Hausdorff-Dimension and Weak Truth-Table Reducibility*

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Abstract

It is shown that in some relativized world B there is a set A such that the effective Hausdorff-dimension of the wtt-degree and of the lower wtt-cone of A are both strictly between 0 and 1.

Hausdorff [2] introduced a general theory of dimension which also covered the case of intermediate fractal dimensions. Starting from Hausdorff's original definition, one introduced an effective variant of the dimension [1, 4, 12, 14]. Reimann [9] investigated the important question which type of degrees can have intermediate Hausdorff-dimension properly between 0 and 1. Reimann and Terwijn [9, Theorem 3.10] found positive results for many-one and bounded truth-table reducibilities. Later Reimann and Slaman [11] announced an extension of the result to truth-table reducibility. In the present work, an oracle B defining a relativized world is constructed together with a set A such that the wtt-degree of A in this relativized world has an intermediate Hausdorff-dimension between $\frac{1}{4}$ and $\frac{1}{2}$. The same applies for the lower wtt-cone of A. Before proving the main result, some more details are provided and basic notation is introduced.

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Notation. The reader should refer to the books of Li and Vitányi [5], Odifreddi [8] and Soare [13] for unexplained facts and notations from recursion theory and Kolmogorov complexity. The following conventions are a bit unusual but simplify the presentation of the proofs.

It is convenient to identify the string x with the number n such that $n+1$ has the binary representation $1x$, so the string 11 stands for 6 since $6+1$ has the binary representation 111 and 000 stands for 7 since 1000 represents $7+1$. This permits to extend the notion of Kolmogorov complexity to numbers. Given a number x, $|x|$ denotes the length of the string identified with x. That is, $|6| = 2$, $|7| = 3$ and, in general, $|x|$ is the down-rounded value of the dual logarithm of $x+1$.

Also, given a set A and a finite set $\{a_1, a_2, \ldots, a_n\}$ of natural numbers with $a_1 < a_2 < \ldots < a_n$, let $A(\{a_1, a_2, \ldots, a_n\})$ stand for the string $A(a_1)A(a_2)\ldots A(a_n)$ where $A(x) = 1$ if $x \in A$ and $A(x) = 0$ if $x \notin A$.

Hausdorff [2] formulated a measure-theoretic approach to dimension. His notion of dimension turned out to be invariant under bi-Lipschitz transformations. This approach permits to study classes of noninteger dimension. It is a natural question how such classes look like.

A further characteristic of Hausdorff-dimension is countable stability: the dimension of a countable union of sets is the supremum of the dimensions of each member of the union. When effectivizing the dimension, it does no longer measure the dimension of the class but the effective dimension of its most unapproximable members. That is, a class is as difficult as its most difficult members and mathematically this means that effective Hausdorff-dimension has uncountable stability. So every class S of sets of natural numbers satisfies $\dim_H(S) = \sup\{\dim_H(\{A\}) : A \in S\}$. Since the classical Hausdorff-dimension of a class is a lower bound for its effective Hausdorff-dimension, there are be singletons having positive effective Hausdorff-dimension in every class of positive classical Hausdorff-dimension. There are several characterizations of the effective Hausdorff-dimension of a singleton $\{A\}$ which, for convenience, is from now on just written as $\dim_H(A)$. An elegant characterization is the one in terms of the Kolmogorov complexity C [1, 4, 12, 14] which is here taken as a definition.

Definition 1. The effective Hausdorff-dimension of a set A is given as

$$\dim_H(A) = \liminf_{n\to\infty} \tfrac{1}{n} \cdot C(A(0)A(1)\ldots A(n-1)).$$

Since the present work deals only with the effective version and so there is no danger of confusing the terms, this notion is just called "Hausdorff-dimension" from now on.

Recall that the Kolmogorov complexity $C(x)$ of a string x is the length of the shortest program generating x. These programs are measured with respect of a universal machine and the actual value of the Kolmogorov complexity depends only up to a constant on this machine. It is convenient to define the machine such that it is universal for any oracle E. So one chooses U to be a unary universal partial-recursive function accessing an oracle with the property that for every partial-recursive unary function V with oracle access there is a constant c such that

$$\forall p\, \exists q\, (|q| \leq |p| + c \wedge U^E(q) = V^E(p))$$

where the equality "$U^E(q) = V^E(p)$" means that either both sides are undefined or both sides are defined and equal. Now one defines the relativized Kolmogorov complexity as

$$C^E(x) = \min\{|p| : p \in \{0,1\}^* \wedge U^E(p)\downarrow = x\}$$

for any oracle E where one writes simply $C(x)$ and $U(p)$ instead of $C^\emptyset(x)$ and $U^\emptyset(p)$ for the case where the oracle is the empty set. Kolmogorov complexity can be approximated from above and one assumes that $U_s^E(p)$ does not query the oracle at places $a \geq s$. Therefore, one can write $U_s^\sigma(p)$ for partial functions σ with domain $\{t : t < s\}$.

The notion of relativized Kolmogorov complexity also permits to introduce a relativized version of Hausdorff-dimension which follows the straightforward definition

$$\dim_H^E(A) = \liminf_{n\to\infty} \tfrac{1}{n} \cdot C^E(A(0)A(1)\ldots A(n-1))$$

and is extended to classes using stability. These relativized notions will be the major tool for the main result of the paper that in some relativized world there is a wtt-degree of intermediate Hausdorff-dimension.

Recursively enumerable sets have Hausdorff-dimension 0. The reason is that one can compute $A(0)A(1)\ldots A(n-1)$ just from n and from the number m of elements which A has below n: Having the numbers m and n, one can enumerate A until a stage s where A_s has m elements below n is reached and then output $A_s(0)A_s(1)\ldots A_s(n-1)$. This output is then correct and the Kolmogorov complexity of this string is bounded by $c+2\log(n)$ for some constant c.

On the other hand, Martin-Löf gave a measure-theoretic criterion for a set R to be algorithmically random [6]. The class of Martin-Löf random sets has measure 1 and every Martin-Loef random set has Hausdorff dimension 1. Then, for every Martin-Löf random set R, $\dim_H(R \oplus \emptyset) = \frac{1}{2}$, so there are sets of intermediate Hausdorff-dimension. Reimann [9, Theorem 3.10] showed that there is a whole many-one degree where all members have intermediate dimension, in the next example it is shown how this can be done.

Example 2 [9]**.** *There is a set A such that $\dim_H(A) > 0$ and $\dim_H(B) < 1$ for all $B \leq_m A$.*

Proof. Let R be a Martin-Löf random set and define

$$A = \{x \in R : C(x) \geq |x|\}.$$

Now it is shown that every $B \leq_m A$ has intermediate Hausdorff-dimension. First this is done for A itself. There is a constant c such that for almost all n, $\{x \in \{0,1\}^n : C(x) < n\}$ has more than 2^{n-c} elements and more than 2^{n-c} nonelements. Thus, given any n and the number m of $x < n$ with $C(x) < |x|$, one can enumerate these m numbers and then knows that $A(x) = 0$ for all these $x < n$ with $C(x) < |x|$. This gives

$$C(A(0)A(1)\ldots A(n-1)) \leq n - m + 3\log(n)$$

for all sufficiently large n where m depends on n as defined above. In particular, $\dim_H(A) \leq 1 - 2^{-c-1}$. On the other hand, one can use that the $n - m$ numbers $x < n$ not enumerated satisfy $R(x) = A(x)$ in order to get that

$$C(R(0)R(1)\ldots R(n-1)) < C(A(0)A(1)\ldots A(n-1)) + m + 3\log(n)$$

for all sufficiently large n. One can conclude that

$$C(A(0)A(1)\ldots A(n-1)) \geq n - m - 5\log(n)$$

for sufficiently large n and using that $n - m \geq n \cdot 2^{-1-c}$, one gets a lower bound for the Hausdorff-dimension of A: $\dim_H(A) \geq 2^{-1-c}$.

Of course the construction of a single set A of intermediate dimension could be done more easily by considering sets like $R \oplus \emptyset$, but the special thing about this construction is that no $B \leq_m A$ has Hausdorff-dimension 1. So assume that $B \leq_m A$ via f. Either one has for infinitely many n that $f(x) \in \{f(0), f(1), \ldots, f(x-1)\}$ for at least a fourth of all $x < n$ resulting in $\dim_H(B) \leq \frac{3}{4}$. Or one has for almost all n that $|f(x)| \geq |x| - 1$ for at least one half of the $x < n$. The latter then implies that there is a constant c' with $C(f(x)) < |f(x)|$ for at least $2^{-c'} \cdot m$ of the m numbers $x < n$ with $|f(x)| \geq |x| - 1$. From this condition one can conclude that at least $2^{-c'-1} \cdot n$ of the numbers $x < n$ satisfy $C(f(x)) < |f(x)|$ and therefore $B(x) = A(f(x)) = 0$. From this, one can then conclude in the same way as above that $\dim_H(B) < 1$. ∎

Actually, Reimann and Terwijn [9, Theorem 3.10] obtained a better result via concepts not introduced in this paper: first, the whole many-one degree has intermediate Hausdorff dimension; second, this Hausdorff dimension coincides with a prechosen rational number. Reimann and Terwijn also generalized their result to bounded truth-table degrees. Reimann and Slaman [11] announced that there is even a truth-table degree of Hausdorff-dimension $\frac{1}{2}$. Reimann asked whether the same result holds for weak truth-table reducibility and Turing reducibility. So he posed the following open problem.

Question 3 [9]. *Is there a set A such that $0 < \dim_H(\{B : B \leq_T A\}) < 1$?*

Reimann and Slaman [9, 10] obtained the following result on the way to attack this question: There is a set D such that no Martin-Löf random set is Turing reducible to D but there is a nontrivial lower bound on the Kolmogorov complexity of D in the sense that there is a computable function f with $C(D(0)D(1)\ldots D(f(n)-1)) > n$ for all n. Kjos-Hanssen, Merkle and Stephan [3] obtained the same result independently. At the time of submitting this paper, Question 3 was still unsolved for wtt-reducibility and the present work obtains a partial result is obtained by settling it for a relativized world by proving that there is a lower wtt-cone of intermediate Hausdorff-dimension in some relativized world. Indeed, when revising his paper, the author learned that Nies and Reimann [7] obtained an unrelativized positive solution to this question for wtt-reducibility.

Theorem 4. *There is a set A and an oracle B such that*

$$\tfrac{1}{4} \leq \dim_H^B(\{D : D \leq_{wtt}^B A\}) \leq \tfrac{1}{2}.$$

Proof. Proposition 5 below states that there are a $\{0,1\}$-valued partial-recursive function and a partition of the complement of its domain into intervals $I_0, I_1, \ldots$ satisfying four properties (a), (b), (c) and (d) listed in the proposition. Proposition 6 below states that one can choose sets A, B such that B extends ψ and $\frac{1}{4} \leq \dim_H^B(\{D : D \leq_{wtt}^B A\}) \leq \frac{1}{2}$. ∎

Recall that B extends ψ iff the characteristic function of B coincides with ψ on its domain: $\forall x\, (\psi(x)\downarrow \Rightarrow \psi(x) = B(x))$. Furthermore, let $\mathrm{use}(\psi, p)$ denote the largest supremum of all numbers s such that there is some B extending ψ for which $U^B(p)$ converges in exactly s steps of computation and therefore queries B only below s. Note that $\mathrm{use}(\psi, p)$ is only infinite iff there is for every t an extension B of ψ such that $U^B(p)$ is defined but needs more than t steps. Furthermore, one assumes without loss of generality for every oracle B and every $p \in \{0,1\}^*$ that $U^B(p)$ needs always at least $|p|$ computation steps before termination, that for every $l \geq 2$ there is $q \in \{0,1\}^l$ such that $U^B(q)$ terminates and that $U^B(\lambda)$, $U^B(0)$, $U^B(1)$ are undefined for all oracles B.

Proposition 5. *There are a partial-recursive $\{0,1\}$-valued function ψ and a partition of its complement in intervals $I_0, I_1, \ldots$ such that the following conditions hold:*

(a) *each I_n is an interval with $\max(I_n) < \min(I_{n+1})$ and $2^{|I_n|} < |I_{n+1}|$;*

(b) $\min(I_n) = \sup\{use(\psi, p) : |p| < |I_n|/2\}$*;*

(c) *if B extends ψ then $\dim_H^B(A) \geq \frac{1}{4} \cdot \dim_H(A)$ for any set A;*

(d) *if B extends ψ and $D(\{1, \ldots, |I_n|\}) = B(I_n)$ for infinitely many n then $\dim_H^B(D) \leq \frac{1}{2}$.*

Proof. The partial recursive $\{0,1\}$-valued function ψ is defined in stages so that (a), (b), (c) and (d) are satisfied. For each stage s there is a number n such that $I_{0,s}, I_{1,s}, \ldots, I_{n,s}$ are defined and each element of $\{0, \ldots, s\}$ is either in the domain of ψ_s or in one interval $I_{m,s}$ with $m \leq n$ but not in both. In stage s, an interval might be shifted or even be made undefined but in most stages it just remains where it is.

At stage 0, ψ_0 is everywhere undefined. Let $I_{m,s}$ is the interval I_m at stage s whenever it exists at that stage, in particular $I_{0,0} = \{0, 1, 2, 3\}$ and

$I_{m,0}$ does not exist for $m > 0$.

At stage $s+1$, take the first of the following cases to apply. Let n be the maximal number such that the interval I_n exists after stage s.

- There are an extension τ of ψ_s with domain $\{0,1,\ldots,s\}$ and a $p \in \{0,1\}^*$ with $|p| < |I_n|/2$ at stage s such that $U^{\tau}_{s+1}(p)$ but not $U^{\tau}_{s}(p)$ is defined.
  If there are several such (p,τ) then choose the pair with the shortest p; otherwise just take the unique (p,τ). Let m be the minimal number such that $|p| < |I_{m,s}|$. Now update
  $$\psi_{s+1}(x) = \begin{cases} \psi_s(x) & \text{if defined;} \\ \tau(x) & \text{if } \min(I_{m,s}) \le x \le s; \\ \uparrow & \text{otherwise.} \end{cases}$$
  The intervals I_k with $k < m$ remain unchanged ($I_{k,s+1} = I_{k,s}$), the interval I_m is moved beyond s ($I_{m,s+1} = \{s+1, s+2, \ldots, s+|I_{m,s}|\}$) and the intervals I_k with $k > m$ are undefined after stage $s+1$.

- $s+1 \in I_n$ after stage s.
  Then everything remains unchanged including ψ and the intervals.

- Otherwise, that is, no interval is shifted and $s = \max(I_n)$ after stage s. Then I_{n+1} is newly defined as $\{s+1, s+2, \ldots, s+2^{s+2}\}$ at stage $s+1$. $I_{m,s+1} = I_{m,s}$ for all $m \le n$ and $\psi_{s+1} = \psi_s$.

It is easy to see that every stage of the construction is finite and recursive. Now it will be shown by induction that every interval becomes only finitely often undefined and moves only finitely often. As I_n becomes only undefined at stage s if some I_m with $m < n$ moves at stage s, it is sufficient to show that each I_n moves only finitely often and is almost always defined.

The interval I_0 never moves as $|I_0| = 4$ and $U^B(p)$ is undefined for all p with $|p| < 2$. So the assumption is true for $n = 0$. Now assume that it is proven for n and that one has to prove it for $n+1$. Let $s = \max(I_n)+1$ where I_n denotes in this phrase the final position of this interval. Then no interval with $m \le n$ does move at stage s or later since otherwise $s \le \min(I_n)$ in contradiction to the choice of s. So I_{n+1} is newly defined at stage $s+1$ and $I_{n+1,s+1} = \{s+1, s+2, \ldots, s+2^{s+2}\}$. From then on, I_{n+1} will never become undefined again but it will only be moved. In particular, I_{n+1} will never change its length again. Now for every $\eta \in \{0,1\}^{\max(I_n)+1}$ and every p with

$|p| < |I_{n+1}|$ it happens at most once that some $t > s$ that I_{n+1} will be moved because of an extension τ of both, η and ψ_t, with $U^\tau(p)$ halting exactly at t steps. Thus one can conclude that I_{n+1} moves at most $2^{\max(I_n)+|I_{n+1}|/2}$ times.

This concludes the proof of the inductive step and now the properties (a), (b), (c) and (d) are verified.

(a): $\max(I_n) < \min(I_{n+1})$ and $2^{|I_n|} < |I_{n+1}|$.

The first condition is easy to see. The second one follows from the above verification. If $s = \max(I_n)$ then $I_{n+1,s+1} = \{s+1, s+2, \ldots, s+2^{s+2}\}$ and $|I_{n+1}| = 2^{s+2}$. Using that $|I_n| \leq s+1$ completes this part of the proof.

(b): $\min(I_n) = \sup\{\mathrm{use}(\psi, p) : |p| < |I_n|/2\}$ for almost all n.

Let I_n be an interval that is defined, for the last time, at some stage s and has length 2^{s+2}. If n is sufficiently large, so is s and there is a program p which computes $B(s)$. This program needs time $O(s)$ and thus the interval I_n is moved at least once. There is a stage t such that I_n is moved at stage t for the last time, that is, $I_{n,t} \neq I_{n,t+1} = I_{n,t'}$ for all $t' > t$. Thus there is a τ extending ψ_t and a p such that $|p| < 2^{s+2}/2$ and $U^B(p)$ halts in exactly $t+1$ steps. As the intervals I_m with $m < n$ have all reached their final positions at s, the string τ is not only consistent with ψ_t and ψ_{t+1} but also with the final function ψ. Thus some extension B of τ is also an extension of ψ and $\mathrm{use}(\psi, p) = t+1$. Furthermore, there is no $q \in \{0,1\}^*$ and no extension B of ψ such that $|q| < |I_n|/2$ and $U^B(q)$ halts after exactly $t'+1$ steps with $t' > t$: otherwise the interval I_n would be moved again because the program q together with the prefix of length $t'+1$ of B would qualify in the search of stage $t'+1$ in contrary to the assumption that t is the last stage where this interval I_n moves.

Although this completes the proof of (b), a short note should be added. This note says that for some constant c and all n, $C(\min(I_n)) \leq |I_n|/2 + 2\log(|I_n|) + c$. To see this, consider the s such that $|I_n| = 2^{s+2}$ and the p of length less than $|I_n|/2$ such that there is an extension B of ψ with $U^B(p)$ halting in exactly $\min(I_n)$ steps. If a place queried by $U^B(p)$ is larger than s then ψ is defined at that place and the simulation of ψ provides the answer. If it is smaller, one might look up the answer in a tabled string containing $s+1$ bits. Thus one can compute $\min(I_n)$ from $0^s1B(0)B(1)\ldots B(s)p$. As $|I_n| = 2^{s+2}$, this additional information is logarithmic in s and one could use some additional bits in order to make the code prefix-free and having the length $2^{s+1}+2s+3$, so one would use the code $0^s1B(0)B(1)\ldots B(s)0^{2^{s+1}-|p|}1p$. So there is a constant c such that both, the prefix-free and the plain Kolmogorov complexity of $\min(I_n)$ is bounded by $|I_n|/2 + 2\log(|I_n|) + c$ for all n where this constant c is so large that it also covers the case $n = 0$ where $\min(I_n)$ where the interval is not moved and therefore the p with $|p| < |I_n|/2$ and $\min(I_n) = \mathrm{use}(\psi, p)$ does not exist.

(c): If B extends ψ then $\dim_H^B(A) \geq \frac{1}{4} \cdot \dim_H(A)$.

Let n be any number and p be the shortest program with $U^B(p) = A(\{0, 1, \ldots, n\})$. Let m be the largest number such that $U^B(p)$ queries B at some element of I_m. Due to (b), $|p| \geq |I_m|/2$. Let s be the number with $|I_m| = 2^{s+2}$, that is, $s+1$ is the stage where I_m is newly defined for the last time; from then on I_m is only moved and never becomes undefined again. Now one can compute $U^B(p)$ from the inputs p, s, $B(\{0, 1, \ldots, s\})$, $B(I_m)$ and $\min(I_m)$. Let q be the prefix-free coding for $\min(I_m)$ and s from the end of part (b) of length $2^{s+1}+2s+3$. Then $qB(\{0, 1, \ldots, s\})B(I_m)p$ is a description for $U^B(p)$ in some machine (not necessarily U) of length $3 \cdot 2^{s+1} + 3s + 4 + |p|$. Using that $2^{s+1} \leq |p|$, one gets that $C(A(\{0, 1, \ldots, n\})) \leq 4|p| + 4\log(|p|) + c' = 4(C^B(A(\{0, 1, \ldots, n\})) + \log(C^B(A(\{0, 1, \ldots, n\})))) + c'$ for some constant c' independent of n, m, p. It follows that $\dim_H(A) \leq 4\dim_H^B(A)$ and $\dim_H^B(A) \geq \frac{1}{4} \cdot \dim_H(A)$.

(d): If B extends ψ and $D(\{1, \ldots, |I_n|\}) = B(I_n)$ for infinitely many n then $\dim_H^B(D) \leq \frac{1}{2}$.

Recall that $D(\{1, \ldots, |I_n|\}) = B(I_n)$ means that $D(1) = B(\min(I_n))$, $D(2) = B(\min(I_n) + 1)$, $\ldots$, $D(|I_n|) = B(\max(I_n))$. By (b) there is a constant c such that $C(\min(I_n)) \leq |I_n|/2 + 2\log(|I_n|) + c$ for all n. Then $D(\{1, \ldots, |I_n|\})$ can be computed relative to B by reading out the oracle B on the interval I_n. Recall that the description for $\min(I_n)$ also revealed s and therefore $|I_n| = 2^{s+2}$. Thus, up to an additive constant, the relation $C^B(D(\{1, \ldots, |I_n|\})) \leq |I_n|/2 + 2\log(|I_n|)$ holds for infinitely many n and so the Hausdorff-dimension of D relative to B is at most $\frac{1}{2}$. ∎

Proposition 6. *Assume that a partial-recursive $\{0, 1\}$-valued function ψ and a partition of its complement into intervals $I_0, I_1, \ldots$ satisfy the following four conditions from Proposition 5:*

(a) *each I_n is an interval with* $\max(I_n) < \min(I_{n+1})$ *and* $2^{|I_n|} < |I_{n+1}|$*;*

(b) $\min(I_n) = \sup\{$*use*$(\psi, p) : |p| < |I_n|/2\}$*;*

(c) *if* $\tilde{B}$ *extends* ψ *then* $\dim_H^{\tilde{B}}(\tilde{A}) \geq \frac{1}{4} \cdot \dim_H(\tilde{A})$ *for any set* $\tilde{A}$*;*

(d) *if* $\tilde{B}$ *extends* ψ *and* $D(\{1,\ldots,|I_n|\}) = \tilde{B}(I_n)$ *for infinitely many* n *then* $\dim_H^{\tilde{B}}(D) \leq \frac{1}{2}$.

Then there are sets A, B *such that* B *extends* ψ *and the lower wtt-cone of* A *relative to* B *has nonintegral dimension:* $\frac{1}{4} \leq \dim_H^B(\{D : D \leq_{wtt}^B A\}) \leq \frac{1}{2}$.

Proof. The sets A and B are constructed by a finite extension method. The set A will be an infinite branch of a recursive tree T where T is chosen such that every infinite branch of T has Hausdorff-dimension 1. The tree T is constructed as some type of complement of the class O of a universal Martin-Löf test; O has Lebesgue measure less than one half, O is effectively open and O contains all sets of Hausdorff-dimension less than 1. Indeed $O = \cup_s \vartheta_s \cdot \{0,1\}^\infty$ for a recursive enumeration $\vartheta_0, \vartheta_1, \ldots$ of binary strings with $\sum_s 2^{-|\vartheta_s|} < \frac{1}{2}$ and $\tau \in T$ iff there is no $s \leq |\tau|$ such that τ extends ϑ_s.

A itself will be build by a finite extension method inside T. B will be chosen by inductively by constructing inductively extensions θ_e of ψ which are defined on the domain of ψ plus finitely many intervals I_n. These extensions θ_e are still partial-recursive, but the choice of the extensions is not. At the beginning, one starts with $\sigma_0 = \lambda$ as the empty string and θ_0 being the partial-recursive function ψ. Then all wtt-reductions are addressed inductively, note that every wtt-reduction has infinitely many indices.

In stage e, the e-th wtt-reduction is considered. The construction follows the first case of the three below which applies.

Case 1: There is an extension $\tilde{A}$ of σ_e inside T and $\tilde{B}$ of θ_e and an x such that $\varphi_e^{\tilde{B}}(x; \tilde{A})$ is undefined because the computation of the query bound fails.

Then one chooses a program p for the universal Turing machine U such that $U^E(p)$ computes the bound of the e-th wtt-reduction on input x relative to an oracle E whenever this bound is defined. By condition (b), for all intervals I_n with $|I_n| > 2 \cdot |p|$ the convergence of the computation $U^E(p)$ does not depend on the choice of the values of E on I_n since the computation either converges within $\min(I_n)$ computation steps or diverges.

Let θ_{e+1} be the extension of θ_e which takes the value $\tilde{B}(z)$ whenever z is in the domain of θ_e or z is in an interval I_n with $|I_n| \leq 2 \cdot |p|$. θ_{e+1} is undefined on all other inputs z. The string σ_{e+1} is just σ_e as the oracle $\tilde{A}$ was never queried.

Case 2: There is an extension $\tilde{A}$ of σ_e inside T and $\tilde{B}$ of θ_e and an x such that $\varphi_e^{\tilde{B}}(x;\tilde{A})$ is undefined although the computation of the query bound terminates.

Let r be the query bound. Then there is a program p that codes r, e and the oracle answers of $\tilde{A}$ up to r such that $U^E(p) = \varphi_e^E(x;\tilde{A}(0)\tilde{A}(1)\ldots\tilde{A}(r))$ whenever E is an oracle extending θ_e. Again one can argue using U as above and justify the following extension.

Let θ_{e+1} be the extension of θ_e which takes the value $\tilde{B}(z)$ whenever z is in the domain of θ_e or z is in an interval I_n with $|I_n| \leq 2 \cdot |p|$. θ_{e+1} is undefined on all other inputs z. The string σ_{e+1} is the prefix of $\tilde{A}$ of length $|\sigma_e| + r$.

Case 3: For every extension $\tilde{A}$ of σ_e inside T and $\tilde{B}$ of θ_e and every x the wtt-reduction $\varphi_e^{\tilde{B}}(x;\tilde{A})$ terminates and outputs some value.

Now one can define the reduction Θ based on φ_e as follows: $\Theta^{E,F}(x)$ takes $\varphi_e^E(x;F)$ in the case that this reduction terminates and 0 in the case that E does not extend θ_e or F does not extend σ_e or F is not an infinite branch of T; if both cases apply then the output follows that case which is determined first to apply. Since Θ is defined for all oracles E, F, Θ is actually a truth-table reduction. There is a recursive function $\tilde{p}$ mapping any input k to a program of size up to $\log(k) + c$ for some constant c independent of k such that $U(\tilde{p}(k))$ computes the use of the tt-reduction when requested to compute $\Theta^{E,F}(0)\ldots\Theta^{E,F}(k)$ and needs also this use as computation time. Thus, by the construction of the intervals I_n, whenever $|I_n| > 2 \cdot |\tilde{p}(k)|$ then the use of $\Theta^{E,F}(l)$ is below $\min(I_n)$ for $l = 0, 1, \ldots, k$.

Since $|\tilde{p}(|I_n|)| \leq \log(|I_n|) + c$ for all n, there is an interval I_n outside the domain of θ_e where $\tilde{p}(|I_n|) < |I_n|/2$. Now let $\tilde{A}$ be any extension of σ_e inside T. There is an extension $\tilde{B}$ of θ_e such that $\tilde{B}(I_n) = \Theta^{\tilde{A},\tilde{B}}(\{1, \ldots, |I_n|\})$; this can be obtained by first choosing $\tilde{B}$ arbitrarily and then changing $\tilde{B}_n$ on I_n which does not affect the value of $\Theta^{\tilde{A},\tilde{B}}(\{1, \ldots, |I_n|\})$ since $\tilde{B}$ is not queried on I_n for these computations.

Let θ_{e+1} be the extension of θ_e which takes the value $\tilde{B}(z)$ whenever z is in the domain of θ_e or $z \leq \max(I_n)$ for the I_n determined above. The string σ_{e+1} is the prefix of $\tilde{A}$ of length $|\sigma_e| + \max(I_n)$.

This completes the distinction of Cases 1,2,3 to build σ_{e+1} and θ_{e+1} in stage e in the simultaneous construction of A and B. The following properties of the construction can be verified.

- Every σ_e can be extended to an infinite branch of T.
- Each θ_e is a finite variant of ψ and thus a partial-recursive function.

- For every x there is an e such that x is in the domain of σ_e and θ_e: This follows from the fact that there are infinitely many indices e for the wtt-reduction mapping every set A to A and for each x there is such an index e with the additional property that the I_n picked satisfies $\max(I_n) \geq x$. The construction which has to go through the third case, causes that $\theta_{e+1}(x)$ and $\sigma_{e+1}(x)$ is defined.

- The limits A of the σ_e and B of the θ_e exist and extend all σ_e and θ_e. Furthermore A is an infinite branch of T and thus $\dim_H(A) = 1$. It follows from (c) that $\dim_H^B(A) \geq \frac{1}{4}$.

- If $D \leq_{wtt}^B A$ then this reduction has infinitely many indices e and the construction goes through Case 3 for these e. For each such e there is an interval I_n outside the domain θ_e on which B takes the value-sequence $D(\{1, \ldots, |I_n|\})$. It follows from condition (d) that $\dim_H^B(D) \leq \frac{1}{2}$.

These properties complete the proof of the theorem. ∎

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