Presentations of K-trivial reals and Kolmogorov Complexity

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Abstract. For given real $\alpha \in \{0,1\}^\infty$, a presentation V of α is a prefix-free and recursively enumerable subset of $\{0,1\}^*$ such that $\alpha = \sum_{\sigma \in V} 2^{-|\sigma|}$. So, α has a presentation iff α is a left-r.e. real. Let $\mathcal{A}$ be the class of all reals which have only computable presentations. Downey and LaForte proved that $\mathcal{A}$ has an incomputable member. Call α strongly Kurtz-random if there does not exist any recursive function f with $K(\alpha(0)\ldots\alpha(f(n)-1)) < f(n) - n$ for all n. It is shown that every $\alpha \in \mathcal{A}$ is either computable or strongly Kurtz-random. In particular, all K-trivial members of $\mathcal{A}$ are computable where α is K-trivial iff there is a c such that, for all n, $K(\alpha(0)\ldots\alpha(n-1)) < K(n) + c$. Thus there is a natural and nontrivial Turing ideal of left-r.e. α not containing any incomputable member of $\mathcal{A}$.

1 Introduction

In this paper, we work with machines with input and output alphabets $\{0,1\}$. A Turing machine M is *prefix-free* if $M(\tau)\downarrow \Rightarrow M(\tau')\uparrow$ for all finite binary strings $\tau' \succeq \tau$ where $\succeq$ denotes string extension. It is *universal* if for each prefix-free machine N there is a constant c such that, for all binary strings τ, if $N(\tau)\downarrow$ then there is some σ such that $|\sigma| \le |\tau| + c$ and $M(\sigma)\downarrow = N(\tau)$. c is called the *coding constant* of N. For a prefix-free machine M and a binary string τ, $K_M(\tau)$ denotes the length of the shortest binary string σ such that $M(\sigma)\downarrow = \tau$, if such σ exists and let $K_M(\tau)$ be undefined otherwise. In the following we fix a universal prefix-free machine U and let $K(\tau) = K_U(\tau)$, which is called the

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Kolmogorov complexity of τ. We remark here that the choice of U does not affect the Kolmogorov complexity by more than an additive constant.

For convenience, we identify subsets of the natural numbers with the reals between 0 and 1. So $\alpha \in \{0,1\}^\infty$ stands for both, for the set $\{n : \alpha(n) = 1\}$ and for the real number $\sum_{n=0,1,\ldots} \alpha(n) \cdot 2^{-n-1}$. We write $\alpha < \beta$ iff α is less than β as a real number. By $\alpha[m]$ we denote the string $\alpha(0)\alpha(1)\ldots\alpha(m-1)$ which we identify with the finite set $\{n \in \alpha : n < m\}$ and the number $\sum_{n<m} \alpha(n) \cdot 2^{-n-1}$. α is called recursively enumerable iff it is recursively enumerable as a set and it is called left-r.e. iff it has a recursive approximation $\alpha_0, \alpha_1, \ldots$ such that $\alpha_n \leq \alpha$ for all n; the left-r.e. sets are also known as "nearly computable" and in the case that they are more viewed upon as real numbers than as sets, they are often just referred to as "r.e. reals". In order to keep the notation between numbers and sets consistent, we do not follow this notation and our "r.e. reals" correspond to those which some people call "strongly r.e." in order to distinguish them from the left-r.e. reals. Note that we can choose the approximation such that $\alpha_0 < \alpha_1 < \ldots < \alpha$ and $\alpha_n \subset \{0, 1, \ldots, h(n)\}$ for some computable function h and all n.

Calude, Hertling, Khoussainov and Wang [4] proved that α is left-r.e. if and only if there is an infinite *recursively enumerable* prefix-free set of strings V such that $\alpha = \sum_{\sigma \in V} 2^{-|\sigma|}$ if and only if there is an infinite computable prefix-free set of strings V such that $\alpha = \sum_{\sigma \in V} 2^{-|\sigma|}$, where a set V is *prefix-free* means that if $\sigma \in V$, then no extension of σ can be in V. So, based on the convention that $\alpha = \sum_{n=0,1,\ldots} 2^{-1-n}\alpha(n)$, Downey and LaForte [10] introduced the following notion.

Definition 1. *V is a presentation of α iff V is a recursively enumerable and prefix-free set with $\alpha = \sum_{\sigma \in V} 2^{-|\sigma|}$.*

Downey and LaForte [10] proved that there are incomputable sets having only computable presentations.

In this paper, we will give a characterization for the existence of an incomputable presentation in terms of the existence of a certain type of approximation. This characterization will be the basis for showing that the Kolmogorov complexity of any incomputable left-r.e. α without incomputable presentations cannot be bounded for any recursively selected set of prefixes of α, neither from below nor from above. More precisely, if α is incomputable and left-r.e. set which has only computable presentations and f is a computable function then neither $\forall n\,(K(\alpha[f(n)]) > n)$ nor $\forall n\,(K(\alpha[f(n)]) < f(n) - n)$.

The first condition can be interpreted as α being unable to wtt-compute fixed-point free functions, that is, of α being wtt-incomplete. This condition follows already from the result of Downey and LaForte [10] that every left-r.e. set of promptly simple degree has an incomplete presentation.

So the really interesting condition is the second one. The second notion is a stronger version of Kurtz-randomness and thus we assign to it the corresponding name.

Definition 2. *An $\alpha \in \{0,1\}^\infty$ is* strongly Kurtz-random *iff there is no computable function f such that $K(\alpha[f(n)]) < f(n) - n$ for all n.*

Normally Kurtz-random is defined in terms of computable martingales. We drop the computability constraint and consider the martingale M given as the sum over all $M_{\tau,\sigma}$ with $U(\tau) = \sigma$ and

$$M_{\tau,\sigma}(\eta) = \begin{cases} 2^{|\sigma|-|\tau|} & \text{if } \sigma \prec \eta; \\ 2^{|\eta|-|\tau|} & \text{if } \eta \preceq \sigma; \\ 0 & \text{otherwise.} \end{cases}$$

Then α is strongly Kurtz-random if there is no computable function f with $M(\alpha[f(n)]) \geq n$ for all n. Since M is a universal martingale, this condition is stronger than the following one: $(*)$ There is no computable martingale $\widetilde{M}$ and no computable function f such that $\widetilde{M}(\alpha[f(n)]) \geq n$ for all n. It follows from Wang's martingale characterization [16] that $(*)$ is equivalent to Kurtz-randomness. This justifies our notion "strongly Kurtz-random" which we use to state the first main result that every left-r.e. set is either computable or has an incomputable presentation or is strongly Kurtz-random.

Theorem 3. *If α is left-r.e., incomputable and every presentation of α is computable then α is strongly Kurtz-random.*

As a direct corollary, every such α is also Kurtz-random with respect to the original definition. Theorem 3 is also the basis for the other main result of the paper. We say that α is *K-trivial* if the Kolmogorov complexity of each initial segment of α is minimal, that is, if there is a constant c such that for all n, $K(\alpha[n]) \leq K(n)+c$. K-trivial reals are interesting due to the connection between algorithmic complexity and effective randomness. We, for example, exploit the fact that no K-trivial set is Kurtz-random. Chaitin [5] proved that if an α is K-trivial then α must be Δ_2. In 1974, Solovay [15] proved that there is an incomputable K-trivial real. Downey, Hirschfeldt, Nies and Stephan [9] proved the existence of an incomputable recursively enumerable K-trivial real. Recently, Nies [12] proved that all the K-trivials are low and that α is K-trivial if and only if α is low for random if and only if there is a constant c such that for all x, $K(x) \leq K^\alpha(x) + c$, that is, adding the oracle α does not change the prefix-free complexity by more than a constant. In this paper, we show that every incomputable left-r.e. K-trivial has always an incomputable presentation and that thus Downey and LaForte's reals cannot be K-trivial.

Theorem 4. *Every K-trivial and incomputable left-r.e. real α has an incomputable presentation.*

Our notation is standard and generally follows the books of Cutland [7], Downey and Hirschfeldt [8], Odifreddi [13] and Soare [14]; but we differ from the notation in [7] by requiring that computable functions are total. Since we identify subsets of the natural numbers with reals, we use the names of notions in the version which is defined for sets.

2 A Characterization

The existence of an r.e. but incomputable presentation can be obtained from the following characterization.

Theorem 5. *Let α be left-r.e. and incomputable. Then α has an incomputable presentation iff there is a recursive approximation $\alpha_0, \alpha_1, \ldots$ such that*

(1) $\alpha_0 < \alpha_1 < \ldots < \alpha$;
(2) *there is a recursive function h such that $\alpha_n \subseteq \{0, 1, \ldots, h(n)\}$ for all n;*
(3) *for every computable function g and every m there exist $n \geq m$ and $s \geq g(n)$ with $\alpha_{s+1} \geq \alpha_s + 2^{-n}$.*

Proof. Assume that α has an incomputable presentation V. Now let $\sigma_0, \sigma_1, \ldots$ be a one-one recursive enumeration of V; this sequence is infinite since V is not computable. Now define an approximation α_s by the equation

$$\alpha_s = \sum_{t<s} 2^{-|\sigma_t|}$$

where $\alpha_0 = 0$. Clearly (1) is satisfied. (2) is obtained by defining

$$h(s) = \max\{|\sigma_t| : 0 \leq t \leq s\}.$$

Now consider any computable function g and any m. Since V is not computable there is a length $n > m$ and an element of V of length n which is not enumerated into V within $g(n)$ steps. It follows that this element is of the form σ_s for some $s \geq g(n)$. Thus $\alpha_{s+1} \geq \alpha_s + 2^{-n}$. So (3) is satisfied.

For the converse direction, let $\alpha_0, \alpha_1, \ldots$ be a recursive approximation of α satisfying the three conditions together via the computable function h. Now define recursive sets T_s to consist of all strings τ such that

(a) $\alpha_s \leq \tau$, where τ is interpreted as the real number $\sum_{k<|\tau|} 2^{-1-k}\tau(k)$;
(b) the reals given by k and τ differ at a digit and the least position k of such a digit satisfies $k < |\tau|$, $\tau(k) = 0$ and $\alpha_{s+1}(k) = 1$;
(c) every proper prefix $\tau' \prec \tau$ violates either (a) or (b).

Here $|\tau|$ is the length of τ and τ can end with a 0. Note that the $\tau \in T_s$ are the shortest prefixes of reals between β with $\alpha_s \leq \beta < \alpha_{s+1}$ such that every binary extension γ of τ, including the extension $\tau 1111\ldots$, also satisfies $\alpha_s \leq \gamma < \alpha_{s+1}$. Thus every set T_s is prefix-free and satisfies $\sum_{\tau \in T_s} 2^{-|\tau|} = \alpha_{s+1} - \alpha_s$. Let V be the union of all T_s. It is easy to see that V is also prefix-free. Since the α_s are monotonically increasing, the sets T_s are not empty and the equations

$$\alpha = \sum_s (\alpha_{s+1} - \alpha_s) = \sum_s \sum_{\tau \in T_s} 2^{-|\tau|} = \sum_{\tau \in V} 2^{-|\tau|}$$

hold. By the existence of h, the sets T_s are uniformly recursive and thus V is recursively enumerable. So V is a presentation for α.

For any given s, let $l(s)$ be the length of the shortest string of T_s. It is easy to see that $-2 - \log(\alpha_{s+1} - \alpha_s) < l(s) < 2 - \log(\alpha_{s+1} - \alpha_s)$. For example, assume that $\alpha_s = \{2, 3, 5, 7\}$ and $\alpha_{s+1} = \{2, 3, 4, 5, 6, 9\}$. Then α_s and α_{s+1} are the real numbers 0.00110101 and 0.0011111001 in binary notation. The value $\alpha_{s+1} - \alpha_s$ is 0.0000100101 and $-\log(\alpha_{s+1} - \alpha_s)$ is between 5 and 6. The shortest string in T_s is 001110 since the binary numbers since $\alpha_s \leq 0.001110000\ldots$ and $0.0011101111\ldots = 0.001111000\ldots < \alpha_{s+1}$. The value of $l(s)$ is $|001110| = 6$.

To complete the proof, assume by way of contradiction that V is recursive. One can compute for any given n the value $g(n) = \max\{t : t = 0 \text{ or } t = s + 1 \text{ and } \exists \tau \in T_s \, (|\tau| \leq n + 2)\}$ by simply taking all $\tau \in V$ with $|\tau| \leq n + 2$ and then searching for the s with $\tau \in T_s$, which is possible since the T_s are uniformly recursive. It follows that for every $s \geq g(n)$, T_s does not contain any string of length $n+2$ or less and thus $\alpha_{s+1} < \alpha_s + 2^{-n}$. This contradicts the third property of the approximation $\alpha_0, \alpha_1, \ldots$ from the statement of the theorem. □

An application of this result is that every α which has only computable presentations is either recursive or hyperimmune. The proof of the following result is a model for the proof of the first main result that every left-r.e. and incomputable set without incomputable presentation is Kurtz-random.

Theorem 6. *Let α be left-r.e., incomputable and not hyperimmune. Then α has an incomputable presentation.*

Proof. Since α is not hyperimmune, there is a computable function f such that $\alpha \cap \{n+1, n+2, \ldots, f(n)\}$ is not empty for every n. Let $\beta_0, \beta_1, \ldots$ be a recursive approximation of α from below. Now one defines inductively a new approximation $\alpha_0, \alpha_1, \ldots$ and a recursive function h such that $\alpha_n = \beta_s[f(t) + 1]$ and $h(n) = t$ for the first pair (s, t) in some fixed enumeration of all pairs of natural numbers with

(d) $s > h(m)$ and $t > h(m)$ for all $m < n$;
(e) $\alpha_m < \beta_s[f(t) + 1]$ for all $m < n$;
(f) β_s intersects $\{m + 1, m + 2, \ldots, f(m)\}$ for all $m \leq t$.

Then the resulting approximation $\alpha_0, \alpha_1, \ldots$ of α has the following properties.

(g) $\alpha_0 < \alpha_1 < \ldots < \alpha$;
(h) $h(n + 1) > h(n)$ and $\alpha_n \subseteq \{0, 1, \ldots, f(h(n))\}$ for all n;
(i) the intersection $\alpha_n \cap \{m + 1, m + 2, \ldots, f(m)\}$ is not empty for any n and any $m \leq h(n)$.

It remains to show that (3) of Theorem 5 is true for every computable function g and m. Suppose not. Let g be a computable function with $g(x) > x$ for all x and m be a number at which (3) is false. Then, for any $n \geq m$ and $s \geq g(n)$, $\alpha_{s+1} < \alpha_s + 2^{-n}$. Thus, for any given x, $f(x) + m + 2 > m$, and for any $s > g(f(x) + m + 2)$, $\alpha_{s+1} < \alpha_s + 2^{-f(x)-m-2}$.

Now fix x. Then by (h), $h(x) > x$. Let $t(x) = \max\{h(x), g(f(x) + m + 2)\}$. We prove that for any $s > t(x)$, $\alpha_s(x) = \alpha_{t(x)}(x)$ and hence $\alpha(x) = \alpha_{t(x)}(x)$.

Suppose not, and let $s > t(x)$ be a stage with $\alpha_{s+1}(x) \neq \alpha_s(x)$. Then $h(s+1) > h(x) > x$. By the choice of the approximation $\alpha_0, \alpha_1, \ldots$ and (i), there is some y in $\{x+1, x+2, \ldots, f(x)\}$ such that $\alpha_{s+1}(y) = 1$. As a consequence,

$$\alpha_{s+1} \geq \alpha_s + 2^{-y} \geq \alpha_s + 2^{-f(x)} > \alpha_s + 2^{-f(x)-m-2},$$

a contradiction.

Thus, for any x, $\alpha(x) = \alpha_{t(x)}(x)$ and α would be computable, contradicting the assumption of α. Thus (3) of Theorem 5 is also satisfied and hence α has an incomputable but recursively enumerable presentation. $\square$

Since every wtt-complete left-r.e. set has promptly simple degree, it has an incomputable presentation, by a result of Downey and LaForte [10]. An application of Theorem 6 gives an alternative proof for this fact. The connection to Kolmogorov complexity is based on the fact that a left-r.e. α is wtt-complete iff it can wtt-compute an DNR function [2, 3]. Note that Arslanov's criterion also applies to left-r.e. sets since those are tt-equivalent to r.e. ones. Furthermore, Kjos-Hanssen, Merkle and Stephan [11] proved that a set wtt-computes a DNR function iff there is a computable function f such that $K(\alpha[f(n)]) > n$ for all n. Theorem 7 is given in terms of Kolmogorov complexity and not in terms of wtt-completeness.

Theorem 7. (Downey and LaForte [10]) *If α is left-r.e. and there is a computable function f such that, for all n, $K(\alpha[f(n)]) > n$ then α has an incomputable presentation.*

Proof. Without loss of generality one can take f to be monotonically increasing; so assume that for α there is a monotonically increasing function f such that $K(\alpha[f(n)]) > n$ for all n. Now let $g(0) = 1$ and $g(n+1) = f(4^{g(n)})$. Given a string $\tau \in \{0,1\}^{g(n)}$, one can compute from τ the number n, the value $g(n+1)$ and the string $\tau 0^{g(n+1)-g(n)}$. Since there is a constant c with $K(\tau) < 3^{|\tau|} + c$ for all $\tau \in \{0,1\}^*$, it holds for almost all n that

$$K(\alpha[g(n+1)]) = K(\alpha[f(4^{g(n)})]) > 4^{g(n)} \geq K(\alpha[g(n)] \cdot 0^{g(n+1)-g(n)}).$$

Thus, $\alpha(g(n))\alpha(g(n)+1)\ldots\alpha(g(n+1)-1)$ differs from $0^{g(n+1)-g(n)}$ for almost all n and α cannot be a hyperimmune set. It follows from Theorem 6 that α has an incomputable presentation. $\square$

3 The Main Results

We now prove that every α which has a recursively enumerable but not incomputable presentation is either computable or strongly Kurtz-random. That is, the following second main result is proven on the way to get the first one. Since strongly Kurtz-random sets are Kurtz-random, a direct corollary is that every left-r.e. and incomputable α where every presentation of α is computable, is also Kurtz-random.

Theorem 3. *If α is left-r.e., incomputable and every presentation of α is computable then α is strongly Kurtz-random.*

Proof. Assume that α is incomputable, left-r.e. and not strongly Kurtz-random. We will prove that α has an incomputable presentation. This is done by showing that there is an approximation $\alpha_0, \alpha_1, \ldots$ satisfying the three conditions of Theorem 5.

Let f be a computable function witnessing that α is not strongly Kurtz-random: $\forall n\,(K(\alpha[f(n)]) < f(n) - n)$. Furthermore, let $\beta_0, \beta_1, \ldots$ be an approximation of α from below. Now one defines inductively a new approximation $\alpha_0, \alpha_1, \ldots$ and a recursive function h such that $\alpha_n = \beta_s[f(t)]$ and $h(n) = t$ for the first pair (s, t) with

(j) $s > h(m)$ and $t > h(m)$ for all $m < n$;
(k) $\alpha_m < \beta_s[f(t)]$ for all $m < n$;
(l) $K_s(\beta_s[f(m)]) < f(m) - m$ for all $m \leq t$.

The resulting approximation $\alpha_0, \alpha_1, \ldots$ of α has the following properties:

(m) $\alpha_0 < \alpha_1 < \ldots < \alpha$;
(n) $h(n+1) > h(n)$ and $\alpha_n \subseteq \{0, 1, \ldots, f(h(n))\}$ for all n;
(o) $K_s(\alpha_s[f(m)]) < f(m) - m$ for all n and $m \leq h(n)$.

It remains to show that (3) of Theorem 5 is also true.

Assume by way of contradiction that this is false for a function g and an m; without loss of generality, $g(n) > n$ for all n. Now for any $n \geq m$, search for the first pair (o, t) found such that

(p) $o > n$;
(q) $h(t) > o + 1$;
(r) $t > g(f(o+1))$;
(s) $\alpha_t(o) = 0$.

This search terminates since there α has the digit 0 infinitely often and the search condition is satisfied whenever $\alpha(o) = 0$ and t is sufficiently large. So the search for o and t can be realized by a computable function.

For every $s \geq t$, *by (q)*, $K(\alpha_s[f(o+1)]) < f(o+1) - o - 1$. Thus, by the choice of m and $o > n \geq m$, there are less than $2^{f(o+1)-o-1}$ many stages $s \geq t$ with $\alpha_{s+1}[f(o+1)] \neq \alpha_s[f(o+1)]$. Each of them satisfies that $\alpha_{s+1} - \alpha_s < 2^{-f(o+1)}$ by $s \geq g(f(o+1))$. Therefore, $\alpha_{s+1}[f(o+1)] = 2^{-f(o+1)} + \alpha_s[f(o+1)]$. Since there are less than $2^{f(o+1)-o-1}$ many places $s \geq t$ where $\alpha_{s+1}[f(o+1)] \neq \alpha_s[f(o+1)]$ and since the value goes up each time by exactly $2^{-f(o+1)}$, one has

$$\alpha[f(o+1)] < \alpha_t[f(o+1)] + 2^{f(o+1)-o-1} \cdot 2^{-f(o+1)} = \alpha_t[f(o+1)] + 2^{-o-1}.$$

Since $\alpha_t(o) = 0$, the finite $\beta = \{o\} \cup \alpha_t[f(o+1)]$ satisfies $\alpha_s[f(o+1)] \leq \beta$ for all s. It follows that $\alpha(n) = \alpha_t(n)$ and one can compute $\alpha(n)$ by searching for (o, t) as above and taking the approximation $\alpha_t(n)$ as the desired value. Since α is incomputable, the assumed g does not exist. Thus the conditions on the approximation $\alpha_0, \alpha_1, \ldots$ from Theorem 5 are all satisfied and hence α has an incomputable presentation. $\square$

One might ask whether also randomness notions stronger than Kurtz-randomness are implied. The answer is negative. Kurtz-randomness is something special since for each not Kurtz-random set there is a computable martingale M and a recursive function f such that $M(\alpha[f(n)]) > n$ for all n. The other randomness notions require the corresponding condition only for infinitely many n. Thus randomness happens "much less frequently" and no hyperimmune set is Schnorr-random or Martin-Löf-random. So it follows from Theorem 6 that every Schnorr-random and every Martin-Löf-random left-r.e. set has an incomputable presentation.

In order to see the other main result, note that there is a constant c with $K(n) < c + n/2$ for all n. If α is K-trivial then there is a constant c' such that $K(\alpha[n]) < K(n) + c'$. Taking $f(n) = 2n + c + c'$, one has $K(\alpha[f(n)]) < f(n) - n$ for all n. Thus a K-trivial set is not strongly Kurtz-random and so one has the following.

Theorem 4. *Every K-trivial and incomputable left-r.e. real α has an incomputable presentation.*

4 A Degree-Theoretic Conclusion

Theorem 4 has some degree-theoretic consequences since the K-trivial sets are closed downward under Turing reduction and since below every incomputable r.e. degree there is an incomputable, K-trivial r.e. degree. Let $\mathbf{R}$ be the set of nonzero r.e. Turing degrees and $\mathbf{A}$ be the subset of those degrees in $\mathbf{R}$ which contain an left-r.e. α without an incomputable presentation. Then every $\mathbf{a}$ in $\mathbf{R}$ bounds an $\mathbf{b} \in \mathbf{R}$ not bounding any degree in $\mathbf{A}$. That is,

$$\forall \mathbf{a} \in \mathbf{R}\, \exists \mathbf{b} \in \mathbf{R}\, \forall \mathbf{c} \in \mathbf{R}\, (\mathbf{b} \leq \mathbf{a} \wedge (\mathbf{c} \leq \mathbf{b} \Rightarrow \mathbf{c} \notin \mathbf{A})).$$

In particular, every $\mathbf{a} \in \mathbf{R}$ bounds a degree in $\mathbf{R} - \mathbf{A}$. Nies [12] showed that there is a K-trivial and promptly simple set, let $\mathbf{s}$ be its degree. As seen, no $\mathbf{a} \leq \mathbf{s}$ is in $\mathbf{A}$. Ambos-Spies, Jockusch, Shore and Soare [1] showed that the degrees of promptly simple sets form a filter in the r.e. Turing degrees. Following the result of Downey and LaForte [10] that every left-r.e. of promply simple degree has an incomputable presentation, one has that the degrees of incomputable left-r.e. sets without incomputable presentation are incomparable to $\mathbf{s}$:

$$\forall \mathbf{a} \in \mathbf{A}\, (\mathbf{a} \not\leq \mathbf{s} \wedge \mathbf{a} \not\geq \mathbf{s}).$$

So, the Turing degrees of the incomputable left-r.e. sets having only computable presentations are disjoint from a filter and from a ideal in the Turing degrees where this filter and this ideal intersect. Both the ideal and the filter are important: Nies [12] showed that the K-trivial sets are closed under Turing reduciblity and that they coincide with several other natural notions being around for many years [12]; Ambos-Spies, Jockusch, Shore and Soare [1] showed that the degrees of promptly simple set coincide with the non-cappable degrees and the low cuppable degrees, see [14, Chapter XIII] for a discussion.

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