VAN LAMBALGEN'S THEOREM AND HIGH DEGREES

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Abstract. We show that van Lambalgen's Theorem fails for some recursively random real and therefore some Schnorr random real in every high degree and provide a full characterization of the Turing degrees for which van Lambalgen's Theorem can fail for Kurtz random reals. However, we also show that there is a recursively random real that is not Martin-Löf random for which van Lambalgen's Theorem holds.

1. Introduction

Martin-Löf randomness is the most frequently studied randomness notion. It can be easily defined in terms of unpredictability, measure theory or initial-segment complexity [15], and in each of these frameworks, there is a universal test; i.e., a single martingale, test or oracle machine that can be used to determine whether a real is Martin-Löf random [8].

However, there are other well-known notions of randomness, such as recursive randomness, Schnorr randomness and Kurtz randomness. As we compare and contrast these notions with Martin-Löf randomness and with each other, it is instructive to investigate the extent to which results that are true of one notion are true of the others. We present here some comparative results concerning van Lambalgen's Theorem, a classic theorem in the study of Martin-Löf randomness.

We now give the background of our work. Our notation is standard and generally follows [13, 14] and [16]. For background on effective randomness, please refer to [3] or [11].

In this paper, we will primarily use the unpredictability approach to randomness. In the most general sense, a real, or an element of the Cantor space $\{0,1\}^\omega$, is considered to be random if no algorithm of the appropriate computational strength can predict its $(n+1)^{st}$ bit given its first n bits. We formalize this notion using martingales.

Definition 1.1. A *martingale* is a function $m : \{0,1\}^{<\omega} \to \mathbb{R}^{\geq 0}$ such that for all $\sigma \in \{0,1\}^{<\omega}$,

$$m(\sigma) = \frac{m(\sigma 0) + m(\sigma 1)}{2}.$$

We say that a martingale m is *r.e.* if the values $m(\sigma)$ are uniformly left-r.e. reals and *recursive* if the values $m(\sigma)$ are uniformly recursive reals.

Definition 1.2. A martingale m *succeeds* on a real A if $\limsup_n m(A{\upharpoonright}n) = \infty$.

In other words, a martingale succeeds on a real if there is no bound on the amount of capital the martingale attains by "betting" on the real.

Martin-Löf, recursive and Schnorr randomness can all be defined in terms of martingales as below. Recall that an order function is simply a recursive function that is nondecreasing and unbounded.

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Theorem 1.3. [15] *Let A be a real.*

(1) *A is Martin-Löf random if no r.e. martingale succeeds on it.*
(2) *A is recursively random if no recursive martingale succeeds on it.*
(3) *A is Schnorr random if there is no recursive martingale m such that $m(A\restriction n) \geq h(n)$ for infinitely many n for some order function h.*

It is easy to see that every Martin-Löf random real is recursively random and that every recursively random real is Schnorr random. However, neither of the reverse implications hold [15, 18].

In addition to the properties mentioned earlier, Martin-Löf randomness also satisfies several of our intuitions about randomness. For instance, it makes sense that the halves of any random real should not only be random themselves, but also random with respect to each other. This was proved for Martin-Löf randomness by van Lambalgen in [17].

Theorem 1.4 (van Lambalgen's Theorem)**.** *Let A_0 and A_1 be reals. Then $A_0 \oplus A_1$ is Martin-Löf random if and only if A_0 is Martin-Löf random and A_1 is Martin-Löf random with respect to A_0.*

Since $A_0 \oplus A_1$ is Martin-Löf random exactly when $A_1 \oplus A_0$ is Martin-Löf random, we can see that A_0 is Martin-Löf random with respect to A_1 if and only if A_1 is Martin-Löf random with respect to A_0.

We say that a real $A = A_0 \oplus A_1$ satisfies van Lambalgen's Theorem for a particular randomness notion $\mathcal{R}$ precisely when A is $\mathcal{R}$-random and A_i is $\mathcal{R}$-random with respect to A_{1-i} for each $i \leq 1$. If there is a real A that is $\mathcal{R}$-random for which this statement does not hold, we say that van Lambalgen's Theorem fails for A with respect to $\mathcal{R}$-randomness.

2. Previous work

As mentioned in [19], it is not hard to see that one direction of van Lambalgen's Theorem holds for both recursive and Schnorr randomness.

Theorem 2.1. *If A_0 is recursively (Schnorr) random and A_1 is recursively (Schnorr) random relative to A_0, then $A_0 \oplus A_1$ is recursively (Schnorr) random.*

However, the other direction of van Lambalgen's Theorem does not hold for recursive randomness or Schnorr randomness. In [10], Merkle et al. showed that there is a recursively random real $A_0 \oplus A_1$ such that A_i is not Schnorr random relative to A_{1-i} for some $i \leq 1$. In addition, the following result and proof appear in [11].

Theorem 2.2 (Kjos-Hanssen)**.** *There is a Schnorr random real that does not satisfy van Lambalgen's Theorem.*

Proof. There is a high minimal degree by the Cooper Jump Inversion Theorem [2]. Since every high degree contains a Schnorr random real that is not Martin-Löf random [12], there is a Schnorr random real $A_0 \oplus A_1$ in this high degree and the only possible Turing degrees for A_0 and A_1 are $\mathbf{0}$ and that of $A_0 \oplus A_1$. Since A_0 and A_1 are Schnorr random, they cannot be recursive, so $A_0 \equiv_T A_1 \equiv_T A_0 \oplus A_1$. Clearly, A_0 cannot be Schnorr random relative to A_1 (and vice versa), so van Lambalgen's Theorem does not hold for this particular Schnorr random real. □

We note that this proof can also be used to show that there is a recursively random real that does not satisfy van Lambalgen's Theorem, since every high degree also contains a recursively random real that is not Martin-Löf random [12]. More recently, Yu has shown that van Lambalgen's Theorem fails for a particular class of Schnorr random reals [19].

Theorem 2.3 (Yu [19, Theorem 0.9])**.** *Let $B <_T 0'$ be an r.e. set. If $A = A_0 \oplus A_1 \leq_T B$ is recursively (Schnorr) random but not Martin-Löf random, then A_i is not A_{1-i}-recursively (Schnorr) random for $i \leq 1$.*

Kjos-Hanssen's and Yu's results only apply to very selective classes of reals: those in high minimal degrees and those bounded away from $\mathbf{0}'$ by an r.e. set. This leads to a very natural question: are there any degrees in which van Lambalgen's Theorem holds for all recursively random reals or even all Schnorr random reals?

We begin by noting that all three of these randomness notions coincide in the nonhigh Turing degrees [12], so van Lambalgen's Theorem holds trivially for the Schnorr random reals in these degrees.

First, we consider the reals $A \oplus B$ such that $A \oplus B$ is recursively (Schnorr) random and A and B are each recursively (Schnorr) random relative to each other. In Section 3, we show that there is at least one recursively random real $A \oplus B$ in every high degree for which van Lambalgen's Theorem fails; that is, such that A is not recursively random relative to B or vice versa. However, we go on to show in Section 4 that this failure is not universal in the high degrees; i.e., that van Lambalgen's Theorem does hold for some recursively random reals that are not Martin-Löf random. Finally, in Section 5, we will consider a recursively (Schnorr) random real A and attempt to describe the class of reals B that are recursively (Schnorr) random relative to A such that A is also recursively (Schnorr) random relative to B. This will help us characterize those reals B for which $A \oplus B$ is a recursively (Schnorr) random real that satisfies van Lambalgen's Theorem.

3. When van Lambalgen's Theorem fails

We will construct a real in an arbitrary high degree on which no recursive martingale succeeds in such a way that it can be recursively decomposed into two parts, one of which is not recursively random with respect to the other. We will need the following result, which follows easily from a theorem in [5].

Fact 3.1. If $\mathbf{a}$ is a high Turing degree, then there is an $A \in \mathbf{a}$ such that A *wtt*-computes a dominating function.

Theorem 3.2. *Let $\mathbf{a}$ be a high Turing degree. Then there is a $B \in \mathbf{a}$ such that B is recursively random and van Lambalgen's Theorem does not hold for B.*

Proof. We let A be an element of $\mathbf{a}$ that *wtt*-computes a dominating function g. We must now produce a real B such that B is recursively random, $B \equiv_T A$ and B does not satisfy van Lambalgen's Theorem.

To satisfy the first of these conditions, we will use g to construct a martingale d that subsumes all recursive martingales and then construct B so that d cannot succeed on it. This will be enough to ensure that B is computable from A and that B is recursively random. However, we must also ensure that A is computable from B. To do this, we will construct B in segments of lengths determined by a strictly increasing recursive function f. These segments will increase in length and code more and more of A. Finally, the bits of B at indices in the range of f will be computable from those that are not, so we will have an infinite, recursively identifiable portion of B that is computable from the rest of B. Therefore, van Lambalgen's Theorem will fail for B.

At this point, we will not define f, and the reader should simply bear in mind that f is a recursive function.

We begin by describing a sequence of martingales $\langle d_k \rangle_{k\in\omega}$ that will contain all recursive martingales. We define each martingale d_k from the k^{th} partial recursive function φ_k, a recursive bijection $r : \omega \to (0,2) \cap \mathbb{Q}$ and the dominating function g and recursive function f mentioned above.

To define d_k for a string σ such that $f(n) \leq |\sigma| < f(n+1)$, we first ask whether $\varphi_k(\tau)$ converges within $g(n)$ steps for all τ such that $f(n) \leq |\tau| < f(n+1)$. If the answer is yes and $n > k$, we extend d_k as follows.

$$\begin{aligned} d_k(\sigma 0) &= r(\varphi_k(\sigma))d_k(\sigma) \\ d_k(\sigma 1) &= (2 - r(\varphi_k(\sigma)))d_k(\sigma) \end{aligned}$$

Otherwise, we set $d_k(\sigma 0) = d_k(\sigma 1) = d_k(\sigma)$. Without loss of generality, we can define the initial value of each d_k to be 1.

Now we can define d as a weighted sum of the d_k by setting

$$d(\sigma) = \sum_k \frac{1}{2^{k+1}} d_k(\sigma)$$

for all $\sigma \in \{0,1\}^{<\omega}$. Note that d is recursive in A.

Now we define f. We begin by defining an auxiliary function u, where $u(n)$ is one plus the upper bound on the use of the calculation of $g(n+1)$ from A. Since $g \leq_{wtt} A$, we may take u to be recursive.

We will now make use of the following fact from [9].

Fact 3.3. Given a rational $\delta > 1$ and a $k \in \omega$, we can recursively compute a length $l(\delta, k)$ such that for any martingale m and any $\sigma \in \{0,1\}^{<\omega}$, the following inequality holds.

$$|\{\tau \in \{0,1\}^{l(\delta,k)} \mid m(\sigma\tau) \leq \delta m(\sigma)\}| \geq k$$

This fact allows us to find a recursive function w such that for each σ of length $f(n)+1$, there are $2^{u(n)}$ strings τ of length $(f(n)+1)+w(n)$ extending σ such that

$$d(\tau) \leq \left(1 + \frac{1}{2^n}\right) d(\sigma).$$

Now we can finally define f recursively by letting $f(0) = 0$ and

$$f(n+1) = f(n) + 1 + w(n).$$

At this point, we are ready to construct B. Rather than define it bit by bit, we will define B in intervals; in particular, the intervals of natural numbers k such that $f(n) < k \leq f(n+1)$.

For each $n \in \omega$, we suppose that we are given a string σ_n in $\{0,1\}^{f(n)}$. We first extend it by either 0 or 1 to obtain a τ_n in $\{0,1\}^{f(n)+1}$ as follows.

$$\tau_n = \begin{cases} \sigma_n 0 & \text{if } d(\sigma_n 0) \leq d(\sigma_n 1) \\ \sigma_n 1 & \text{else} \end{cases}$$

We can now find a subset α_{n+1} of $\{0,1\}^{f(n+1)}$ such that each $\rho \in \alpha_{n+1}$ extends τ_n and $d(\rho) \leq \left(1 + \frac{1}{2^n}\right) d(\tau_n)$. We know from Fact 3.3 that there will be at least $2^{u(n)}$ elements of α_{n+1}. Therefore, we may code all binary strings of length $u(n)$ as one of the leftmost $2^{u(n)}$ elements of α_{n+1}. We now define σ_{n+1} to be the element of α_{n+1} that codes the first $u(n)$ bits of A.

Let $B = \lim \sigma_n$.

Lemma 3.4. *If the martingale d does not succeed on a real X, then no recursive martingale will succeed on X.*

Proof. Suppose that X is a real on which d does not succeed and assume for a contradiction that m is a recursive martingale that succeeds on X. We note that if a recursive martingale that succeeds on X exists, then there is also a recursive martingale that succeeds on X whose range consists solely of positive rationals [15]. Therefore, without loss of generality, we can say that m is definable as

$$\begin{aligned} m(\sigma 0) &= r(\varphi_k(\sigma))m(\sigma) \\ m(\sigma 1) &= (2 - r(\varphi_k(\sigma)))m(\sigma) \end{aligned}$$

for some k, where r is the bijection from ω to $(0, 2) \cap \mathbb{Q}$ defined above. However, when we defined each martingale d_k, we did not simply use the relationship above. Instead, we put constraints on the definition based on whether $\varphi_k(\sigma)$ converged for a certain set of σ within $g(n)$ steps. Since g is a dominating function, there is some index n such that $d_k = d_n$ and $m = d_n$. Since d is simply the weighted sum of the d_k, it is clear that if d does not succeed on the real X, neither will d_k or m. This gives us a contradiction, so no recursive martingale cannot succeed on X. □

We can see easily that d does not succeed on B since the value of d on B is bounded by $\prod_n (1 + \frac{1}{2^n})$. This product converges, so by Lemma 3.4, no recursive martingale succeeds on B and B is therefore recursively random.

Now we must show that $B \equiv_T A$. It is clear from the construction that $B \leq_T A$. To compute $A(n)$ from B, we begin by finding a k such that $n \leq u(k)$. Now we consider $B \upharpoonright f(k+1)$. We recall that we have coded the first $u(k)$ bits of A into the segment of B of bits with indices between $f(k) + 1$ and $f(k+1)$. Only the first $u(k-1)$ bits of A and recursive functions were used in this coding, and this information can be obtained from a shorter initial segment of B (namely, $B \upharpoonright f(k)$), so we can see that we can recursively compute $A(n)$ from B for any n.

Finally, we must show that B does not satisfy van Lambalgen's Theorem. We will not do this in the customary way by decomposing B into two reals, one consisting of its bits with even indices and one of its bits with odd indices. Instead, we will consider a different recursive decomposition of B. We introduce the following notation to simplify matters.

Notation 3.5. Let Z be an infinite recursive subset of ω. For any real X, we may write $X = X_0 \oplus_Z X_1$, where $X_0(n)$ is the bit of X whose index is the n^{th} element of Z and $X_1(n)$ is the bit of X whose index is the n^{th} element of $\overline{Z}$.

Let $F = \text{ran}(f)$. Since f is a strictly increasing recursive function, F will be infinite and recursive. Therefore, we can express B as $B_0 \oplus_F B_1$. We can see that B_0 is recursive in B_1, since we can calculate $B(f(n))$ from $B \upharpoonright f(n)$ for all n. Therefore, B_0 is clearly not recursively random relative to B_1 and B does not satisfy van Lambalgen's Theorem for recursive randomness. □

We now briefly consider van Lambalgen's Theorem in the context of Kurtz randomness. Kurtz randomness was first defined in [7] and is a weaker notion than either Schnorr or recursive randomness. Recall that the hyperimmune Turing degrees are precisely those that compute a function that is not majorized by any recursive function.

Definition 3.6. A real A is Kurtz random if $A \in U$ for every r.e. open set U of measure 1.

We first observe that one direction of van Lambalgen's Theorem holds for Kurtz randomness.

Theorem 3.7. *If A_0 is Kurtz random and A_1 is Kurtz random relative to A_0, then $A_0 \oplus A_1$ is Kurtz random.*

Proof. We assume that A_0 is Kurtz random and let U be an arbitrary r.e. open set of measure 1. Now, for a rational $r < 1$, we define the class $U_r = \{P \mid \mu(\{P \oplus Q \in U\}) > r\}$. Each U_r can be enumerated recursively, so each U_r is an r.e. open set. Furthermore, $\mu(U_r) = 1$ for each r since otherwise the measure of U would be strictly less than 1. Since A_0 is Kurtz random, $A_0 \in U_r$ for each r, and therefore the set $T = \{Q \mid A_0 \oplus Q \in U\}$ has measure 1 and is r.e. relative to A_0. Since A_1 is assumed to be Kurtz random relative to A_0, A_1 must be an element of T and therefore $A_0 \oplus A_1$ must be an element of our arbitrary r.e. open set U. □

We now show that van Lambalgen's Theorem fails for Kurtz randomness in a larger class of Turing degrees than Schnorr or recursive randomness.

Theorem 3.8. *Every hyperimmune Turing degree contains a Kurtz random real that does not satisfy van Lambalgen's Theorem for Kurtz randomness.*

Proof. Let E be a hyperimmune Turing degree and let $f \leq_T E$ be such that f is not majorized by any recursive function; that is, there is no recursive function g such that $g(n) \geq f(n)$ for all n. We will build a real $A \oplus B$ by finite extensions such that $A \oplus B$ is Kurtz random, but neither A nor B is Kurtz random relative to the other.

We begin by dividing the natural numbers into a sequence of intervals $\langle I_k \rangle_{k \in \omega}$ such that $I_k = \{2^k, 2^k + 1, \ldots, 2^{k+1} - 1\}$ and say that an extension function is a partial recursive function that, given a finite binary string σ, will output a finite binary string τ extending σ of length 2^k for some k. To ensure that the real we build is Kurtz random, we will require that it meet all extension functions φ_e that are not constant on either the even half (the bits with even indices) or on the odd half (the bits with odd indices) of any of these intervals. This will be enough, since any extension function that is constant on either half of one of these intervals will produce an r.e. open set with measure strictly less than 1.

We let σ_0 be the string with domain I_0 that takes the value $E(0)$ on all bits with indices in I_0. At stage $n \geq 1$, we assume that we have defined σ_{n-1} and that its domain is the union of the intervals $I_0, I_1, \ldots, I_m$ for some m. Now we list those $e \leq n$ such that the extension function $\varphi_e(\sigma_{n-1})$ converges within $f(n)$ steps and let e' be the least index for which such a σ can be found and for which $\varphi_{e'}$ has not already been utilized at a previous stage. If such an extension exists, we set $\tau = \varphi_{e'}(\sigma_{n-1})$; otherwise, we let $\tau = \sigma_{n-1}$. Suppose that τ is defined on the intervals $I_0, I_1, \ldots, I_k$. Now we define σ_n to be the string that extends τ to the interval I_{k+1} by giving it the value $E(n)$ everywhere on this interval.

Let $A \oplus B = \lim_n \sigma_n$. It is clear that $A \oplus B \leq_T E$. We can see that $A \oplus B$ meets all extension functions of the appropriate type, since f is not majorized by any recursive function, so $A \oplus B$ is Kurtz random. Furthermore, we can calculate E from A. Since the even bits of $A \oplus B$ are only constant on the intervals that code the bits of E, all we need to do to find the k^{th} bit of E is to consider the intervals of A that correspond to intervals I_m of $A \oplus B$ and have constant values. The only value A will take on the k^{th} such interval is $E(k-1)$, so $E \leq_T A$. Similarly, we can see that $E \leq_T B$, so $A \oplus B \equiv_T E \equiv_T A \equiv_T B$, and A and B cannot be Kurtz random relative to each other. □

We note that since Kurtz randomness coincides with Martin-Löf randomness in the hyperimmune-free Turing degrees [12], this gives us a full characterization of the Turing degrees that contain Kurtz random reals for which van Lambalgen's Theorem fails for Kurtz randomness.

4. When van Lambalgen's Theorem holds

We have seen that every high degree contains a recursively random (and therefore Schnorr random) real for which van Lambalgen's Theorem fails. One obvious question remains: is there any recursively random or Schnorr random real that is not Martin-Löf random for which van Lambalgen's Theorem holds? In this section, we answer this question positively. We further produce an example of such a Schnorr random real.

Theorem 4.1. *There is a recursively random real that is not Martin-Löf random and satisfies van Lambalgen's Theorem for recursive, Schnorr and Kurtz randomness.*

Proof. In [6], a recursively random real R in a high but incomplete r.e. Turing degree was constructed that is recursively random relative to all elements of a class of measure 1. The existence of such a real can also be deduced easily from Theorem 1.2 in [1]. This real is clearly not Martin-Löf random. Now we observe that the class of 2-random reals has measure 1, so there is a 2-random real A such that R is recursively random relative to A. Furthermore, since A is 2-random, A is Martin-Löf random relative to R.

Since both reals are recursively random relative to each other, $A \oplus R$ is recursively random by Theorem 2.1. However, since R is not Martin-Löf random, $A \oplus R$ is not either. □

We also construct a particular Schnorr random real that is not Martin-Löf random. This construction involves Ω, the halting probability of a universal prefix-free Turing machine.

Example 4.2. To construct a Schnorr random real that is not Martin-Löf random for which van Lambalgen's Theorem holds, we will first define two reals based on Ω that are Schnorr random with respect to each other. We begin by fixing an approximation $\langle \Omega_s \rangle_{s\in\omega}$ to Ω. We also use the convergence modulus of Ω, the function defined as $c_\Omega(n) = \min\{s \mid \Omega_s \restriction n = \Omega \restriction n\}$, to define the sequence $\langle a_n \rangle_{n\in\omega}$, where $a_0 = 0$ and $a_{n+1} = c_\Omega(a_n) + 1$ for all n. We use this sequence to partition the natural numbers into infinitely many intervals $\langle I_n \rangle_{n\in\omega}$, where $|I_0| = 1$, $|I_{2k+1}| = 1$ and $|I_{2k+2}| = a_{k+1} - a_k$ for all k.

We can now define a real A that is equal to 0 on odd intervals, $\Omega(0)$ on I_0 and $\Omega(a_k+1)\Omega(a_k+2) \dots \Omega(a_{k+1})$ on I_{2k+2}. Note that $A \leq_T \Omega$. Now consider Ω^Ω, which is the halting probability relative to Ω. This real is Martin-Löf random (and thus Schnorr random) relative to Ω, and since $A \leq_T \Omega$, it is also Martin-Löf random relative to A. Furthermore, A is Schnorr random relative to Ω^Ω, so $A \oplus \Omega^\Omega$ is Schnorr random by Theorem 2.1.

However, A is not Martin-Löf random relative to Ω^Ω, since Ω is Martin-Löf random relative to Ω^Ω and c_Ω dominates every Ω^Ω-recursive function. Therefore, $A \oplus \Omega^\Omega$ is not Martin-Löf random.

5. Another approach

As mentioned before, there are two primary ways in which we may consider a random real X in the context of van Lambalgen's Theorem. First, we may ask, as we have above, whether X can be recursively decomposed into an X_0 and X_1 such that one X_i is not random with respect to the other. However, we may also ask whether X is half of a real for which van Lambalgen's Theorem fails; i.e., whether there is another real Y such that $X \oplus Y$ is random but van Lambalgen's Theorem fails for $X \oplus Y$. It turns out that every Schnorr random real is half of another Schnorr random real for which van Lambalgen's Theorem fails.

Theorem 5.1. *If A is a Schnorr random real, then there is a real B such that $A \oplus B$ is Schnorr random and van Lambalgen's Theorem fails for $A \oplus B$ for Schnorr randomness.*

We will need the following result to prove this theorem. Recall that a maximal set is an r.e. set E that has an infinite complement for which there is no r.e. set W such that $|W \cap \overline{E}| = |\overline{W} \cap \overline{E}| = \infty$.

Proposition 5.2 (Fitzgerald [4])**.** *If R is Schnorr random and E is maximal, then $R \cap E$ is Schnorr random.*

Proof of Theorem 5.1. We have to consider the case in which A Turing computes K and the case in which A does not Turing compute K separately.

If A is Schnorr random and $A \not\geq_T K$, by Theorem 2.1 in [6], there is a real $B \geq_T A$ such that B is Schnorr random relative to A precisely when $A \not\geq_T K$. If A is Schnorr random, then by Theorem 2.1, $A \oplus B$ must be Schnorr random. However, since A is not Schnorr random relative to B, van Lambalgen's Theorem fails for $A \oplus B$.

If A is Schnorr random and $A \geq K$, we may let R be Martin-Löf random with respect to A and consider $B = R \cap E$ for some maximal set E. We note that $A \oplus R$ is Schnorr random by Theorem 2.1 and that $\omega \oplus E$ is maximal. By Proposition 5.2, we can see that $A \oplus B = (A \oplus R) \cap (\omega \oplus E)$ is Schnorr random. However, since $E \leq_T A$, we can create a martingale recursive in A that succeeds on B by computing E and then betting that $B(n) = 0$ whenever n is not in E. We can also find an order function recursive in A that grows sufficiently slowly that this martingale succeeds at the rate indicated by the order function, so we can see that B is not Schnorr random relative to A and, once again, van Lambalgen's Theorem fails for $A \oplus B$. $\square$

We may also ask the following question for those Schnorr random reals which Turing compute K.

Question 5.3. Suppose that $A \geq_T K$ is Schnorr random. Must there be a real $B \leq_T A$ such that B is Schnorr random and A is Schnorr random relative to B?

A positive answer to this question would not only refine the second part of the proof of Theorem 5.1 but also provide a sort of randomness inversion theorem.

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