U-Shaped Learning may be Necessary [3]

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Abstract *U-shaped learning behaviour* in cognitive development involves learning, unlearning and relearning. It occurs, for example, in learning irregular verbs. The prior cognitive science literature is occupied with how humans do it, for example, general rules versus tables of exceptions. This paper is mostly concerned with whether U-shaped learning behaviour may be *necessary* in the abstract mathematical setting of inductive inference, that is, in the computational learning theory following the framework of Gold. All notions considered are learning from text, that is, from positive data. Previous work showed that U-shaped learning behaviour is necessary for behaviourally correct learning but not for syntactically convergent, learning in the limit (= explanatory learning). The present paper establishes the necessity for the whole hierarchy of classes of vacillatory learning where a behaviourally correct learner has to satisfy the additional constraint that it vacillates in the limit between at most k grammars, where $k \geq 1$. Non U-shaped vacillatory learning is shown to be restrictive: Every non U-shaped vacillatorily learnable class is already learnable in the limit. Furthermore, if vacillatory learning with the parameter $k = 2$ is possible then non U-shaped behaviourally correct learning is also possible. But for $k = 3$, surprisingly, there is a class witnessing that this implication fails.

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1 Introduction and Motivation

U-shaped learning is a learning behaviour in which the learner first learns the correct behaviour, then abandons the correct behaviour and finally returns to the correct behaviour once again. This pattern of learning behaviour has been observed by cognitive and developmental psychologists in a variety of child development phenomena, such as language learning [9,28,40] understanding of temperature [40,41], understanding of weight conservation [8,40], object permanence [8,40] and face recognition [10]. The case of language acquisition is paradigmatic. In the case of the past tense of english verbs, it has been observed that children learn correct syntactic forms (call/called, go/went), then undergo a period of overregularization in which they attach regular verb endings such as 'ed' to the present tense forms even in the case of irregular verbs (break/breaked, speak/speaked) and finally reach a final phase in which they correctly handle both regular and irregular verbs. This example of U-shaped learning behaviour has figured so prominently in the so-called "Past Tense Debate" in cognitive science that competing models of human learning are often judged on their capacity for modeling the U-shaped learning phenomenon [28,35,42].

The prior literature is typically concerned with modeling *how* humans achieve U-shaped behaviour, while, in the present paper, we are mostly interested in *why* humans exhibit this seemingly inefficient behaviour. Is it a mere harmless evolutionary accident or is it *necessary* for full human learning power? Specifically, are there some learning tasks for which U-shaped behaviour is logically necessary? In the present paper we present some new theorems in the context of Gold's formal model of language learning from positive data [22] of answering this latter question. To explain our results, we informally review the main notions of inductive inference, that is Gold-style learning theory, and refer to the next section for precise mathematical definitions.

A learner $\mathbf{M}$ reads an infinite sequence consisting of the elements of any language L in arbitrary order with possibly some pause symbols # in between. During this process the learner outputs a corresponding sequence $e_0\, e_1\, \ldots$ of hypotheses (grammars) which may generate the language L to be learned. Sometimes, especially when numerically coded, we call also call these hypotheses *indices*. The learning criteria (regarding successful learning of L) considered herein can each be defined by constraining this sequence of hypotheses. The main requirement is that the learner is behaviourally correct, i.e., almost all e_n generate the language L to be learned. The criterion **TxtBc** postulates just this and nothing more, but the further variants require that the learner $\mathbf{M}$ satisfies additional requirements which have to be met by the sequence of its hypotheses: for $b \geq 1$, $\mathbf{TxtFex}_b$ requires that the e_n eventually vacillate between at most b hypotheses. **TxtEx** requires syntactic convergence to one hypothesis and thus coincides with $\mathbf{TxtFex}_1$. *Non U-shaped learning for L* would require that e_{n+1} generates L whenever e_n does.

Baliga, Case, Merkle, Stephan and Wiehagen [4] initiated the Gold style learning theoretic study of U-shaped learning behaviour and showed that it is circumventable for **TxtEx**-learning, see Theorem 2.6. In contrast to this, Fulk, Jain and Osherson's proof of [20, Theorem 4] shows that U-shaped learning behaviour is necessary for the full learning power of **TxtBc**-learning. We show in Theorem 3.1 below that U-shaped learning behaviour is also necessary for full learning power for the whole hierarchy of the learning criteria $\mathbf{TxtFex}_b$ *strictly* between **TxtEx** and **TxtBc**. While Case [11] proved that the $\mathbf{TxtFex}_b$ criteria form a hierarchy of more and more powerful learning criteria, Theorem 3.1 of the present paper shows that *non* U-shaped $\mathbf{TxtFex}_b$ learners are not more powerful than **TxtEx**-learners and the hierarchy collapses to $\mathbf{TxtFex}_1$ if U-shaped behaviour is forbidden. In other words, there are classes of languages that can be $\mathbf{TxtFex}_n$-identified, for $n > 1$, but these learners *must* be U-shaped on some texts. What if we consider the more liberal criterion **TxtBc**? Our Theorem 5.1 strengthens the collapse result of Theorem 3.1 considerably by showing that there are classes in $\mathbf{TxtFex}_3$ (and therefore in $\mathbf{TxtFex}_n$ for all $n > 3$) that cannot be **TxtBc**-learned by a non U-shaped learner. This means that U-shaped learning behaviour cannot be dispensed with for learning such classes, even if we only require behavioural convergence and permit convergence to possibly infinitely many syntactically different correct hypotheses. By contrast, our last main result, Theorem 4.3, shows that every class of languages that can be $\mathbf{TxtFex}_2$-identified can be **TxtBc**-identified by a non U-shaped learner. Hence, for only this early stage of the hierarchy, the cases in which $\mathbf{TxtFex}_2$-identification necessitates U-shaped learning behaviour can be circumvented by shifting to **TxtBc**-identification.

In Section 2 we introduce the notation and the basic definitions for the rest of the paper. We also include the basic results from [4] which are relevant for the present paper. In Section 3 we show that $\mathbf{NUShTxtFex}_b$ actually coincides with **TxtEx** so that on one hand every **TxtEx**-learner can be replaced by a non U-shaped one while on the other hand vacillatory non U-shaped learners do not have more power than non U-shaped **TxtEx**-learners; that is, $\mathbf{NUShTxtFex}_b = \mathbf{TxtEx}$ for all $b \in \{1, 2, 3, \ldots, *\}$. In the subsequent sections we investigate the question whether one can obtain a non U-shaped learner from a $\mathbf{TxtFex}_b$ learner if one is willing to give up some constraints on the number of hypotheses considered. The positive result $\mathbf{TxtFex}_2 \subseteq \mathbf{NUShTxtBc}$ is presented in Section 4 and the negative result $\mathbf{TxtFex}_3 \not\subseteq \mathbf{NUShTxtBc}$ is presented in Section 5. Section 6, as well as some previous sections, provides some results on non U-shaped team learning. In Section 7 we summarize our main results and provide a brief discussion on their relevance to U-shaped learning behaviour in cognitive science.

2 Preliminaries

$\mathbb{N}$ denotes the set of natural numbers, $\{0,1,2,\ldots\}$. $\mathbb{N}^+$ denotes the set of positive natural numbers. Unless otherwise specified the variables a, c, d, e, i, j, k range over $\mathbb{N}$. D, P, S range over *finite* sets of natural numbers. The cardinality function is denoted by $\mathrm{card}(\cdot)$. $\mathrm{card}(D) \leq *$ means that $\mathrm{card}(D)$ is finite. The symbol $*$ is used to denote the 'finite with no preassigned bound'. Unless otherwise stated, b will range over $\mathbb{N} \cup \{*\}$. The symbols $\subseteq, \subset, \supseteq, \supset$ respectively denote the subset, superset, proper subset and proper superset relation between sets. The union $\cup$ and the intersection $\cap$ are defined as usual; $\in$ denotes set-theoretic membership, $-$ denotes set-theoretic difference and $\boldsymbol{\Delta}$ denotes the symmetric difference of sets, that is, $A \boldsymbol{\Delta} B = (A \cup B) - (A \cap B)$. The quantifiers $\forall^\infty$ and $\exists^\infty$ mean 'for all but finitely many' and 'there exists infinitely many', respectively.

A pair $\langle i, j \rangle$ stands for an arbitrary, computable one-to-one encoding of all pairs of natural numbers onto $\mathbb{N}$ [36]. Similarly we can define $\langle \cdot, \ldots, \cdot \rangle$ for encoding n-tuples of natural numbers, for $n > 1$, onto $\mathbb{N}$.

φ denotes a fixed *acceptable* programming system for the partial computable functions [36]. φ_e denotes the partial computable function computed by the program with code number e in the φ-system. We will unambiguously refer to programs using their code number in the φ-system. We let H, I, J, L range over recursively enumerable sets and $\mathcal{L}$ range over classes of recursively enumerable sets. K denotes the diagonal halting problem which is the standard example for a nonrecursive r.e. set.

Furthermore, we fix a uniformly recursive enumeration of all r.e. sets such that

- W_e is the domain of φ_e and $W_e = \bigcup_s W_{e,s}$;
- $W_{e,0} = \emptyset$ and $W_{e,s} \subseteq W_{e,s+1} \subseteq \{0, \ldots, s\}$ for all e, s;
- $\{(e, s, x) : x \in W_{e,s}\}$ is recursive;
- $\{(x, s) : x \in W_{e,s}\}$ is primitive recursive for all e;
- For every primitive-recursive enumeration A_s of some set A with $A_0 = \emptyset$ $\wedge\, (\forall s)\, [A_s \subseteq A_{s+1} \subseteq \{0, \ldots, s\}]$ there is an index e with $(\forall s)\, [W_{e,s} = A_s]$; furthermore, e can be computed from an index of the enumeration for A_s.

Of course, $W_{e,s}$ can be easily based on a specifically designed Blum complexity measure from [7]. Any unexplained recursion-theoretic notions are from [30,36].

We now introduce the basic definitions of Gold-style computational learning theory.

Definition 2.1 A *sequence* σ is a mapping from an initial segment of $\mathbb{N}$ into $\mathbb{N} \cup \{\#\}$. An *infinite sequence* is a mapping from $\mathbb{N}$ into $\mathbb{N} \cup \{\#\}$. The content of a finite or infinite sequence σ is the set of natural numbers occurring in σ and is denoted by $\mathrm{content}(\sigma)$. The length of a sequence σ is the number of elements in the domain of σ and is denoted by $|\sigma|$. For a subset L of $\mathbb{N}$, $\mathrm{seg}(L)$ denotes the set of sequences σ with $\mathrm{content}(\sigma) \subseteq L$. An infinite sequence T is a *text for* L iff $L = \mathrm{content}(T)$.

The symbol # is mainly introduced to uniformly deal with the empty language and intuitively represents a pause in the presentation of the language to the learner. Concatenation of two sequences σ and τ is denoted by $\sigma\tau$. If $x \in (\mathbb{N} \cup \{\#\})$, then σx means $\sigma\tau$ where τ is the sequence consisting of exactly one element which is x. $\sigma \subseteq \tau$ means that σ is an initial segment of τ and $\sigma \subset \tau$ means that σ is a proper initial segment of τ.

Intuitively, a *text* for a language L is an infinite stream or sequential presentation of *all* the elements of the language L in any order and with the #'s representing pauses in the presentation of the data. For example, the only text for the empty language is an infinite sequence of #'s. Technically, a text is a mapping from $\mathbb{N}$ into $(\mathbb{N} \cup \{\#\})$. We let T, with possible subscripts and superscripts, range over texts. $T[n]$ denotes the finite initial segment of T with length n. Observe that the domain of $T[n]$ is $\{x : x < n\}$. So $T(n)$ is not a member of the sequence $T[n]$. $\sigma \subset T$ denotes the fact that σ is an initial segment of T. Observe that in this case we have $\sigma = T[|\sigma|]$.

A learner will map sequences to hypotheses. These are represented by natural numbers and interpreted as codes for programs in the φ-system. $\mathbf{M}$, with possible superscripts and subscripts, is intended to range over language learning machines.

Definition 2.2 [4,5,11,14,15,22,33] A *language learning machine* $\mathbf{M}$ is a computable mapping from seg($\mathbb{N}$) into $\mathbb{N}$. $\mathbf{M}$ **TxtBc**-learns a class $\mathcal{L}$ of r.e. languages iff for every $L \in \mathcal{L}$ and every text T for L, almost all hypotheses $\mathbf{M}(T[n])$ are indices for the language L to be learned.

A **TxtBc**-learner $\mathbf{M}$ for $\mathcal{L}$ is a $\mathbf{TxtFex}_*$-learner iff for every $L \in \mathcal{L}$ and every text T for L the set $\{T[n] : n \in \mathbb{N}\}$ is finite.

A $\mathbf{TxtFex}_*$-learner $\mathbf{M}$ for $\mathcal{L}$ is a $\mathbf{TxtFex}_b$-learner for a $b \in \{1, 2, \ldots\}$ iff there are for every $L \in \mathcal{L}$ and every text T for L at most b indices which $\mathbf{M}$ outputs infinitely often, that is, $|\{e : (\exists^\infty n)\,[e = \mathbf{M}(T[n])]\}| \leq b$.

A **TxtBc**-learner $\mathbf{M}$ for $\mathcal{L}$ is a **TxtEx**-learner iff for every $L \in \mathcal{L}$ and every text T for L almost all hypotheses $\mathbf{M}(T[n])$ are the same grammar for L.

A **TxtBc**-learner $\mathbf{M}$ for $\mathcal{L}$ is non U-shaped iff for every $L \in \mathcal{L}$ and every text T for L there are no three numbers k, m, n such that $k < m < n$ and $W_{\mathbf{M}(T[k])} = L$, $W_{\mathbf{M}(T[m])} \neq L$ and $W_{\mathbf{M}(T[n])} = L$. Furthermore, $\mathbf{NUShTxtFex}_b$-learners and **NUShTxtEx**-learners (of a class $\mathcal{L}$) are those learners which are non U-shaped and at the same time a $\mathbf{TxtFex}_b$-learner and **TxtEx**-learner, respectively, (of $\mathcal{L}$).

The criteria **TxtBc**, $\mathbf{TxtFex}_b$, **TxtEx**, **NUShTxtBc**, $\mathbf{NUShTxtFex}_b$, **NUShTxtEx** are the sets consisting of all those classes which are learnable by a learner satisfying the respective above defined requirements.

The part "**Txt**" in the acronyms relates to the fact that all notions considered in the present work are learning from text, that is, from positive data. Intuitively, the notion **TxtBc** captures what could be called learning in the most general sense. All other notions are restrictions. The historically (from [22]) most important one is **TxtEx** where the learner has to converge syntactically to a single index of the language to be learned.

Intuitively a class $\mathcal{L}$ of r.e. languages is $\mathbf{TxtFex}_b$ identified by a machine $\mathbf{M}$ iff when $\mathbf{M}$ is given as input *any* listing T of *any* $L \in \mathcal{L}$, it outputs a sequence of grammars such that, past some point in this sequence, no more than b *syntactically* different grammars occur and each of them is a grammar for L. The acronym **TxtFex** stands for 'finite explanatory identification from text'. $\mathbf{TxtFex}_1$ is equivalent to Gold's original notion of *identification*, also denoted by **TxtEx** and called 'explanatory identification from text'. Osherson and Weinstein [31] first studied the case with $b = *$, later Case [11] studied the whole hierarchy with $b \in \mathbb{N}^+$.

We say σ is a *stabilizing sequence* [19] for a learner $\mathbf{M}$ on a set L iff $\sigma \in \text{seg}(L)$ and $\mathbf{M}(\sigma\tau) = \mathbf{M}(\sigma)$ for all $\tau \in \text{seg}(L)$. Furthermore, σ is called a *locking sequence* [6] for $\mathbf{M}$ on L iff $W_{\mathbf{M}(\sigma)} = L$. Note that stabilizing and locking sequence definitions can be generalized to other learning criteria such as **TxtFex** and **TxtBc**.

Smith [38] studied learning by teams of machines, and, we can show vacillatory learning can be characterized by teams as follows.

Theorem 2.3 *A class $\mathcal{L}$ has a $\mathbf{TxtFex}_b$-learner iff there is a team of b machines $\mathbf{N}_1, \ldots, \mathbf{N}_b$ such that for every $L \in \mathcal{L}$ and for every text T for L, each machine $\mathbf{N}_a$ converges to a single index e_a and at least one of these indices e_a is an index for L.*

Proof. If $\mathcal{L}$ is $\mathbf{TxtFex}_b$ learnable by $\mathbf{M}$ and $L \in \mathcal{L}$ and T is a text for L, then one can define b new machines $\mathbf{N}_1, \ldots, \mathbf{N}_b$ such that at every input σ one computes a list $e_1, \ldots, e_b$ of the most recent conjectures of $\mathbf{M}$ ordered by size and each machine $\mathbf{N}_a$ outputs e_a; in the case that there are less than b hypotheses in total, the list is made to have size b by using arbitrary constants. Since $\mathbf{M}$ outputs only finitely many hypotheses on T at all, the considered list $e_1, \ldots, e_b$ stabilizes after finite time and from then on all machines $\mathbf{N}_a$ output the same hypothesis. Since $\mathbf{M}$ vacillates eventually between b or fewer correct indices, some machine $\mathbf{N}_a$ converges to a correct hypothesis.

For the other way round, consider that $\mathbf{N}_1, \ldots, \mathbf{N}_b$ is a list of machines such that each converges on every text T for any language $L \in \mathcal{L}$ to some index and one of these indices is correct. Then $\mathbf{M}$ can be defined such that $\mathbf{M}(T[n])$ takes one of those hypotheses $\mathbf{N}_a(T[n])$ for which the quality of the hypothesis measured as

$$q_{T[n],a} = \max(\{x \leq n : (\forall y < x)\,[y \in W_{\mathbf{N}_a(T[n]),n} \Leftrightarrow y \in \text{content}(T[n])\,]\})$$

is maximal. Since all b machines converge to an index, the learner $\mathbf{M}$ clearly vacillates between at most b hypotheses. Furthermore, if $\mathbf{N}_a$ converges on T to an index e_a of L then $q_{T[n],a}$ converges to ∞; if $\mathbf{N}_a$ converges on T to an index e_a of a set different from L then $q_{T[n],a}$ converges to a finite number in $\mathbb{N}$. Thus, from some time on, $\mathbf{M}$ will only consider correct indices and thus $\mathbf{M}$ $\mathbf{TxtFex}_b$-learns $\mathcal{L}$. ∎

Case [11] showed that the criteria $\mathbf{TxtFex}_b$, $b = 1, 2, 3, \ldots, *$, form a hierarchy of more and more powerful learning criteria, that is, $\mathbf{TxtFex}_1 \subset \mathbf{TxtFex}_2 \subset \ldots \subset \mathbf{TxtFex}_*$, as stated in the following Theorem.

Theorem 2.4 [11] *For each $b \in \mathbb{N}^+$, the separation $\mathbf{TxtFex}_{b+1} \not\subseteq \mathbf{TxtFex}_b$ is witnessed by the self-referential class*

$$\{W_e : e \in W_e \wedge |W_e \cap \{0, \ldots, e\}| \leq b + 1\}$$

consisting of all nonempty sets which have an index among its smallest $b+1$ elements. Furthermore consider the class

$$\{W_e : W_e \neq \emptyset \wedge e \leq \min(W_e)\}$$

consisting of all nonempty sets where the smallest index is bounded by their smallest element. This class witnesses that $\mathbf{TxtFex}_* \not\subseteq \mathbf{TxtFex}_b$ for any $b \in \mathbb{N}^+$.

Proposition 2.5 $\mathbf{NUShTxtBc} \not\subseteq \mathbf{TxtFex}_*$.

Proof. Let $K = \{e : e \in W_e\}$ and $\mathcal{K} = \{K \cup E : E \text{ is finite}\}$. We will show that $\mathcal{K} \in \mathbf{NUShTxtBc} - \mathbf{TxtFex}_*$. The $\mathbf{NUShTxtBc}$-learner is given by a machine which outputs an hypothesis for $K \cup \mathrm{content}(\sigma)$ for every input σ. It is easy to see that for every finite set E and every text T for $K \cup E$, after the learner has seen all the elements of E, it outputs hypotheses for $K \cup E$. Furthermore, the learner is strongly monotone, that is, $W_{\mathbf{M}(\sigma)} = K \cup \mathrm{content}(\sigma) \subseteq K \cup \mathrm{content}(\sigma\tau) = W_{\mathbf{M}(\sigma\tau)}$ for all $\sigma, \tau \in \mathrm{seg}(\mathbb{N})$. Thus, $\mathbf{M}$ $\mathbf{NUShTxtBc}$-identifies $\mathcal{K}$.

On the other hand, suppose by way of contradiction that $\mathbf{M}$ $\mathbf{TxtFex}_*$-identifies $\mathcal{K}$. Then there is a locking sequence σ for $\mathbf{M}$ on K in the sense that $\sigma \in \mathrm{seg}(K)$ and for some finite set F of indices of K, $\mathbf{M}(\sigma\tau) \in F$ for all $\tau \in \mathrm{seg}(K)$. Here is why. If such σ, F would not exist, then one could construct a text for K on which $\mathbf{M}$ outputs either infinitely many indices or infinitely often an index for a language different from K — on this text, $\mathbf{M}$ would not $\mathbf{TxtFex}_*$-learn K.

Now, if $x \notin K$, then there is a sequence $\tau \in \mathrm{seg}(K)$ such that $\mathbf{M}(\sigma x \tau) \notin F$ since no index in F is an index for $K \cup \{x\}$. Thus, having F and σ, the following Π^0_1-predicate defines the set K:

$$x \in K \Leftrightarrow (\forall \tau \in \mathrm{seg}(K))\, [\mathbf{M}(\sigma x \tau) \in F].$$

But, then, K would be recursive. By this contradiction, $\mathcal{K}$ is not $\mathbf{TxtFex}_*$-learnable. ∎

The next theorem is from [4] and states that being non U-shaped is not restrictive for $\mathbf{TxtEx}$-learning. Its proof can also be obtained by letting $b = 1$ in Theorem 3.6 below. Its fundamental equality will be extended to all classes $\mathbf{NUShTxtFex}_b$ in Theorem 3.1.

Theorem 2.6 [4] $\mathbf{NUShTxtFex}_1 = \mathbf{TxtEx}$.

Hence, for **TxtEx**, U-shaped behaviour is not necessary for full learning power. By contrast, by an easy adaptation of the proof of Theorem 4 in [20], we have the following.

Theorem 2.7 [4,20] $\mathbf{NUShTxtBc} \subset \mathbf{TxtBc}$.

Hence, for **TxtBc**, U-shaped behaviour is necessary for full learning power.

3 U-shaped vacillatory learning

We first show, by a simple counting argument, that the hierarchy $\mathbf{TxtFex}_1 \subset \mathbf{TxtFex}_2 \subset \ldots \subset \mathbf{TxtFex}_*$ of vacillatory learning criteria collapses to $\mathbf{TxtFex}_1$ if U-shaped behaviour is forbidden.

Theorem 3.1 $\mathbf{NUShTxtFex}_* \subseteq \mathbf{TxtFex}_1$.

Proof. Let $\mathcal{L} \in \mathbf{NUShTxtFex}_*$ and let $\mathbf{M}$ be a learner witnessing this fact. We define a new learner $\mathbf{N}$ witnessing that $\mathcal{L} \in \mathbf{TxtFex}_1$ as follows.

On a text T, $\mathbf{N}$ keeps a list of all of $\mathbf{M}$'s conjectures in order of appearance and without repetitions and outputs the most recent entry in the list.

If T is a text for a language $L \in \mathcal{L}$, then $\mathbf{M}$ outputs on T only finitely many different hypotheses and at least one of them, say e, infinitely often, and $W_e = L$. Furthermore, $\mathbf{N}$ converges to that hypothesis e' which goes into the list last. If $e = e'$ then $\mathbf{N}$ is correct on T. If $e \neq e'$ then $\mathbf{M}$ has output e' the first time after having already output e at least once. Since $\mathbf{M}$ is not U-shaped and e is correct, so is e'. Thus $\mathbf{N}$ is correct on T again. ∎

Theorems 2.6 and 3.1 give $\mathbf{NUShTxtFex}_1 = \mathbf{TxtFex}_1$ and $\mathbf{NUShTxtFex}_* \subseteq \mathbf{NUShTxtFex}_1$. Furthermore, the definition of $\mathbf{NUShTxtFex}_b$ immediately gives $\mathbf{NUShTxtFex}_1 \subseteq \mathbf{NUShTxtFex}_b \subseteq \mathbf{NUShTxtFex}_*$. Thus all these criteria coincide.

Corollary 3.2 $(\forall b \in \{1, 2, \ldots, *\})\, [\mathbf{NUShTxtFex}_b = \mathbf{TxtFex}_1]$.

The result $\mathbf{NUShTxtFex}_1 = \mathbf{TxtFex}_1$ stands in contrast to the fact that $\mathbf{TxtFex}_1 \subset \mathbf{TxtFex}_2 \subset \ldots \subset \mathbf{TxtFex}_*$. Thus we have that the following inclusions are proper.

Corollary 3.3 $(\forall b \in \{2, 3, \ldots, *\})\, [\mathbf{NUShTxtFex}_b \subset \mathbf{TxtFex}_b]$.

Corollaries 3.2 and 3.3 show that U-shaped learning behaviour is *necessary* for the full learning power of $\mathbf{TxtFex}_b$-identification for $b > 1$ in a strong sense: if U-shaped learning behaviour is forbidden, the hierarchy collapses to $\mathbf{TxtFex}_1$. Hence, for $b > 1$, the $\mathbf{TxtFex}_b$-learnability of any class in $(\mathbf{TxtFex}_b - \mathbf{TxtFex}_1)$ *requires* U-shaped learning behaviour. Recall from Theorem 2.4 that for any $b \in \mathbb{N}^+$ the class

$$\mathcal{L}_{b+1} = \{W_e : e \in W_e \wedge |W_e \cap \{0, \ldots, e\}| \leq b + 1\}$$

is in $\mathbf{TxtFex}_{b+1} - \mathbf{TxtFex}_b$. This class is then the example in the following corollary which is an easy consequence of Theorem 2.4 and Corollary 3.2.

Corollary 3.4 *Let $b \in \{2, 3, \ldots\}$. Then any $\mathbf{M}$ witnessing $\mathcal{L}_b \in \mathbf{TxtFex}_b$ necessarily employs U-shaped learning behaviour on $\mathcal{L}_b$.*

A non U-shaped learner does not make a mind change from a correct to an incorrect hypothesis since it cannot learn the set otherwise. This property is enforced on all machines for the case of team learning.

Definition 3.5 A team learning a class $\mathcal{L}$ is *non U-shaped* iff no machine in the team on any text for any language $L \in \mathcal{L}$ ever makes a mind change from an index for L to an index for a different language. In particular, the class $\mathcal{L}$ is in $[a, b]\mathbf{NUShTxtEx}$ iff there are b machines such that on any text for any language in $\mathcal{L}$ at least a machines converge to an index for that language and no machine makes a mind change from a correct to an incorrect hypothesis. For any learning criterion I, $[a, b]I$ is the corresponding team variant of this criterion.

The next result shows that Theorem 2.3 can be extended such that every class in $\mathbf{TxtFex}_b$ is learnable from a non U-shaped team. So the restriction $\mathbf{NUShTxtFex}_b = \mathbf{TxtFex}_1$ is caused by the fact that the hypothesis of the learner have to be brought into an ordering and cannot be done in parallel as in the case of the team below. Actually Theorem 3.6 enables us to achieve more properties of the team than that it is just non U-shaped.

Theorem 3.6 *Let $b \in \mathbb{N}^+$ and $\mathcal{L} \in \mathbf{TxtFex}_b$. Then there is a team of b learners $\mathbf{M}_1, \ldots, \mathbf{M}_b$ such that for all $L \in \mathcal{L}$ and all texts T for L there is an $n \in \mathbb{N}$ such that,*

(1) *$T[n]$ is a stabilizing sequence of all members of the team on L, in particular $\mathbf{M}_a(T[m]) = \mathbf{M}_a(T[n])$ for all $m \geq n$;*
(2) *there is an $a \in \{1, \ldots, b\}$ such that $\mathbf{M}_a(T[n])$ is an index for L;*
(3) *if $a \in \{1, \ldots, b\}$ and $\mathbf{M}_a(T[m])$ is an index for L then $m \geq n$.*

In particular, $\mathbf{M}_1, \ldots, \mathbf{M}_b$ $[1, b]\mathbf{NUShTxtEx}$-learns $\mathcal{L}$.

Proof. By Theorem 2.3 there is a team $\mathbf{N}_1, \ldots, \mathbf{N}_b$ of $\mathbf{TxtEx}$-learners for $\mathcal{L}$ such that for every $L \in \mathcal{L}$ and every text T for L, every machine converges on T to some hypothesis and at least one of the hypotheses is an index for L. The basic idea of the proof is to apply the locking-sequence hunting construction [23, Proposition 5.29] to a team.

If $\mathbb{N}$ is not in $\mathcal{L}$ then let $E = \mathbb{N}$. Otherwise let E be an infinite and coinfinite superset of the content of some $\mathbf{TxtFex}_b$-locking sequence on $\mathbb{N}$ for some $\mathbf{TxtFex}_b$-learner of $\mathcal{L}$. Note that $E \cup \tilde{E} \notin \mathcal{L}$ for any finite set $\tilde{E}$.

Let $\sigma \sqsubseteq \tau$ denote that $\mathrm{content}(\sigma) \subseteq \mathrm{content}(\tau)$ and $|\sigma| \leq |\tau|$. Furthermore, let T_e be the canonical text for W_e.

As long as the content of the input is $\emptyset$ or no σ is found in the below algorithm, all machines $\mathbf{M}_1, \ldots, \mathbf{M}_b$ output the least index of $\emptyset$.

The σ searched for on input $T[t]$ has to satisfy the following conditions:

(a) $\sigma \sqsubseteq T[t]$ and $\mathrm{content}(\sigma) \neq \emptyset$;
(b) $\mathbf{N}_a(\sigma\eta) = \mathbf{N}_a(\sigma)$ for all $a \in \{1, \ldots, b\}$ and $\eta \sqsubseteq T[t]$.

Once having σ, this is only replaced by a σ' on a future input $T[t']$ iff σ' but not σ satisfies (a) and (b) with respect to $T[t']$. If there are several choices to replace σ, the first one with respect to some fixed recursive enumeration of $\mathrm{seg}(\mathbb{N})$ is taken. Having σ, define D as follows.

(c) $D = \{a \in \{1, ..., b\} : (\forall \eta \sqsubseteq T_{\mathbf{N}_a(\sigma)}[t])\, (\forall c \in \{1, ..., b\})\, [\mathbf{N}_c(\sigma\eta) = \mathbf{N}_c(\sigma)]\}$.

Having σ and D, $\mathbf{M}_a(\tau) = F(\sigma, D, a)$ where $W_{F(\sigma,D,a)}$ is the set of all x for which there is an s such that the conditions (d) and either (e) or (f) below hold.

(d) $a \in D$ and $\sigma \sqsubseteq \mathrm{content}(T_{\mathbf{N}_a(\sigma)}[s])$;
(e) $x \in \mathrm{content}(T_{\mathbf{N}_a(\sigma)}[s])$ and $\mathbf{N}_c(\sigma\eta) = \mathbf{N}_c(\sigma)$ for all $c \in \{1, \ldots, b\}$, $d \in D$ and $\eta \sqsubseteq T_{\mathbf{N}_d(\sigma)}[s]$;
(f) $x \in E$ and $\mathbf{N}_c(\sigma\eta) \neq \mathbf{N}_c(\sigma)$ for some $c \in \{1, \ldots, b\}$, $d \in D$ and $\eta \sqsubseteq T_{\mathbf{N}_d(\sigma)}[s]$.

It is easy to see that the sets $W_{F(\sigma,D,a)}$ are uniformly recursively enumerable and thus the specified function F can be taken to be recursive. Thus also the learners $\mathbf{M}_1, \ldots, \mathbf{M}_b$ are recursive.

Now the properties (1), (2) and (3) are verified. If $L = \emptyset$ then all machines output on every prefix of every text for L the least index of $\emptyset$ and thus all four conditions are satisfied with the parameter $n = 0$. So let $L \in \mathcal{L} - \{\emptyset\}$ and T be a text for L.

(1) Since all machines converge on every text for L, using locking sequence argument, one can construct by induction sequences $\eta_1, \eta_2, \ldots, \eta_b \in \mathrm{seg}(L)$ such that $\eta_1 \subseteq \eta_2 \subseteq \ldots \subseteq \eta_b$ and $\mathbf{N}_c(\eta_a) = \mathbf{N}_c(\eta_a\vartheta)$ for all $a \in \{1, \ldots, b\}$, $c \in \{1, \ldots, a\}$ and $\vartheta \in \mathrm{seg}(L)$. Since η_b satisfies (a) and (b) with respect to input $T[t]$ for all sufficiently long t, one can verify that the parameter σ is eventually a stabilizing sequence on L and that the team then never changes this parameter again. Furthermore, there is a least number t such that the parameters σ and D for T have their final values σ_{fl} and D_{fl} on input $T[t]$; let n be this number t.

(2) Since σ is a stabilizing sequence on L of the learners $\mathbf{N}_1, \ldots, \mathbf{N}_b$ of $\mathcal{L}$, there is an index a with $\mathbf{N}_a(\sigma) = L$. It follows from (c) that $a \in D$. Furthermore, $\sigma \sqsubseteq T_{\mathbf{N}_a(\sigma)}$ since $\sigma \sqsubseteq T$ and $T, T_{\mathbf{N}_a(\sigma)}$ are both texts for the same language L. So (d) is satisfied for all sufficiently large s. If (f) would hold for some sufficiently large s and $d \in D_{fl}$ then d would be out of D at all sufficiently large inputs $T[t]$ in contradiction to D_{fl} being the final value which D takes. So one can conclude that for every $x \in L$ the condition (e) is satisfied for all sufficiently large s and thus $W_{\mathbf{M}_a(T[n])} = L$.

(3) Assume that $\mathbf{N}_{a'}(T[m]) = L$. Let σ, D be the corresponding parameters on input $T[m]$. Since L is not empty, $a' \in D$. Note that $W_{F(\sigma,D,a')}$ is different from L if the case (f) in the definition of $W_{F(\sigma,D,a')}$ is satisfied

for any $x \in E$ and $d \in D$: then $W_{F(\sigma,D,a')}$ is the union of E and a finite set and not in $\mathcal{L}$ by the choice of E. So, σ is a stabilizing sequence on any $W_{\mathbf{N}_d(\sigma)}$ with $d \in D$ for $\mathbf{N}_1, \ldots, \mathbf{N}_b$. Since L is among these $W_{\mathbf{N}_d(\sigma)}$, one can conclude that σ is never replaced by another parameter and $\sigma = \sigma_{fl}$. Furthermore, all members of D satisfy the selection condition (c) for all t and $D \subseteq D_{fl}$. Since D_{fl} is the final value and D can only loose but not gain members while the parameter σ does not change, $D = D_{fl}$. Since n is the first number where the parameters (σ, D) are equal to (σ_{fl}, D_{fl}), $m \geq n$.

So one has that every class in $\mathbf{TxtFex}_b$ is learnable by a team of b machines which abstains from any further mind changes whenever one of the team members outputs a correct hypothesis. In particular the team $\mathbf{M}_1, \ldots, \mathbf{M}_b$ is non U-shaped. ∎

Theorem 3.7 $\mathbf{TxtFex}_b \subseteq [2, b+1]\mathbf{NUShTxtEx}$ *for all* $b \in \mathbb{N}^+$.

Proof. For a $\mathbf{TxtFex}_b$-learnable class, construct the learners $\mathbf{M}_1, \ldots, \mathbf{M}_b$ and the set E as in the proof of Theorem 3.6. A further learner $\mathbf{M}_{b+1}$ is added which combines the hypotheses of the other learners. Let T be the text for some language L fed into the team. On input $T[m]$, $\mathbf{M}_{b+1}$ does the following where C_m denotes content$(T[m])$.

- Check whether there is an $x \leq m$ such that exactly one $a \in \{1, \ldots, b\}$ satisfies $W_{\mathbf{M}_a(T[m]),m} \cap \{0, \ldots, x\} = C_m \cap \{0, \ldots, x\}$.
- If not, output an index for E and halt.
- If so, let a_m be this unique a and x_m be the least such x.
- Compute from a_m, x_m an index k_m such that

$$W_{k_m} = \begin{cases} W_{\mathbf{M}_{a_m}(T[m])} & \text{if } W_{\mathbf{M}_c(T[m]),m}(y) = W_{\mathbf{M}_c(T[m])}(y) \\ & \text{for all } y \leq x_m \text{ and all } c \in \{1, \ldots, b\}; \\ E \cup \tilde{E} & \text{for some finite } \tilde{E}\text{, otherwise.} \end{cases}$$

- Output the hypothesis k_m and halt.

It is easy to see that $\mathbf{M}_{b+1}$ is recursive. Now fix L as a member of $\mathcal{L}$; as before T is a text for L.

Now it is shown that the machine $\mathbf{M}_{b+1}$ and thus the whole team is non U-shaped on T. That is, $\mathbf{M}_{b+1}$ does not make a mind change from a correct to an incorrect hypothesis. Let m be given and assume that $\mathbf{M}_{b+1}(T[m])$ is correct, that is, $L = W_{\mathbf{M}_{b+1}(T[m])}$. Then $W_{\mathbf{M}_{b+1}(T[m])} \neq E \cup \tilde{E}$ for all finite $\tilde{E}$ by the choice of E in the proof of Theorem 3.6. Thus $W_{\mathbf{M}_{b+1}(T[m])} = W_{\mathbf{M}_{a_m}(T[m])}$ and $m \geq n$ for the n from Theorem 3.6. Furthermore, the following conditions hold for all $o \geq m$.

- $W_{\mathbf{M}_c(T[m]),m}(y) = W_{\mathbf{M}_c(T[m])}(y)$ for all $y \leq x_m$ and all $c \in \{1, \ldots, b\}$;
- $L \cap \{0, \ldots, x_m\} = C_m \cap \{0, \ldots, x_m\} = C_o \cap \{0, \ldots, x_m\}$;
- For all $c \in \{1, \ldots, b\} - \{a_m\}$ there is $y \leq x_m$ such that $W_{\mathbf{M}_{a_m},o}(y) = W_{\mathbf{M}_{a_m}(T[m]),m}(y) \neq W_{\mathbf{M}_c(T[m]),m}(y) = W_{\mathbf{M}_c(T[m]),o}(y)$;
- x_o, a_o exist and $x_o = x_m$, $a_o = a_m$, $k_o = k_m$.

So $\mathbf{M}_{b+1}(T[o]) = \mathbf{M}_{b+1}(T[m])$ for all $o \geq m$ and $\mathbf{M}$ does not make a mind change from a correct to an incorrect hypothesis.

It remains to show that the team $[2, b+1]$**TxtEx**-learns L. From Theorem 3.6, one learner $\mathbf{M}_a$ with $a \in \{1, \ldots, b\}$ learns L. Assume now that $\mathbf{M}_a$ is the only one to do so; otherwise nothing has to be shown. Let m be so large that

- $m \geq n$ for the n from Theorem 3.6 and the machines $\mathbf{M}_1, \ldots, \mathbf{M}_b$ have already reached their final hypotheses;
- $x \leq m$ for the least x such that for all $c \in \{1, \ldots, b\} - \{a\}$ there is an $y \leq x$ with $L(y) \neq W_{\mathbf{M}_c(T[m])}(y)$;
- for all $c \in \{1, \ldots, b\}$ and $y \leq x$, $W_{\mathbf{M}_c(T[m]),m}(y) = W_{\mathbf{M}_c(T[m])}(y)$.

Then x_m, a_m exist and $x_m = x$ and $a_m = a$. In particular, k_m is an index for $W_{\mathbf{M}_a(T[m])}$ which is L. This index is only computed from x_m, a_m which are for all sufficiently large m the same, thus both $\mathbf{M}_a$ and $\mathbf{M}_{b+1}$ converge on T to an index for L. ∎

Note that this proof covers the case where $b = 1$, although one can solve this case much easier by taking $\mathbf{M}_2 = \mathbf{M}_1$. Furthermore, for $b = 2$ the following noninclusion is witnessed.

Corollary 3.8 $[2, 3]$**NUShTxtEx** $\not\subseteq$ **NUShTxtEx**.

4 Vacillatory Learning with 2 Indices

As every $\mathbf{NUShTxtFex}_b$-learner can be turned into a **NUShTxtEx**-learner identifying the same class, the restriction to *non* U-shaped learning without loss of learning power is only possible in the least class $\mathbf{TxtFex}_1$ of the $\mathbf{TxtFex}_b$ hierarchy. But, the next, quite surprising result shows that in the case of $\mathbf{TxtFex}_2$ one can avoid U-shaped learning behaviour if one gives up the constraint that the learner has to vacillate between *finitely* many indices. That is, $\mathbf{TxtFex}_2 \subseteq$ **NUShTxtBc**. In Theorem 4.2 it is shown that there is a uniform learner **U** which is given a set E of up to 2 indices and **NUShTxtBc**-identifies every $\{W_e : e \in E\}$ such that every hypothesis is a subset of an W_e with $e \in E$. Then this result is combined with Theorem 3.6 to show the inclusion $\mathbf{TxtFex}_2 \subseteq$ **NUShTxtBc**. But before turning to Theorem 4.2, the following auxiliary proposition gives a method to enforce that the two sets traced are represented by approximations which are different at all relevant stages of the enumeration.

Proposition 4.1 *Given a set $F = \{i', j'\}$ one can compute a set $G(F) = \{i, j\}$ such that*

- *For all s, either $W_{i,s} \cup W_{j,s} = \emptyset$ or $W_{i,s} \neq W_{j,s}$;*
- *$W_i \subseteq W_{i'}$ and if $W_{i'}$ is infinite then $W_i = W_{i'}$;*
- *$W_j \subseteq W_{j'}$ and if $W_{j'}$ is infinite then $W_j = W_{j'}$;*
- *$\{W_{i'}, W_{j'}\} \subseteq \{W_i, W_j\}$.*

This also holds with $j' = i'$ for the case that $F = \{i'\}$.

Proof. We let $W_i = \bigcup_s W_{i,s}$ and $W_j = \bigcup_s W_{j,s}$ for the approximations obtained by the following inductive construction: $W_{i,0} = \emptyset$ and $W_{j,0} = \emptyset$. From stage s to $s+1$, the approximations are updated as follows:

$$W_{i,s+1} = \begin{cases} W_{i',s+1} & \text{if } W_{i',s+1} \neq W_{j',s+1}; \\ W_{i',s} & \text{if } W_{i',s} \neq W_{i',s+1} = W_{j',s+1}; \\ W_{i',s} & \text{if } W_{i',s} = W_{i',s+1} = W_{j',s+1} \neq W_{j',s}; \\ W_{i,s} & \text{if } W_{i',s} = W_{i',s+1} = W_{j',s} = W_{j',s+1}; \end{cases}$$

$$W_{j,s+1} = \begin{cases} W_{j',s+1} & \text{if } W_{i',s+1} \neq W_{j',s+1}; \\ W_{j',s+1} & \text{if } W_{i',s} \neq W_{i',s+1} = W_{j',s+1}; \\ W_{j',s} & \text{if } W_{i',s} = W_{i',s+1} = W_{j',s+1} \neq W_{j',s}; \\ W_{j,s} & \text{if } W_{i',s} = W_{i',s+1} = W_{j',s} = W_{j',s+1}. \end{cases}$$

For the verification, note the following: If $W_{i,s} \neq W_{j,s}$ or $W_{i,s'+1} \neq W_{j',s+1}$ then $W_{i,s+1} \neq W_{j,s+1}$. If at stage $s+1$ either $W_{i'}$ or $W_{j'}$ receives new elements then $W_{i,s+1} \neq W_{j,s+1}$, $W_{i,s+1} \in \{W_{i',s}, W_{i',s+1}\}$ and $W_{j,s+1} \in \{W_{j',s}, W_{j',s+1}\}$. By induction, $W_{i,s+1} \neq W_{j,s+1}$ whenever at least one of them is different from $\emptyset$. Furthermore, $W_i = W_{i'}$ and $W_j = W_{j'}$ whenever one of these sets is infinite or these two sets are different. Only if $W_{i'} = W_{j'}$ and both sets are finite, one of the sets W_i, W_j will be a proper subset of its counterpart which also cannot be avoided due to the goal that $W_{i,s+1} \neq W_{j,s+1}$ whenever at least one of these sets is not empty. ∎

Theorem 4.2 *There is a learner* $\mathbf{U}$ *such that for every r.e. sets* L, H *and every set* F *of indices for* L, H *with* $|F| \leq 2$, $\mathbf{U}$ **NUShTxtBc***-identifies* $\{L, H\}$ *using the additional information* F*. Furthermore, for every* $\sigma \in seg(\mathbb{N})$,

(1) $W_{\mathbf{U}(F,\sigma)} \subseteq L$ *or* $W_{\mathbf{U}(F,\sigma)} \subseteq H$*;*
(2) *if* $L = H$ *and* L *is infinite then* $W_{\mathbf{U}(F,\sigma)} \in \{\emptyset, L\}$.

Note that $L = H$ *is explicitly permitted.*

Proof. Given F, let $G(F)$ as in Proposition 4.1. For a finite string σ, $\mathbf{U}(F, \sigma)$ outputs an index k for a set W_k enumerated by the algorithm given in Figure 1. Note the following: If the length of σ is 0 then its content is $\emptyset$ and the algorithm goes to (Empty). Thus $|\sigma| > 0$ when defining $\tau = \sigma[|\sigma| - 1]$. Furthermore, the goal of the condition (Min) in (Enum) is just to guarantee that the search always terminates and that $t \geq s$ in order to get that the set $W_{k,u}$ is really a recursively enumerable set and that no elements once enumerated are thrown out later.

We first show that the algorithm **TxtBc**-learns $\{\emptyset, L, H\}$. The learning of $\emptyset$ is clear since the algorithm goes to the label (Empty) and conjectures $\emptyset$ whenever the content of the input is empty set.

The role of L, H is symmetric, so assume that $L \neq \emptyset$ and T being a text for L. For input $T[n]$, let k_n, C_n, i_n, j_n, x_n be the corresponding parameters of the

Uniform non U-shaped Behaviourally Correct Learner U
Parameter: F. Input: σ. Output: k, specified implicitly.
Algorithm to enumerate $W_k = \bigcup_u W_{k,u}$.
(Start) Let $u = |\sigma|$, $C = \text{content}(\sigma)$ and $s = 0$.
 Let $W_{k,t} = \emptyset$ for all $t < |\sigma|$.
 If $C = \emptyset$ or $W_{e,u} = \emptyset$ for all $e \in G(F)$ Then go to (Empty).
 Let $\tau = \sigma[|\sigma| - 1]$.
 Select i, j, x such that
  (a) $\{i, j\} = G(F)$;
  (b) $x = \min(W_{i,u} \mathbf{\Delta} W_{j,u})$;
  (c) $x \in W_{i,u} \Leftrightarrow x \in C$.
 Go to (Branch).
(Branch) If $C \cup W_{i,s} \subseteq W_{\mathbf{U}(F,\tau),u}$ and $y \in (W_{i,u} - W_{i,|\sigma|}) \cup (W_{j,u} - W_{j,|\sigma|})$ for some $y \le x$ Then go to (Copy) Else go to (Enum).
(Enum) Let t be the maximal element of $\{s, \dots, u\}$ such that one of the following conditions holds:
  (Min) $t = s$;
  (Inf) $C \subseteq W_{i,t} \subset W_{i,u} \wedge \forall y \le x\, (y \notin (W_{i,u} - W_{i,|\sigma|}) \cup (W_{j,u} - W_{j,|\sigma|}))$;
  (Diff) $C \subseteq W_{i,t} \subset W_{i,u}$ and $W_{j,u} = W_{i,s}$;
  (Equal) $C \subseteq W_{i,t} \cup W_{j,t} \subset W_{i,u} \cap W_{j,u}$;
  (Sub) $C \subseteq W_{i,t} \subseteq W_{\mathbf{U}(F,\tau),u}$;
  (Exact) $C = W_{i,t}$ and $t = |\sigma|$.
 Let $W_{k,u} = W_{i,t}$, update $s = t$, $u = u + 1$ and go to (Branch).
(Copy) Let $W_{k,u} = W_{\mathbf{U}(F,\tau),u}$, update $u = u + 1$ and go to (Copy).
(Empty) Let $W_{k,u} = \emptyset$, update $u = u + 1$ and go to (Empty).

Figure 1. Algorithm to enumerate $W_{\mathbf{U}(F,\sigma)}$ from Theorem 4.2.

variables in the algorithm where i_n, j_n, x_n are only defined if the algorithm does not go to (Empty). One of the following two Cases 1.1, 1.2 applies.

(Case 1.1) $L = H$ and L is infinite.
Let n be the first step such that the algorithm does not go to the label (Empty). Note that $n > 0$; $T[n-1]$ exists and $W_{\mathbf{U}(F,T[n-1])} = \emptyset$. Thus the algorithm does not reach the label (Copy) and the indices i_n, j_n satisfy $W_{i_n} = W_{j_n} = L$. For every t and all sufficiently large u the condition (Equal) in (Enum) is satisfied and thus $W_{i_n,t} \subseteq W_{k_n}$. In particular $W_{i_n} = W_{k_n}$. For $m = n, n+1, \dots$ one can see that either $W_{k_{m+1}} = W_{k_m}$ since the algorithm goes infinitely often through (Copy) or $W_{k_{m+1}} = W_{k_m}$ since the algorithm goes infinitely often through (Enum) and the condition (Sub) is eventually satisfied for every t. In particular, **U TxtBc**-learns L from T.

(Case 1.2) $L \ne H$ or L is finite.
The two sets L', H' enumerated by the indices in $G(F)$ satisfy $L' \ne H'$ and $L \in \{L', H'\}$. Let $x = \min(L' \mathbf{\Delta} H')$. Assume that n is so large such that all $e \in G(F)$ and all y satisfy the following:
 - if L is finite and $y \in L$ then $y \in C_n$ and $W_{e,n}(y) = W_e(y)$;
 - if $y \le x$ then $C_n(y) = L(y)$ and $W_{e,n}(y) = W_e(y)$.
This implies that $W_{e,n} \ne \emptyset$ for some $e \in G(F)$ and the algorithm does not

go to label (Empty). Thus x_n, i_n, j_n are defined and i_n is the unique index in $G(F)$ with $W_{i_n} = L$. Since $W_{i_n,n}, W_{j_n,n}$ coincide with W_{i_n}, W_{j_n} up to x_n, the algorithm will never reach the label (Copy) and only enumerate elements of W_{i_n} into L, thus $W_{k_n} \subseteq W_{i_n}$. If L is finite then $L = C_n$ and the elements of L go into W_{k_n} due to the condition (Exact) in (Enum). If L is infinite then the algorithm will eventually enumerate all elements in any set $W_{i_n,t}$ since for every t there is a u with $W_{i_n,t} \subset W_{i_n,u}$ by condition (Inf). So in both cases, **U TxtBc**-learns L.

Additional property (2) has already been shown in Case 1.1. Now additional property (1) is verified inductively. For $n = 0$ the hypothesis is $\emptyset$ and the property is true. Having the property for n, consider the hypothesis $W_{k_{n+1}}$. If the algorithm for $W_{k_{n+1}}$ goes to label (Empty) then $W_{k_{n+1}} = \emptyset$ and the property holds for $n+1$. If the algorithm for $W_{k_{n+1}}$ reaches the label (Copy) then $W_{k_{n+1}} = W_{k_n}$ and the property is true by induction. Otherwise $W_{k_{n+1}} \subseteq W_{i_{n+1}}$ and $W_{i_{n+1}}$ is a subset of either L or H, so additional property (1) holds also in this case.

Now it is shown that the learner is non U-shaped. Let n be the first index with $W_{k_n} = L$. Below let P_n denote the property:

$W_{i_n} \cap \{0, \ldots, x_n\} = W_{i_n,n} \cap \{0, \ldots, x_n\}$ and $W_{j_n} \cap \{0, \ldots, x_n\} = W_{j_n,n} \cap \{0, \ldots, x_n\}$.

One of the following Cases 2.1, 2.2, 2.3, 2.4 applies.

(Case 2.1) $L = W_{i_n}$ and $L \subset W_{\mathbf{U}(F,T[n-1])}$.
In this case clearly, $W_{i_n} \neq W_{j_n}$ via the additional property (1) proved above. Note that the algorithm on input $T[n]$ never reaches the label (Copy) in this case, since otherwise $W_{k_n} = W_{\mathbf{U}(F,T[n-1])} \neq L$. We now claim that P_n holds. To see this, note that if $L = W_{i_n}$ is finite, then P_n holds (otherwise the procedure would eventually go to (Copy) via (Branch)). If $L = W_{i_n}$ is infinite, then since $W_{i_n} \neq W_{j_n}$, (Diff), (Equal), (Exact) cannot act infinitely often. Furthermore if (Inf) acts infinitely often, then clearly P_n holds. If (Sub) acts infinitely often, then P_n holds (otherwise the procedure would eventually go to (Copy) via (Branch)).
Furthermore $W_{\mathbf{U}(F,T[n-1])} \subseteq W_{j_n}$ and thus $x_n \in W_{j_n} - W_{i_n}$. Thus, for all $m \geq n$, $x_m = x_n$, $x_n \in W_{j_m,m} - W_{i_m,m}$ and $i_m = i_n$.
Now one shows by induction that $W_{i_m} = L$ for all $m \geq n$. It clearly holds for $m = n$. Assume now as inductive hypothesis for $m \geq n$ that $W_{k_m} = L$. If the algorithm for input $T[m+1]$ reaches (Copy) then $W_{k_{m+1}} = W_{k_m}$. If it goes through (Enum) infinitely often then on one hand $W_{k_{m+1}} \subseteq W_{i_{m+1}} = L$ and on the other hand $W_{k_{m+1}} \supseteq W_{i_{m+1},t}$ for all t with $C_m \subseteq W_{i_m,t}$ since for sufficiently large u the condition (Sub) is satisfied. Again $W_{k_{m+1}} = L$.

(Case 2.2) $L = W_{i_n}$ and $L \not\subseteq W_{\mathbf{U}(F,T[n-1])}$ and L is finite.
There is a first u where $W_{k_n,u} = L$. This cannot happen in (Copy), so the algorithm on input $T[n]$ goes to (Enum) infinitely often. In (Enum), W_{k_n} can become equal to W_{i_n} only due to the condition (Exact) since the conditions (Inf), (Diff), (Equal) give only proper subsets of L and (Sub)

does not apply by $L \not\subseteq W_{\mathbf{U}(F,T[n-1])}$. Thus $C_n = L = W_{i_n,n} = W_{i_n}$. Thus, for all $m \geq n$, $C_m = L = W_{i_n,m} = W_{i_n}$, and hence $i_m = i_n$. Then one can show by induction as in the last paragraph of Case 2.1 that $W_{i_m} = L$ for all $m \geq n$.

(Case 2.3) $L = W_{i_n}$ and $L \not\subseteq W_{\mathbf{U}(F,T[n-1])}$ and L is infinite.
If $W_{i_n} = W_{j_n}$ then the condition (Sub) in (Enum) and (Copy) guarantee that $W_{i_{m+1}} = W_{i_m}$ for all $m \geq n$. If $W_{i_n} \neq W_{j_n}$ then W_{k_n} being infinite means that for every t with $C_n \subseteq W_{i,t}$ there is an $u \geq t$ such that one of the conditions (Inf), (Diff), (Equal), (Sub) and (Exact) are satisfied. Due to $W_{i_n} \neq W_{j_n}$, $W_{i_n} \not\subseteq W_{\mathbf{U}(F,T[n-1])}$ the conditions (Diff), (Equal) and (Sub) are satisfied only for finitely many t. Similarly (Exact) is satisfied only for $t = n$ if at all. Thus for almost all t, $W_{i,t}$ is enumerated into $W_{k_n,u}$ by satisfying condition (Inf). Thus all $y \in (W_{i_n} - W_{i_n,n}) \cup (W_{j_n} - W_{j_n,n})$ are strictly larger than x_n. So $x_m = x_n$ for all $m \geq n$. Since $W_{i_n} \neq W_{j_n}$ and the learner identifies L, $i_m = i_n$ for almost all m. Considering sufficiently large m gives $x_n \in L$ iff $x_n \in C_m$ iff $x_n \in W_{i_n,m}$ iff $x_n \in W_{i_n,n}$ iff $x_n \in C_n$ and thus $i_m = i_n$ for all $m \geq n$. Now one can prove inductively for all $m \geq n$ that $W_{k_m} = L$ as in the last paragraph of Case 2.1.

(Case 2.4) $L \subset W_{i_n}$.
Then L is finite since $L = W_{i_n,s}$ for some s. Furthermore $L = W_{j_n}$ since $L \neq W_{i_n}$. Since $W_{k_{n-1}} \neq W_{k_n}$, the algorithm on input $T[n]$ never goes to the label (Copy) and thus goes through (Enum) infinitely often. If u is so large that $W_{i_n,s} \subset W_{i_n,u}$ and $W_{j_n,u} = L$ then there cannot be any t with $W_{i_n,s} \subset W_{i_n,t} \subset W_{i_n,u}$ since otherwise the set W_{k_n} would be increased according to the condition (Diff) in (Enum). It follows that for all $t \geq s$, either $W_{i_n,t} = L$ or $W_{i_n,t} = W_{i_n}$. Now one can show by induction that $W_{k_m} = L$ for all $m \geq n$. Having this for W_{k_m}, consider the following cases for $W_{k_{m+1}}$: if the algorithm reaches label (Copy) then $W_{k_{m+1}} = W_{k_m}$. If it ends up going through (Enum) infinitely often and $i_{m+1} = j_n$ then $W_{k_{m+1}} = W_{j_n} = L$ due to condition (Sub) in (Enum). If it ends up going through (Enum) infinitely often and $i_{m+1} = i_n$ then $W_{k_{m+1}} \supseteq W_{i_n,s}$ by the condition (Sub). But no condition permits to get $W_{k_{m+1}} \supseteq W_{i_{m+1},t}$ for some t with $W_{i_{m+1},t} \supset W_{i_{m+1},s}$ since for this t there is no u with $W_{i_{m+1},t} \subset W_{i_{m+1},u}$ as required in the conditions (Inf), (Diff) and (Equal). Furthermore, (Exact) and (Sub) do not enforce $W_{i_{m+1},t} \subseteq W_{k_{m+1}}$ since $W_{i_{m+1},t}$ is neither a subset of C_{m+1} nor of W_{k_m}. So again $W_{k_{m+1}} = L$.

This completes the proof. ∎

Theorem 4.3 *Every* **TxtFex**$_2$*-learnable class is* **NUShTxtBc***-learnable.*

Proof. Given a **TxtFex**$_2$-learnable class $\mathcal{L}$, there is by Theorem 2.3 a pair of two learners $\mathbf{N}_1, \mathbf{N}_2$ which converge on every language from $\mathcal{L}$ and $[1,2]$**TxtEx**-identify $\mathcal{L}$. Obtain from these two learners the team $\mathbf{M}_1, \mathbf{M}_2$ as done in the proof of Theorem 3.6. Note that one can, for each input σ, check in the limit whether one of the learners $\mathbf{M}_1, \mathbf{M}_2$ outputs an index which enumerates some elements using part (f) of the algorithm in the proof of Theorem 3.6.

Now one builds the following new learner $\mathbf{U}'$:

$$W_{\mathbf{U}'(\tau)} = W_{\mathbf{U}(\{e_1,e_2\},\tau)} \text{ where, for } a = 1,2,$$
$$W_{e_a} = \begin{cases} W_{\mathbf{M}_a(\tau)} & \text{if (f) is not used for } W_{\mathbf{M}_1(\tau)} \text{ or } W_{\mathbf{M}_2(\tau)}; \\ W_{\mathbf{M}_1(\tau)} \cup W_{\mathbf{M}_2(\tau)} & \text{if (f) is used for } W_{\mathbf{M}_1(\tau)} \text{ or } W_{\mathbf{M}_2(\tau)}. \end{cases}$$

Let $L \in \mathcal{L}$ and T be a text for L. Let n be the first number where one of the sets $W_{\mathbf{M}_1(T[n])}, W_{\mathbf{M}_2(T[n])}$ is L. Then, for any $m \geq n$ and any $a \in \{1,2\}$, $W_{\mathbf{M}_a(T[m])} = W_{\mathbf{M}_a(T[n])}$ and $W_{\mathbf{M}_a(T[m])}$ is not of the form $E \cup \tilde{E}$ where $\tilde{E}$ is finite. Then $\mathbf{U}$ is fed with the same parameter set $\{e_1, e_2\}$ for all $m \geq n$ and one of the e_1, e_2 enumerates L. Thus $\mathbf{U}'$ **TxtBc**-learns L on T.

It remains to show that $\mathbf{U}'$ is non U-shaped on T. This is clearly true if L is the empty set. So assume $L \neq \emptyset$. Consider any m with $W_{\mathbf{U}'(T[m])} = L$. If case (f) of the algorithm for $\mathbf{M}_1(T[m])$ and $\mathbf{M}_2(T[m])$ applies, then W_{e_1} and W_{e_2} are the same infinite set $E \cup \tilde{E}$ for some finite set $\tilde{E}$. It follows by the additional property (2) of $\mathbf{U}$ in Theorem 4.2 that $\mathbf{U}'(T[m])$ either outputs an index for the empty set or for $E \cup \tilde{E}$; both sets are different from L, thus case (f) does not apply. Hence, $T[m]$ is a stabilizing sequence for both $\mathbf{M}_1, \mathbf{M}_2$ on those sets $W_{\mathbf{M}_1(T[m])}, W_{\mathbf{M}_2(T[m])}$ which are not empty. Since one of these is a superset of L by the additional property (1) of $\mathbf{U}$ in Theorem 4.2, it follows that $\mathbf{M}_1, \mathbf{M}_2$ do not change mind on T beyond $T[m]$. For $a = 1,2$ the parameter e_a is defined as above for $\tau = T[m]$ and it holds that $W_{e_a} = W_{\mathbf{M}_a(T[m])}$. Thus $\mathbf{U}'(T[o])$ coincides with $\mathbf{U}(\{e_1,e_2\}, T[o])$ for all $o \geq m$ and $\mathbf{U}$ with the parameter set $\{e_1, e_2\}$ is non U-shaped on the text T for L. The same holds for $\mathbf{U}'$. Thus $\mathbf{U}'$ **NUShTxtBc**-learns $\mathcal{L}$. ∎

5 Vacillatory Learning with 3 Indices

From Theorem 3.1 it is already known that U-shaped learning behaviour is necessary for $\mathbf{TxtFex}_b$ $(b > 1)$ identification of any class in $\mathbf{TxtFex}_b - \mathbf{TxtFex}_1$ for all $b > 1$. Theorem 5.1 strengthens this result by showing that, for some classes of languages in $\mathbf{TxtFex}_b$ for $b > 2$, the necessity of U-shaped behaviour cannot be circumvented by allowing infinitely many correct grammars in the limit, that is, by shifting to the more liberal criterion of **TxtBc**-identification. This is one of the rare cases in inductive inference where the containment in a class defined without numerical parameters holds for level 2 but not for level 3 and above of a hierarchy. The proof is a diagonalization proof reminiscent of the proof of Theorem 4 in [20].

Theorem 5.1 $\mathbf{TxtFex}_3 \not\subseteq$ **NUShTxtBc**.

Proof. Let $L_{i,j} = \{\langle i,j,k\rangle : k \in \mathbb{N}\}$, $I_{i,j} = W_i \cap L_{i,j}$ and $J_{i,j} = W_j \cap L_{i,j}$ for $i, j \in \mathbb{N}$. The class

$$\mathcal{L} = \{L_{i,j} : i,j \in \mathbb{N}\} \cup \{I_{i,j}, J_{i,j} : i,j \in \mathbb{N} \wedge I_{i,j} \subset J_{i,j} \wedge |I_{i,j}| < \infty\}$$

witnesses the separation.

To see that $\mathcal{L}$ is in $\mathbf{TxtFex}_3$, consider the following machines $\mathbf{N}_I, \mathbf{N}_J, \mathbf{N}_L$ which initially output indices of the empty set. Each of them waits for the first tuple of the form $\langle i, j, k\rangle$ for some k to come up in the input. From then on, $\mathbf{N}_I$ outputs an index for $I_{i,j}$ forever, $\mathbf{N}_J$ an index for $I_{i,j}$ forever and $\mathbf{N}_L$ an index for $L_{i,j}$ forever. So, for every $i, j \in \mathbb{N}$, $\mathbf{N}_I$ learns the set $I_{i,j}$, $\mathbf{N}_J$ the set $I_{i,j}$ and $\mathbf{N}_L$ the set $L_{i,j}$. The class $\mathcal{L}$ is learnable by a team of three machines which converge on every text for every language in $\mathcal{L}$ to some index. It follows from Theorem 2.3 that $\mathcal{L}$ is in $\mathbf{TxtFex}_3$.

So it remains to show that $\mathcal{L}$ is not in $\mathbf{NUShTxtBc}$, that is, to show that any given $\mathbf{TxtBc}$-learner for $\mathcal{L}$ is U-shaped on some text for some language in $\mathcal{L}$. Given the learner $\mathbf{M}$, one defines the following function F by an approximation from below:

$$F_s(i,j) = \begin{cases} F_{s-1}(i,j) & \text{if } s > 0 \text{ and} \\ & W_{\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,F_{s-1}(i,j)\rangle),s} \subseteq L_{i,j}; \\ k & \text{otherwise where } k \text{ is the first number} \\ & \text{found with } k > F_{s-1}(i+j) + s \text{ and} \\ & \{\langle i,j,0\rangle, \langle i,j,1\rangle, \ldots, \langle i,j,k\rangle\} \subset \\ & \qquad\qquad W_{\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,k\rangle)}. \end{cases}$$

Since $\langle i,j,0\rangle, \langle i,j,1\rangle, \ldots$ is a text for $L_{i,j}$ and $\mathbf{M}$ $\mathbf{TxtBc}$-learns $L_{i,j}$, almost all hypotheses $\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,k\rangle)$ are indices for $L_{i,j}$. Thus the k is always found in the second part of the definition of F_s and F_s is well-defined. Furthermore, if $F_{s-1}(i,j)$ is sufficiently large, the condition $W_{\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,F_{s-1}(i,j)\rangle),s} \subseteq L_{i,j}$ holds for all s and thus $F_s(i,j) = F_{s-1}(i,j)$. So the limit $F(i,j)$ of all $F_s(i,j)$ exists and is approximated from below. By considering the first s where $F(i,j) = F_s(i,j)$ and the fact that it is then no longer updated, one has

$$\{\langle i,j,0\rangle, \langle i,j,1\rangle, \ldots, \langle i,j,F(i,j)\rangle\} \subset W_{\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,F(i,j)\rangle)} \subseteq L_{i,j}.$$

Now there are r.e. sets W_a, W_b such that

$$\begin{aligned} W_a &= \{\langle i,j,l\rangle : i,j \in \mathbb{N} \wedge l \in \{0,1,\ldots,F(i,j)\}\}, \\ W_b &= \{\langle i,j,l\rangle : i,j \in \mathbb{N} \wedge \exists t > l\,(\langle i,j,l\rangle \in W_{\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,F_t(i,j)\rangle),t})\}. \end{aligned}$$

Now fix the parameters i, j such that $i = a$ and $j = b$; the cases where $i \neq a$ or $j \neq b$ are not important in the below considerations.

Assume that $\langle i,j,l\rangle \in W_j$ using a parameter t with $F_t(i,j) \neq F(i,j)$. Let s be the first stage with $F_s(i,j) = F(i,j)$; note that $s > t$. Then by the definition of F_s, $F(i,j) = F_s(i,j) > s > t$ and $\{\langle i,j,0\rangle, \langle i,j,1\rangle, \ldots, \langle i,j,F_s(i,j)\rangle\} \subset W_{\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,F_s(i,j)\rangle)}$. So $\langle i,j,l\rangle$ is in $W_{\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,F(i,j)\rangle)}$ as well.

Thus $\{\langle i,j,0\rangle, \langle i,j,1\rangle, \ldots, \langle i,j,F(i,j)\rangle\} = W_i \cap L_{i,j} = I_{i,j} \subset J_{i,j} = W_j \cap L_{i,j} = W_{\mathbf{M}(\langle i,j,0\rangle\,\langle i,j,1\rangle\,\ldots\,\langle i,j,F(i,j)\rangle)}$ and $I_{i,j}$ is finite. Hence $I_{i,j}, J_{i,j} \in \mathcal{L}$.

There is a text T for $J_{i,j}$ which starts with $\langle i, j, 0\rangle\ \langle i, j, 1\rangle\ \ldots\ \langle i, j, F(i, j)\rangle$ and after which $\mathbf{M}$ outputs an index for $J_{i,j}$. Then a lot of pause symbols follow until $\mathbf{M}$ outputs an index for $I_{i,j}$. After that the remaining elements of $J_{i,j}$ follow and $\mathbf{M}$ has to return to the index $J_{i,j}$. So $\mathbf{M}$ is not a **NUShTxtBc**-learner for $\mathcal{L}$. Since $\mathbf{M}$ was chosen arbitrarily, $\mathcal{L}$ is not **NUShTxtBc**-learnable. ∎

Intriguingly, the proof of Theorem 5.1 above features the contrast between learning a finite table and learning a general rule. This contrast is often invoked in accounts of U-shaped learning behaviour in children [9]. In our proof the finite table is embodied by the set $I_{i,j}$, while the (possibly) infinite set, only specified / learnable by a general rule is the sets $J_{i,j}, L_{i,j}$. The presence of both these types of sets is the key for the impossibility of learning (even in the **TxtBc** sense) the class $\mathcal{L}$ with a non U-shaped learner. Observe that the proof does *not* feature the learning of an incorrect general rule followed by a correct general rule augmented by a finite table, as in most psychological accounts of U-shaped learning behaviour. Actually the proof shows that any learner of $\mathcal{L}$ when confronted with the task of learning $J_{i,j}$ is forced to overgeneralize when fed the finite table $I_{i,j} = \{\langle i, j, 0\rangle, \ldots, \langle i, j, F(i, j)\rangle\} \subset J_{i,j}$ which it also has to learn – $\mathbf{M}$ conjectures $J_{i,j}$ although the data is from the proper subset $I_{i,j}$. After that and seeing long enough only elements from $I_{i,j}$, $\mathbf{M}$ correctly learns this finite table. But then, it eventually returns to the general rule representing the set $J_{i,j}$ when more examples from this set come up.

Since $\mathbf{TxtFex}_3 \subset \mathbf{TxtFex}_4 \subset \ldots \subset \mathbf{TxtFex}_*$, one immediately gets the following corollary.

Corollary 5.2 $(\forall b \in \{3, 4, \ldots, *\})\,[\mathbf{TxtFex}_b \not\subseteq \mathbf{NUShTxtBc}]$.

A further corollary is that the counterpart of Theorem 4.2 does not hold for sets of three indices. Indeed, if such an algorithm would exist, then one could **NUShTxtBc**-learn $\mathcal{L}$ from Theorem 5.1 by conjecturing $\emptyset$ until the first triple $\langle i, j, k\rangle$ comes up and then simulating the uniform learner with a set of three indices for the sets $I_{i,j}, J_{i,j}, L_{i,j}$ from then on without changing this parameter set anymore. But Theorem 5.1 clearly showed that such a learner does not exist.

Corollary 5.3 *No machine uniformly* **NUShTxtBc**-*learns* $\{W_e : e \in F\}$ *with* F *as additional information where* F *is a set of* 3 *indices.*

6 Teams Revisited

Classes in $\mathbf{TxtFex}_2$ are in **TxtBc** and in $[1, 2]$**NUShTxtEx**. The next proposition shows that one cannot weaken the condition of being in $\mathbf{TxtFex}_2$ to the combination of the two consequences in Theorem 4.3. Furthermore the condition that the team members converge on every text for a language in $\mathcal{L}$ is essential in Theorem 2.3.

Proposition 6.1 *The class $\mathcal{L}$ from Theorem 5.1 is* $[1,2]$**NUShTxtEx***-learnable and* $\mathbf{TxtFex}_3$*-learnable but it is not* **NUShTxtBc***-learnable.*

Proof. By Theorem 5.1, $\mathcal{L}$ is $\mathbf{TxtFex}_3$-learnable but not **NUShTxtBc**-learnable. So it remains to show that $\mathcal{L}$ is $[1,2]$**NUShTxtEx**-learnable.

Let L be a language and T be a text for L. The behaviour of the team on T is explained as follows where $C_n = \text{content}(T[n])$. As long as $C_n = \emptyset$, both learners output a fixed index for $\emptyset$. If $C_n \neq \emptyset$ then determine the components i, j of the first data item $\langle i, j, k \rangle$ in $T[n]$ which is not $\#$. The first learner outputs a fixed index for $L_{i,j}$. The second learner considers an index e of $I_{i,j}$ computed from i, j. On input $T[n]$, the learner computes the least m satisfying the following:

- $W_{e,m} = W_{e,n}$;
- $C_m \neq \emptyset$;
- $C_m \subseteq W_{e,m} \Leftrightarrow C_n \subseteq W_{e,n}$.

Then the second learner outputs an hypothesis e_m which is given as

$$W_{e_m} = \begin{cases} L_{i,j} \cup L_{i+1,j+1} & \text{if } W_{e,m} \subset W_e; \\ I_{i,j} & \text{if } C_m \subseteq W_{e,m} \text{ and } W_{e,m} = W_e; \\ J_{i,j} & \text{if } C_m \not\subseteq W_{e,m} \text{ and } W_{e,m} = W_e. \end{cases}$$

Note that this hypothesis depends only on m and C_m but not on n and C_n. Now it is shown that the team correctly $[1,2]$**NUShTxtEx**-learns L from T. There are five cases, in the last four cases it is assumed that $L \in \mathcal{L}$.

$L \notin \mathcal{L}$**:** Then one only has to verify that the algorithm for both learners is recursive which can be done easily. By the way, this is also satisfied for all below cases.

$L = \emptyset$**:** Then it is easy to see that both members of the team always output the same index for $\emptyset$.

$L = I_{i,j}$ **and** $L \neq \emptyset$**:** Then $I_{i,j} \subset J_{i,j}$ and $I_{i,j}$ is finite. The first learner never outputs an hypothesis for $I_{i,j}$ and is thus non U-shaped. The second learner outputs incorrect hypotheses which are either $\emptyset$ or $L_{i,j} \cup L_{i+1,j+1}$ until n is so large that $W_{e,n} = I_{i,j}$ and $C_n \neq \emptyset$. From then on the second learner outputs the index e_m for the first m such that $C_m \neq \emptyset$ and $W_{e,m} = W_{e,n}$; this index e_m indeed enumerates $I_{i,j}$ and is never replaced by another one.

$L = J_{i,j}$ **and** $L \subset L_{i,j}$**:** Then $I_{i,j} \subset J_{i,j}$ and $I_{i,j}$ is finite. The first learner never outputs an hypothesis for $J_{i,j}$ and is thus non U-shaped. The second learner outputs incorrect hypotheses which are either $\emptyset$ or $L_{i,j} \cup L_{i+1,j+1}$ or $I_{i,j}$ until n is so large that $W_{e,n} = I_{i,j}$ and $C_n \not\subseteq I_{i,j}$. From then on the second learner outputs the index e_m for the first m such that $C_m \not\subseteq W_{e,n}$ and $W_{e,m} = W_{e,n}$; this index e_m indeed enumerates $J_{i,j}$ and is never replaced by another one.

$L = L_{i,j}$**:** Then the first learner outputs finitely often an index for $\emptyset$ and then makes exactly one mind change to an index for $L_{i,j}$. If $I_{i,j}$ is infinite then the second learner always outputs an hypothesis for $L_{i,j} \cup L_{i+1,j+1}$ which is incorrect. If $I_{i,j}$ is finite then the second learner outputs finitely often incorrect hypotheses until it makes a final mind change to an hypothesis for $J_{i,j}$, which can be verified as in the previous case. Thus the team is non U-shaped in both subcases, the one where $J_{i,j} = L_{i,j}$ and the one where $J_{i,j} \neq L_{i,j}$.

So the team witnesses that $\mathcal{L}$ is indeed $[1,2]$**NUShTxtEx**-learnable. ∎

A corollary of the result is that $\mathbf{TxtFex}_2 \subset [1,2]$**NUShTxtEx**. This can also be shown in general for all $b > 1$.

Proposition 6.2 $\mathbf{TxtFex}_b \subset [1,b]$**NUShTxtEx** *for all* $b \in \{2,3,\ldots\}$.

Proof. The inclusion is from Theorem 3.6. Its properness is witnessed by the class $\mathcal{L}$ of all sets $\mathbb{N} - F$ where $1 \leq |F| \leq b$. The learner $\mathbf{M}_a$ determines for input σ the least a numbers $x_1, \ldots, x_a \notin \mathrm{content}(\sigma)$ and conjectures the set $\mathbb{N} - \{x_1, \ldots, x_a\}$. If $|F| = a$ then $\mathbf{M}_a$ **NUShTxtEx**-learns $\mathbb{N} - F$. If $|F| \neq a$ then $\mathbf{M}_a$ never outputs an hypothesis for $\mathbb{N} - F$. Thus, the team $\mathbf{M}_1, \ldots, \mathbf{M}_b$ is $[1,b]$**NUShTxtEx**-learning the class $\mathcal{L}$. It is well-known that $\mathcal{L}$ is not in **TxtBc** and thus also not in $\mathbf{TxtFex}_b$ whenever $b \geq 2$. ∎

Remark 6.3 $\mathbf{TxtFex}_* \not\subseteq [1,b]$**TxtEx** *for all* $b \in \mathbb{N}^+$. *This is witnessed by the class* $\{W_e : e \in \mathbb{N} \wedge \{0,1,\ldots,e\} \subseteq W_e \subset \mathbb{N}\}$. *Note that by Proposition 6.1 it can be that a class in* $\mathbf{TxtFex}_{b+1} - \mathbf{TxtFex}_b$ *is already* $[1,b]$**TxtEx**-*learnable.*

A further interesting question is whether one can at least obtain non U-shaped team learning for arbitrary team learnable classes. This is true for $[1,1]$**TxtEx** by Theorem 3.6 but it fails for $[1,2]$**TxtEx**-learning.

Theorem 6.4 *For all* $b \in \{2,3,\ldots\}$, $[1,b]$**NUShTxtEx** $\subset [1,b]$**TxtEx**.

Proof. Let $b \in \{2,3,\ldots\}$. Since the inclusion is obvious from the definition, one only has to show that it is a proper inclusion. Consider a list $team_0, team_1, \ldots$ of all teams of b **TxtEx**-learners. Let $D_0, D_1, \ldots$ be an enumeration of all sets of $b-1$ elements such that $m < n$ whenever $\max(D_m) < \max(D_n)$. We say that $team_m$ *qualifies at* n iff there is an ordering $\mathbf{M}_1, \ldots, \mathbf{M}_b$ of the machines in $team_m$ and a string $\sigma_{m,n} \in \mathrm{seg}(\mathbb{N})$ such that

(1) $m \leq n$ and $|\sigma_{m,n}| \geq z_{b-1}$ and $D_n = \{z_1, \ldots, z_{b-1}\}$ where z_k it the k-th nonelement of $\mathrm{content}(\sigma_{m,n})$;
(2) $\sigma_{m,n}[2z_{b-1}] \in \mathrm{seg}(\{0, \ldots, z_{b-1}\})$;
(3) for $a = 1, \ldots, b-1$ and $L_a = \mathbb{N} - \{z_c : 1 \leq c < a\}$, $\sigma_{m,n}[2z_{b-1}]$ is a stabilizing sequence for $\mathbf{M}_a$ on L_a;
(4) $\mathrm{content}(\sigma_{m,n}) \cup \{z_b, z_{b+1}, \ldots, z_{2b}\} \subseteq W_{\mathbf{M}_b(\sigma_{m,n})}$.

One can determine with oracle K whether $team_m$ qualifies at n; note that while searching for $\sigma_{m,n}$ one can test for all orderings of the b team members and for all $\tau \in \mathrm{seg}(\{0, \ldots, z_{b-1}\})$ of length $2z_{b-1}$ whether they are stabilizing sequences for $\mathbf{M}_a$ on L_a for $a = 1, \ldots, b-1$. For those orderings of the teams and τ which satisfy one can then check for whether there is an extension $\sigma_{m,n}$

such that also the other conditions are satisfied. If $team_m$ does not qualify at n then let $G_{m,n} = \{0,\ldots,\max(D_n)\} - D_n$. Otherwise fix the first string $\sigma_{m,n}$ found, let $H_{m,n} = W_{\mathbf{M}_b(\sigma_{m,n})}$ and let $G_{m,n}$ be the set content$(\sigma_{m,n}) \cup \{z_k\}$ for the first $k \geq b$ where

$$\text{content}(\sigma_{m,n}) \cup \{z_k\} \notin \{W_{\mathbf{M}_1(\sigma_{m,n})},\ldots,W_{\mathbf{M}_b(\sigma_{m,n})}\}.$$

Note that there are only b machines and thus $k \leq 2b$. So $G_{m,n} \subset W_{\mathbf{M}_b(\sigma_{m,n})}$.

The class $\mathcal{L}$ to be learned is constructed using an inductively defined sequence $u_0, u_1, \ldots$ of auxiliary numbers.

(1) Put all subsets of $\mathbb{N}$ into $\mathcal{L}$ which miss at most $b-2$ elements of $\mathbb{N}$;
(2) For every n, put $G_{0,n}, G_{1,n}, \ldots, G_{n,n}$ into $\mathcal{L}$;
(3) For each n search for the least m satisfying the following conditions.
   - $2^m \leq \min(D_n)$;
   - there is a value $h(m,n)$, with $m \leq h(m,n) \leq n$, such that $team_m$ qualifies at $h(m,n)$ and $(\forall x \leq \max(D_n))\,[x \in H_{m,h(m,n)} \Leftrightarrow x \notin D_n]$;
   - $u_{n'} \neq m$ for all $n' < n$.

   If such an m is found and hence $h(m,n)$ defined,
   Then let $u_n = m$ and put $H_{m,h(m,n)}$ into $\mathcal{L}$
   Else let $u_n = \infty$ and put $\mathbb{N} - D_n$ into $\mathcal{L}$.

The symbol ∞ is just used to have a value different from all natural numbers for $u_0, u_1, \ldots$ if needed.

To see that $\mathcal{L}$ is in $[1,b]$**TxtEx**, consider the following learners $\mathbf{N}_1, \ldots, \mathbf{N}_b$. For $a = 1, \ldots, b-1$, the learner $\mathbf{N}_a$ always computes the set E of the least $a-1$ numbers which have not yet shown up in the input and outputs a canonical index for $\mathbb{N} - E$. It is easy to see that all sets which miss at most $b-1$ elements of $\mathbb{N}$ are learned by a member of this team. In particular, $\mathbf{N}_1$ always conjectures $\mathbb{N}$. The last learner $\mathbf{N}_b$ deals with all sets $L \in \mathcal{L}$ having at least $b-1$ nonelements. Let $u_{n,s}, G_{m,n,s}, h_s(m,n)$ be recursive approximations to $u_n, G_{m,n}$ and $h(m,n)$, respectively. Note that these exist as the construction above is recursive in K. The algorithm of $\mathbf{N}_b$ on input τ is the following:

(1) Let n be the number for which D_n is the set of the least $b-1$ nonelements of content(τ);
(2) If content$(\tau) = G_{m,n,|\tau|}$ for some $m \in \{0,\ldots,n\}$
   Then output a canonical index for content(τ)
   Else if $u_{n,|\tau|} = \infty$ then output a canonical index for $\mathbb{N} - D_n$
   Else compute the $|\tau|$-th approximation for the index of $H_{u_{n,|\tau|},h_{|\tau|}(u_{n,|\tau|},n)}$.

If τ is long enough and $L = G_{m,n}$ then $\mathbf{N}_b$ clearly learns the set. If $L \neq G_{m,n}$ for the finitely many sets of this form in $\mathcal{L}$, then $\mathbf{N}_b$ will converge to an index of $H_{m,h(m,n)}$ (for the case that $u_n = m$) or to an index of $\mathbb{N} - D_n$ (for the case that $u_n = \infty$). So $\mathcal{L}$ is in $[1,b]$**TxtEx**.

It remains the show that $\mathcal{L}$ is not in $[1,b]$**NUShTxtEx**. Suppose by way of contradiction that $team_m$ $[1,b]$**NUShTxtEx**-infers $\mathcal{L}$. We claim that there is an n such that $team_m$ qualifies at n and $G_{m,n}, H_{m,n}$ are both in $\mathcal{L}$. So assume by way of contradiction that such an n does not exist. Then there is in particular no n with $u_n = m$ since otherwise $G_{m,h(m,n)}, H_{m,h(m,n)}$ both exist and are in $\mathcal{L}$. The following arguments are now used to get the desired contradiction.

Let $\tau_0 = 0\,1\,\ldots\,m$ and $z_0 = m$ in order to get that $z_1 > m$ below. Do the following for $a = 1, \ldots, b-1$:

(1) Choose a machine $\mathbf{M}_a$ from $team_m$ which has a locking sequence extending τ_{a-1} on $\mathbb{N} - \{z_k : 0 < k < a\}$;
(2) Choose such an extension τ_a of τ_{a-1} satisfying the following additional constraints:
   - $z_1, \ldots, z_{a-1}$ are the least $a-1$ nonelements of content(τ_a);
   - There is c_a such that content$(\tau_a) = \{0, \ldots, c_a\} - \{z_k : 0 < k < a\}$ and $z_{a-1} < c_a$ and $2c_a = |\tau_a|$;
   - τ_a is a locking sequence for $\mathbf{M}_a$ on $\mathbb{N} - \{z_k : 0 < k < a\}$;
   - If $a = b-1$ then the n with $D_n = \{z_1, \ldots, z_{b-1}\}$ satisfies $u_n = \infty$ and $n > n'$ for all n' with $u_{n'} < m$.

Note that the fourth condition in (2) can be satisfied in the case that $a = b-1$ since there is, for each m', at most one n' with $u_{n'} = m'$ and furthermore there are less than $1 + \log(z_1)$ many n with $\min(D_n) \leq z_1 \wedge u_n < \infty$. Since $u_n = \infty$, $\mathbb{N} - D_n$ is a member of $\mathcal{L}$. Let $\mathbf{M}_n$ be the unique machine in $team_m$ different from $\mathbf{M}_1, \ldots, \mathbf{M}_{n-1}$. Since $\mathbf{M}_n$ has to infer $\mathbb{N} - D_n$ on every text extending τ_{b-1} for that language, there is an extension τ_b of τ_{b-1} which is a locking sequence for $\mathbf{M}_n$ on $\mathbb{N} - D_n$.

It is easy to see that τ_b would be a possible choice for $\sigma_{m,n}$, thus $team_m$ qualifies at n and $G_{m,n}, H_{m,n}$ exist. $G_{m,n}$ is in $\mathcal{L}$. If $H_{m,n} \notin \mathcal{L}$ then $H_{m,n}$ must miss out at least $b-1$ elements of $\mathbb{N}$. Let n' be such that $D_{n'}$ is the set of the least $b-1$ nonelements of $H_{m,n}$. Since $G_{m,n} \subseteq H_{m,n}$ and D_n is the set of the least $b-1$ nonelements of $G_{m,n}$, either $D_n = D_{n'} \wedge n = n'$ or $\max(D_n) < \max(D_{n'}) \wedge n < n'$. But then $u_{n'} > m$ although m would qualify as a value of $u_{n'}$, since $H_{m,n}$ exists and n would be a possible value for $h(m, n')$ — that is $h(m, n')$ would be defined and $G_{m,h(m,n')}, H_{m,h(m,n')} \in \mathcal{L}$ in contradiction to the assumption on the nonexistance of these sets in $\mathcal{L}$.

So for the $team_m$ there is an n such that $team_m$ qualifies at n and $H_{m,n}$ is in $\mathcal{L}$. Let T be a text for $G_{m,n}$ extending $\sigma_{m,n}$. Since $G_{m,n} \subseteq \mathbb{N} - D_n$, $\sigma_{m,n}$ is a stabilizing sequence for the first $b-1$ machines on $G_{m,n}$. Furthermore, all machines output a set different from $G_{m,n}$ on input $\sigma_{m,n}$. Finally $G_{m,n} \subset H_{m,n}$ and the last machine $\mathbf{M}_b$ conjectures $H_{m,n}$ on $\sigma_{m,n}$. Either it is giving up a hypothesis on a set from $\mathcal{L}$ on data consistent with this set or $team_m$ does not learn $G_{m,n}$ from the text T. This contradicts the assumption that $team_m$ $[1, b]$**NUShTxtEx**-learns $\mathcal{L}$ and completes the proof. ∎

Note that in the case that $b = 1$ the class $\mathcal{L} - \{\mathbb{N}\}$ is **TxtEx**-learnable but not decisively **TxtEx**-learnable. Indeed the construction of $\mathcal{L}$ from the previous theorem is a generalization of the construction of Baliga, Case, Merkle, Stephan and Wiehagen [4] of a class which separates explanatory learning from its decisive variant.

Proposition 6.5 $[a, b]\mathbf{TxtEx} \subseteq [a, a+b]\mathbf{NUShTxtEx}$.

Proof. Angluin [1] defined that a learner is conservative iff every mind change from an hypothesis e to a new hypothesis e' is justified in the sense that some data already seen are not contained in W_e at the time of the mind change. So every conservative learner is non U-shaped since it never abandons a correct hypothesis. The new team will consist of conservative learners and thus be non U-shaped.

Given now a team $\mathbf{M}_1, \ldots, \mathbf{M}_b$ which $[a, b]\mathbf{TxtEx}$-identifies some class $\mathcal{L}$, there is for every $c \in \{1, \ldots, b\}$ a machine $\mathbf{N}_c$ which is conservative and $\mathbf{TxtEx}$-learns every infinite language $\mathbf{TxtEx}$-learned by $\mathbf{M}_c$ [24,32]. So every infinite language in $\mathcal{L}$ is learned by at least a of the conservative machines $\mathbf{N}_1, \ldots, \mathbf{N}_b$. Furthermore, one assigns to the machines $\mathbf{N}_{b+1}, \ldots, \mathbf{N}_{b+a}$ the algorithm which outputs on every input σ a canonical index for content(σ). These machines are conservative as well and each of them learns every finite set. Thus the new team $\mathbf{N}_1, \ldots, \mathbf{N}_{a+b}$ $[a, a+b]\mathbf{NUShTxtEx}$-learns $\mathcal{L}$. ∎

7 Summary and Final Discussion

The following results were obtained.

- $\mathbf{TxtFex}_b \subset [1, b]\mathbf{NUShTxtEx}$ for all $b \in \{2, 3, \ldots\}$.
- $\mathbf{TxtFex}_1 = \mathbf{TxtEx} = \mathbf{NUShTxtEx} = [1, 1]\mathbf{NUShTxtEx}$, see also [4].
- $[1, b]\mathbf{NUShTxtEx} \subset [1, b]\mathbf{TxtEx}$, for all $b \in \{2, 3, \ldots\}$.
- $\mathbf{NUShTxtFex}_b = \mathbf{NUShTxtEx}$ for all $b \in \{1, 2, \ldots, *\}$.
- $\mathbf{TxtFex}_2 \subseteq \mathbf{NUShTxtBc}$.
- $\mathbf{TxtFex}_3 \not\subseteq \mathbf{NUShTxtBc}$.

These results and the facts known from previous work [4,11] are summarized in Figure 2. Single-headed arrows in the diagram denote proper inclusions. Double-headed arrows denote equality. All transitive closures of the inclusions displayed are valid and no other inclusions hold between language learning criteria in the diagram.

Some of the results obtained herein and mentioned just above are suggestive for what may be true for the case of human cognition. For example, perhaps the class of tasks humans must learn to be competitive in the genetic marketplace like, say, our class $\mathcal{L} \in (\mathbf{TxtFex}_3 - \mathbf{NUShTxtBc})$, *necessitates* U-shaped learning behaviour. In the human context, U-shaped learning *may* be necessary.

We note that our proof that $\mathbf{TxtFex}_3 \not\subseteq \mathbf{NUShTxtBc}$ intriguingly features learning finite tables versus general rules, but does *not*, as might be expected from some models of the human case of U-shaped learning, feature, among other things, learning an incorrect general rule followed by learning a general

$$
\begin{tikzcd}
\mathbf{TxtBc} & \mathbf{TxtFex}_* \arrow[l] & & \\
& \vdots \arrow[u] & \vdots & \vdots \\
& \mathbf{TxtFex}_5 \arrow[r] & {[1,5]}\mathbf{NUShTxtEx} \arrow[r] & {[1,5]}\mathbf{TxtEx} \\
& \mathbf{TxtFex}_4 \arrow[r] \arrow[u] & {[1,4]}\mathbf{NUShTxtEx} \arrow[r] \arrow[u] & {[1,4]}\mathbf{TxtEx} \arrow[u] \arrow[ul] \\
& \mathbf{TxtFex}_3 \arrow[r] \arrow[u] & {[1,3]}\mathbf{NUShTxtEx} \arrow[r] \arrow[u] & {[1,3]}\mathbf{TxtEx} \arrow[u] \arrow[ul] \\
\mathbf{NUShTxtBc} \arrow[uuuuu] & \mathbf{TxtFex}_2 \arrow[l] \arrow[r] \arrow[u] & {[1,2]}\mathbf{NUShTxtEx} \arrow[r] \arrow[u] & {[1,2]}\mathbf{TxtEx} \arrow[u] \arrow[ul] \\
\mathbf{NUShTxtFex}_b \arrow[u] \arrow[r, leftrightarrow] & \mathbf{TxtFex}_1 \arrow[r, leftrightarrow] \arrow[u] & {[1,1]}\mathbf{NUShTxtEx} \arrow[r, leftrightarrow] \arrow[u] & {[1,1]}\mathbf{TxtEx} \arrow[u] \arrow[ul]
\end{tikzcd}
$$

Figure 2. Summary of the results for $b \in \{1, 2, 3, 4, 5, *\}$.

rule augmented by a correcting finite table. This difference may be significant or, more likely, nothing more than an artifact of our particular proof.

Not explored herein, but very interesting to investigate in the future, is the possible necessity of U-shaped learning in complexity-bounded learning contexts such as those explored in [12,13,16,39,46].

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