal{B}'$, we set $K_T = \mathrm{cnt}(\sigma_T)$. So we have shown that if $\mathcal{B}$ is positively classifiable with less than β mind changes following φ, then $(*)$ holds.

Conversely, assume that there exists a subset $\mathcal{B}'$ of $\mathcal{B}$, a sequence $(\beta_T)_{T\in\mathcal{B}'}$ of ordinals smaller than β, and a sequence $(K_T)_{T\in\mathcal{B}'}$ of finite sets of formulas such that conditions (i)–(iv) in $(*)$ hold. Define a classifier f as follows. Let $\sigma \in (\mathcal{L} \cup \{\sharp\})^\star$ be given. Suppose that there exists a least ordinal γ such that for some $T \in \mathcal{B}'$, $\beta_T = \gamma$ and $K_T \subseteq \mathrm{cnt}(\sigma)$. Let $T \in \mathcal{B}'$ be such that $\beta_T = \gamma$ and $K_T \subseteq \mathrm{cnt}(\sigma)$. Then either $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$ and $f(\sigma) = 1$, or $T \models^{\mathcal{D}}_{\mathcal{W}} \neg\forall\varphi$ and $f(\sigma) = 0$. If such an ordinal γ does not exist, then $f(\sigma) =\uparrow$. Clause (iv) in $(*)$ implies immediately that f is well defined. It is easy to verify that the length of R_f is defined and smaller than β. So to complete the proof of the proposition, it suffices to show that f positively classifies $\mathcal{B}$ in the limit following φ. Let $T \in \mathcal{B}$ and an environment e for T be given. If $T \notin \mathcal{B}'$ then it is easily verified using (iv) that $f(e_{|k}) =\uparrow$ for all $k \in \mathbb{N}$. Suppose that $T \in \mathcal{B}'$. Choose $k \in \mathbb{N}$ such that $K_T \subseteq \mathrm{cnt}(e_{|k})$. Clause (iv) again then implies that $f(e_{|k'}) = f(e_{|k})$ for all $k' \geq k$, $f(e_{|k}) = 1$ if $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$, and $f(e_{|k}) = 0$ otherwise. Together with clauses (i) and (ii), this implies that f positively classifies $\mathcal{B}$ in the limit following φ, as wanted.

We now examine the connections between classification in the limit and the logical hierarchies. One direction complements one half of Proposition 46, also with no assumption on $\mathcal{P}$.

Proposition 47 *For all formulas φ, if φ is $[\Sigma^{\mathcal{P}}_2, \Pi^{\mathcal{P}}_2, \Delta^{\mathcal{P}}_2]$ then $\mathcal{B}$ is [positively classifiable, negatively classifiable, classifiable] in the limit following φ.*

Proof. Let a formula φ be $\Sigma^{\mathcal{P}}_2$. Let $(\sigma_i)_{i\in\mathbb{N}\setminus\{0\}}$ be an enumeration of all derivations of φ in $\mathfrak{D}$ whose height bound is equal to $(2, 0)$ at most. Given a nonnull $i \in \mathbb{N}$, denote by K_i the first member of σ_i (hence K_i is a finite set of formulas). Define a $g : (\mathcal{L} \cup \{\sharp\})^\star \to \mathbb{N}$ as follows. Let $\sigma \in (\mathcal{L} \cup \{\sharp\})^\star$ be given. Suppose that there exists a least $i > 0$ such that for some $T \in \mathcal{B}$, $K_i \subseteq \mathrm{cnt}(\sigma) \subseteq T$ and σ_i is from T. Then $g(\sigma) = i$. If there is no such i then $g(\sigma) = 0$. Finally define a classifier f as follows. Set $f(()) =\uparrow$. Let $\sigma \in (\mathcal{L} \cup \{\sharp\})^\star$ and $x \in \mathcal{L} \cup \{\sharp\}$ be given. If $g(\sigma) = g(\sigma \star x) > 0$ then $f(\sigma \star x) = 1$; otherwise $f(\sigma \star x) =\uparrow$. We show that f positively classifies $\mathcal{B}$ in the limit following φ. Let $T \in \mathcal{B}$ and an environment e for T be given. It is easy to verify that if $\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$ then $f(e_{|k}) = 1$ for cofinitely many $k \in \mathbb{N}$. Suppose that $\varphi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$. To complete the proof it suffices to show that for all $i > 0$, there exists $k \in \mathbb{N}$ such that for all $k' \geq k$ and $T' \in \mathcal{B}$, either $K_i \not\subseteq \mathrm{cnt}(e_{|k'})$, or $\mathrm{cnt}(e_{|k'}) \not\subseteq T'$, or σ_i is not from T'. So let a nonnull $i \in \mathbb{N}$ be given. Suppose that $K_i \subseteq T$ and σ_i is from some member of $\mathcal{B}$ (otherwise, we are done). Clearly there exists $\psi \in \mathcal{L}$, $H \subseteq \mathcal{L}$, and a nonnull

$\beta \in \mathrm{Ord}$ such that $\psi \in H$, $(1, \beta, H)$ occurs in σ_i, and $\psi \notin \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$. We can then choose a derivation τ of $\neg\forall\psi$ in $\mathfrak{D}$ such that τ is from T and the height bound of τ is at least equal to $(1, 0)$ and smaller than $(1, \beta)$. Let K denote the first member of τ. Let $k \in \mathbb{N}$ be such that $K \cup K_i \subseteq \mathrm{cnt}(e_{|k})$. Then for all $k' \geq k$ and $T' \in \mathcal{B}$, either $\mathrm{cnt}(e_{|k'}) \nsubseteq T'$ or $\neg\forall\psi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')$. We conclude that for all $k' \geq k$ and $T' \in \mathcal{B}$, either $\mathrm{cnt}(e_{|k'}) \nsubseteq T'$ or σ_i is not from T', as wanted.

Proposition 47 can be used to provide an easy proof that if $\mathcal{D} = \emptyset$, then all logical hierarchies collapse to the first layer of the first level. We noted that when $\mathcal{P}$ is the paradigm of classical first-order logic, this result follows immediately from the compactness theorem. If $\mathcal{D} = \emptyset$, the notion $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$—equivalent to $\varphi \in \mathrm{Mod}_{\mathcal{W}}(T)$—is not necessarily compact, but still cannot be stratified on the basis of the property of β-weak compactness.

Proposition 48 *Suppose that $\mathcal{D} = \emptyset$. For all formulas φ, if $\mathcal{B}$ is [positively classifiable, negatively classifiable, classifiable] in the limit following φ, then $\mathcal{B}$ is [positively classifiable, negatively classifiable, classifiable] with no mind change following φ.*

Proof. Let a formula φ and a classifier f be such that f positively classifies $\mathcal{B}$ in the limit following φ. Let $T \in \mathcal{B}$ be such that $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. By Lemma 43, choose a locking sequence $\sigma \in (T \cup \{\sharp\})^\star$ for f and T. Since $\mathcal{D} = \emptyset$, $\mathrm{cnt}(\sigma)$ is a member of $\mathcal{B}$ and $\mathrm{Mod}^{\mathcal{D}}_{\mathcal{W}}(\mathrm{cnt}(\sigma)) = \mathrm{Mod}_{\mathcal{W}}(\mathrm{cnt}(\sigma))$, which together with the choice of σ, implies that $\mathrm{cnt}(\sigma) \models_{\mathcal{W}} \varphi$. It follows immediately that $\varphi \in \Lambda^{\mathcal{P}}_{1,0}(T)$, and we conclude with Proposition 46 that $\mathcal{B}$ is positively classifiable with no mind change following φ.

Directly from Propositions 47 and 48:

Proposition 49 *If $\mathcal{D} = \emptyset$ then for all $T \in \mathcal{B}$, $\bigcup_{\alpha,\beta \in \mathrm{Ord}\setminus\{0\}} \Lambda^{\mathcal{P}}_{\alpha,\beta}(T) = \Lambda^{\mathcal{P}}_{1,0}(T)$.*

Our last task for this paper is to investigate the converse to Proposition 47. Basically, the converse holds provided that the language is rich enough. So given a formula φ, if φ is positively classifiable in the limit following φ, then φ might not be $\Sigma^{\mathcal{P}}_2$; but it is possible to enrich $\mathcal{L}$ with a countable set of infinitary statements, resulting in a new paradigm $\mathcal{P}'$, such that φ is $\Sigma^{\mathcal{P}'}_2$. The next proposition also shows how the topological notions defined in Section 4 can be used to naturally characterize classification in the limit.

Proposition 50 *For all countable fragments L of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$, there exists a countable fragment L' of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ that extends L with the following property. Suppose that $\mathcal{L} = L'$. For all formulas φ, the three clauses below are equivalent.*

- *φ is [$\Sigma^{\mathcal{P}}_2$, $\Pi^{\mathcal{P}}_2$].*
- *$\mathcal{B}$ is [positively classifiable, negatively classifiable] in the limit following φ.*
- *For all $T \in \mathcal{B}$ with [$\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$, $\neg\forall\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T)$], there exists finite $K \subseteq T$ and $\chi \in \mathcal{L}$ such that $T \models^{\mathcal{D}}_{\mathcal{W}} \chi$, χ is $\Pi^{\mathcal{L}}_1$ Borel in $\mathbb{W}$ and for all $T' \in \mathcal{B}$:*

$$\text{if } K \subseteq T' \text{ and } T' \models^{\mathcal{D}}_{\mathcal{W}} \chi \text{ then } [\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T'), \neg\forall\varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T')].$$

Proof. By Proposition 47, the first clause in the statement of the proposition implies the second one. By Proposition 32, the third clause implies the first one. Let a classifier f be such that f positively classifies $\mathcal{B}$ in the limit following φ. To complete the proof of the proposition, it suffices to exhibit a countable set Z of infinitary formulas such that if $\mathcal{L}'$ is the countable fragment of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ generated by $\mathcal{L} \cup Z$, then the following holds.

- All members of Z are $\Pi^{\mathcal{L}'}_1$ Borel in $\mathbb{W}$.
- For all $T \in \mathcal{B}$ with $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$, either $\varphi \in \Lambda^{\mathcal{P}}_{1,0}(T)$ or there exists a finite $K \subseteq T$ and $\chi \in Z$ such that for all $T' \in \mathcal{B}$:

$$\text{if } K \subseteq T' \text{ and } T' \models^{\mathcal{D}}_{\mathcal{W}} \chi \text{ then } T' \models^{\mathcal{D}}_{\mathcal{W}} \varphi.$$

Let X be the set of all $\sigma \in (\mathcal{L} \cup \{\sharp\})^\star$ that are locking sequences for f and some possible knowledge base. We can assume that $X \neq \emptyset$ (otherwise, it follows from Lemma 43 that $T \not\models^{\mathcal{D}}_{\mathcal{W}} \varphi$ for all $T \in \mathcal{B}$, and the result is trivial). Given $\sigma \in X$, let X_σ be the set of all $\tau \in (\mathcal{D} \cup \{\sharp\})^\star$ such that $f(\sigma \star \tau) \neq 1$. For all $\sigma \in X$ with $X_\sigma \neq \emptyset$, let ψ_σ denote the formula $\bigwedge_{\tau \in X_\sigma} \neg \bigwedge \mathrm{cnt}(\tau)$. Define Z as $\{\psi_\sigma : \sigma \in X,\ X_\sigma \neq \emptyset\}$. Clearly, every member of Z is $\Pi^{\mathcal{L}'}_1$ Borel in $\mathbb{W}$. Let $\mathcal{L}'$ be the countable fragment of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ generated by $\mathcal{L} \cup Z$. Let $T \in \mathcal{B}$ be such that $T \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. By Lemma 43, let $\sigma \in X$ be a locking sequence for f and T. So $\mathrm{cnt}(\sigma)$ is a finite subset of T. If $X_\sigma = \emptyset$, then it is immediately verified that φ is a $\mathcal{D}$-minimal logical consequence in $\mathcal{W}$ of every possible knowledge base that contains $\mathrm{cnt}(\sigma)$, which implies that $\varphi \in \Lambda^{\mathcal{P}}_{1,0}(T)$. Assume that $X_\sigma \neq \emptyset$. Obviously, $T \models^{\mathcal{D}}_{\mathcal{W}} \psi_\sigma$. Let $T' \in \mathcal{B}$ be such that $\mathrm{cnt}(\sigma) \subseteq T'$ and $T' \models^{\mathcal{D}}_{\mathcal{W}} \psi_\sigma$. To complete the proof of the proposition, it suffices to show that $T' \models^{\mathcal{D}}_{\mathcal{W}} \varphi$. Suppose for a contradiction that $T' \not\models^{\mathcal{D}}_{\mathcal{W}} \varphi$. Set $T'' = \mathrm{cnt}(\sigma) \cup (T' \cap \mathcal{D})$. Since T' and T'' have the same $\mathcal{D}$-diagram, $T' \subseteq T''$, and using the fact that ψ_σ is $\Pi^{\mathcal{L}'}_1$ Borel in $\mathbb{W}$, it is easily verified that $T'' \models^{\mathcal{D}}_{\mathcal{W}} \psi_\sigma$ and $T'' \not\models^{\mathcal{D}}_{\mathcal{W}} \varphi$. Let e be a member of $(\mathcal{D} \cup \{\sharp\})^{\mathbb{N}}$ such that $\sigma \star e$ is an environment for T''. Since f positively classifies $\mathcal{B}$ in the limit following φ, there exists $k \in \mathbb{N}$ such that $f(\sigma \star e_{|k}) \neq 1$. Hence $e_{|k} \in X_\sigma$ and $T'' \models^{\mathcal{D}}_{\mathcal{W}} \bigwedge \mathrm{cnt}(e_{|k})$. Hence $T'' \not\models^{\mathcal{D}}_{\mathcal{W}} \psi_\sigma$. Contradiction.

Corollary 51 *For all countable fragments L of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$, there exists a countable fragment L' of $\mathcal{L}^{\mathcal{V}}_{\omega_1\omega}$ that extends L with the following property. Suppose that $\mathcal{L} = L'$. For all formulas φ, the three clauses below are equivalent.*

- *φ is $\Delta^{\mathcal{P}}_2$.*
- *$\mathcal{B}$ is classifiable in the limit following φ.*
- *For all $T \in \mathcal{B}$ that $\mathcal{D}$-minimally decide φ in $\mathcal{W}$, there exists a finite $K \subseteq T$ and $\chi \in \mathcal{L}$ such that $T \models^{\mathcal{D}}_{\mathcal{W}} \chi$, χ is $\Pi^{\mathcal{L}}_1$ Borel in $\mathbb{W}$, and for all $T' \in \mathcal{B}$:*

$$\text{if } K \subseteq T' \text{ and } T' \models^{\mathcal{D}}_{\mathcal{W}} \chi, \text{ then } \varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T) \text{ iff } \varphi \in \mathrm{Cn}^{\mathcal{D}}_{\mathcal{W}}(T').$$

6 Conclusion

Though many connections have been established between logic, learning and topology, there has been no attempt to unify the fields, or at least, part of them. Still the connections are tight and natural at a fundamental level, and can result in a logic with parameters that needs topological notions for its early developments, and notions from learning theory for interpretation and applicability. It is essential to observe the same kind of connection when the framework presented here is cast in a computable setting that can also address complexity issues. This is work in progress, and results have already been obtained in this direction, with relativizations of the properties that can be proved in the more general setting at the heart of which is the notion of β-weak compactness described in this paper.

One of our goals is to provide a theoretical foundation for a declarative programming language for applications in Artificial intelligence, whose nature could be purely deductive or inductive, but more often, would combine deductive and inductive aspects. A prototype of a system—whose development is driven by the theoretical considerations presented here— that employs such a programming language has been designed and implemented [18].

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