# EFFICIENT APPROXIMATION ALGORITHMS FOR SPANNING CENTRALITY

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# SPANNING CENTRALITY

- Input:
  - an undirected and connected graph $G$
- Spanning centrality $s(e_{i,j}) \in (0,1]$ of an edge $e_{i,j}$:
  - the fraction of spanning trees of $G$ that contains $e_{i,j}$
- A higher SC $s(e_{i,j})$:
  - $e_{i,j}$ is more crucial for $G$ to ensure connectedness.

# SPANNING CENTRALITY

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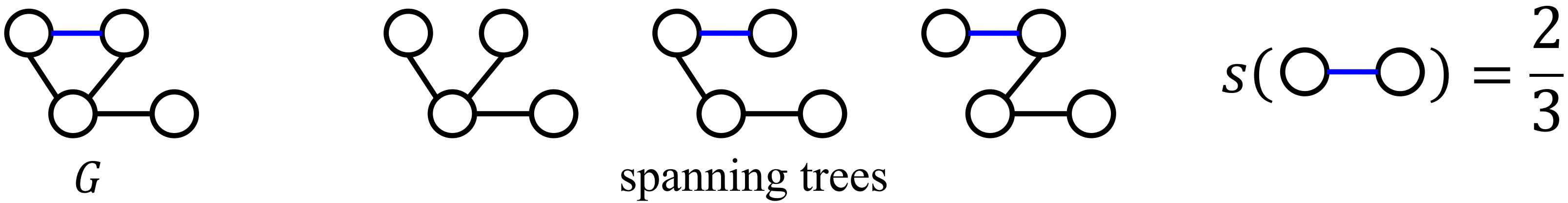
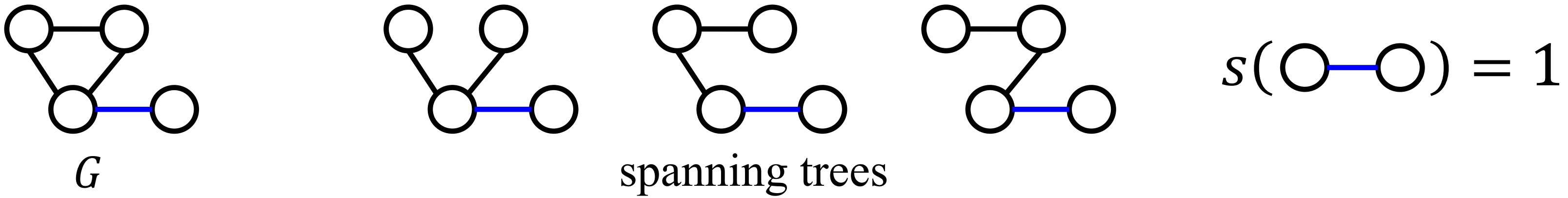
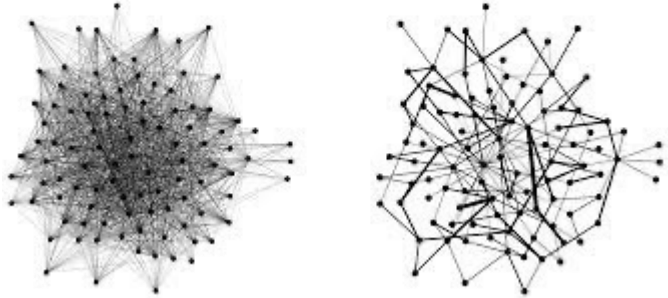
  - the fraction of spanning trees of $G$ that contains $e_{i,j}$

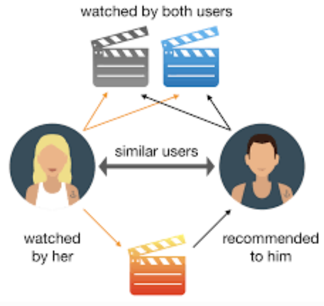


# SPANNING CENTRALITY

- Applications:
  - stability and robustness analysis
  - information propagation analysis
  - graph sparsification
  - collaborative recommendation
  - image segmentation
  - etc.






# PROBLEM DEFINITION

- All Edge Spanning Centrality (AESC)
  - input:
    - an undirected & connected graph $G$ with $n$ nodes and $m$ edges
  - output:
    - $s(e_{i,j})$ for every edge $e_{i,j}$ in $G$
  - time complexity:
    - $O\left(mn^{\frac{3}{2}}\right)$

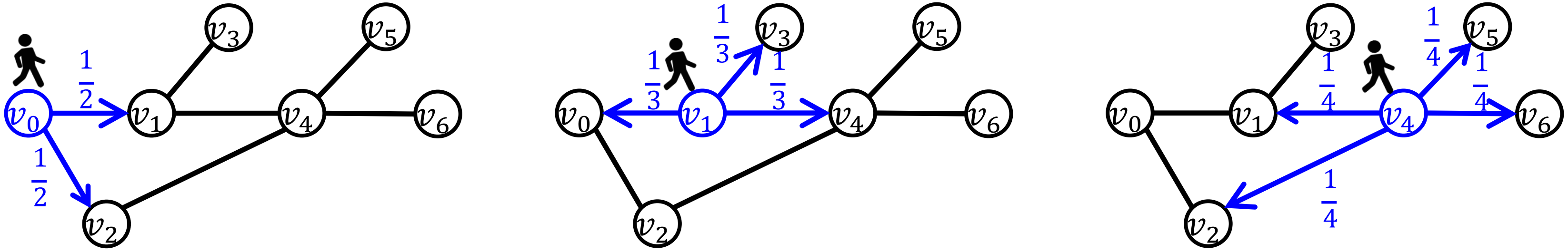
# PROBLEM DEFINITION

- $\epsilon$-approximate AESC
  - input:
    - an undirected & connected graph $G$
    - an absolute error $\epsilon$
  - output:
    - the estimated SC $\hat{s}(e_{i,j})$ for every edge $e_{i,j}$ in $G$ satisfying

$$|s(e_{i,j}) - \hat{s}(e_{i,j})| \leq \epsilon$$

# SOTA FOR $\epsilon$-APPROXIMATE AESC

- Simple random walk from node $v_0$:




$p_\ell(v_i, v_j) = \Pr[\text{A simple random walk from } v_i \text{ visits } v_j \text{ at the } \ell\text{-th step}]$

- SC in a view of simple random walk [Peng et al. KKD'21]:

$$s(e_{i,j}) = \sum_{\ell=0}^{+\infty} \frac{p_\ell(v_i, v_i)}{d(v_i)} + \frac{p_\ell(v_j, v_j)}{d(v_j)} - \frac{p_\ell(v_i, v_j)}{d(v_j)} - \frac{p_\ell(v_j, v_i)}{d(v_i)}$$

$d(v_i) \rightarrow \frac{1}{\#\text{neighbors of } v_i}$

# SOTA FOR $\epsilon$-APPROXIMATE AESC

- [Peng et al. KKD'21] uses the random walk interpretation

$$s(e_{i,j}) = \sum_{\ell=0}^{\tau} \frac{p_\ell(v_i,v_i)}{d(v_i)} + \frac{p_\ell(v_j,v_j)}{d(v_j)} - \frac{p_\ell(v_i,v_j)}{d(v_j)} - \frac{p_\ell(v_j,v_i)}{d(v_i)} + \sum_{\ell=\tau+1}^{+\infty} \frac{p_\ell(v_i,v_i)}{d(v_i)} + \frac{p_\ell(v_j,v_j)}{d(v_j)} - \frac{p_\ell(v_i,v_j)}{d(v_j)} - \frac{p_\ell(v_j,v_i)}{d(v_i)}$$

Estimate this by simple random walk sampling with at most an $\epsilon/2$ error

Derive a random walk length threshold $\tau$ s.t. the error of ignoring this part is at most $\epsilon/2$

- Expensive computational overhead
  - Large random walk length threshold $\tau$
  - Large number of random walks

# OUR TECHNICAL CONTRIBUTIONS

$$s(e_{i,j}) = \sum_{\ell=0}^{\tilde{\tau}} \frac{p_\ell(v_i, v_i)}{d(v_i)} + \frac{p_\ell(v_j, v_j)}{d(v_j)} - \frac{p_\ell(v_i, v_j)}{d(v_j)} - \frac{p_\ell(v_j, v_i)}{d(v_i)}$$

Compute the first $\tilde{\tau}$ steps by deterministic graph traversal

$$+ \sum_{\ell=\tilde{\tau}+1}^{\tau} \frac{p_\ell(v_i, v_i)}{d(v_i)} + \frac{p_\ell(v_j, v_j)}{d(v_j)} - \frac{p_\ell(v_i, v_j)}{d(v_j)} - \frac{p_\ell(v_j, v_i)}{d(v_i)}$$

Estimate the rest $\tau - \tilde{\tau}$ steps by random walk sampling

$$+ \sum_{\ell=\tau+1}^{+\infty} \frac{p_\ell(v_i, v_i)}{d(v_i)} + \frac{p_\ell(v_j, v_j)}{d(v_j)} - \frac{p_\ell(v_i, v_j)}{d(v_j)} - \frac{p_\ell(v_j, v_i)}{d(v_i)}$$

Tighten the random walk length threshold $\tau$

# OUR TECHNICAL CONTRIBUTIONS

- Tightened length threshold $\tau_{i,j}$ personalized to each $e_{i,j}$

$$s(e_{i,j}) = \sum_{\ell=0}^{+\infty} \frac{p_\ell(v_i, v_i)}{d(v_i)} + \frac{p_\ell(v_j, v_j)}{d(v_j)} - \frac{p_\ell(v_i, v_j)}{d(v_j)} - \frac{p_\ell(v_j, v_i)}{d(v_i)}$$

endpoints with larger degrees
↓
smaller SC values
↓
smaller $\tau$ to satisfy $\epsilon/2$
↓
utilize the degree information of two endpoints

can be decomposed by graph spectral property
↓
utilize the eigenvectors and eigenvalues of $G$

# OUR TECHNICAL CONTRIBUTIONS

$$s_\tau(e_{i,j}) = \sum_{\ell=0}^{\tilde{\tau}} \frac{p_\ell(v_i,v_i)}{d(v_i)} + \frac{p_\ell(v_j,v_j)}{d(v_j)} - \frac{p_\ell(v_i,v_j)}{d(v_j)} - \frac{p_\ell(v_j,v_i)}{d(v_i)} + \sum_{\ell=\tilde{\tau}+1}^{\tau} \frac{p_\ell(v_i,v_i)}{d(v_i)} + \frac{p_\ell(v_j,v_j)}{d(v_j)} - \frac{p_\ell(v_i,v_j)}{d(v_j)} - \frac{p_\ell(v_j,v_i)}{d(v_i)}$$

deterministic graph traversal

random walk sampling

Switch to sampling when the cost of the former exceeds the latter

# OUR TECHNICAL CONTRIBUTIONS

- Deterministic graph traversal

$$\sum_{\ell=0}^{\tilde{\tau}} \frac{p_\ell(v_i, v_i)}{d(v_i)} + \frac{p_\ell(v_j, v_j)}{d(v_j)} - \frac{p_\ell(v_i, v_j)}{d(v_j)} - \frac{p_\ell(v_j, v_i)}{d(v_i)}$$

# OUR TECHNICAL CONTRIBUTIONS

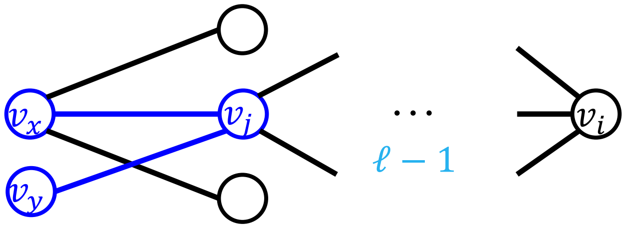
- Deterministic graph traversal

$$\sum_{\ell=0}^{\tilde{\tau}} \frac{p_\ell(v_i,v_i)}{d(v_i)} - \frac{p_\ell(v_j,v_i)}{d(v_i)} + \frac{p_\ell(v_j,v_j)}{d(v_j)} - \frac{p_\ell(v_i,v_j)}{d(v_j)}$$

rely on $p_\ell(v_*, v_i)$ of $v_i$,
where $v_*$ is $v_i$ and its neighbors

↓

deterministic graph traversal
in a reverse manner




$$p_\ell(v_x, v_i) \mathrel{+}= \frac{p_{\ell-1}(v_j,v_i)}{d(v_x)} = \frac{p_{\ell-1}(v_j,v_i)}{3}$$

$$p_\ell(v_y, v_i) \mathrel{+}= \frac{p_{\ell-1}(v_j,v_i)}{d(v_y)} = p_{\ell-1}(v_j,v_i)$$

# OUR TECHNICAL CONTRIBUTIONS

- Random walk sampling

$$\sum_{\ell=\tilde{\tau}+1}^{\tau} \frac{p_\ell(v_i,v_i)}{d(v_i)} + \frac{p_\ell(v_j,v_j)}{d(v_j)} - \frac{p_\ell(v_i,v_j)}{d(v_j)} - \frac{p_\ell(v_j,v_i)}{d(v_i)}$$

$$\downarrow$$

$$\sum_{v_x} \frac{p_{\tilde{\tau}}(v_x,v_i)}{d(v_i)} \left( \sum_{\ell=1}^{\tau-\tilde{\tau}} p_\ell(v_i,v_x) - p_\ell(v_j,v_x) \right)$$

all $p_{\tilde{\tau}}(v_*,v_i)$ are known by traversal

estimate by generating random walks from $v_i$ and $v_j$

# EXPERIMENTS

- Dataset statistics

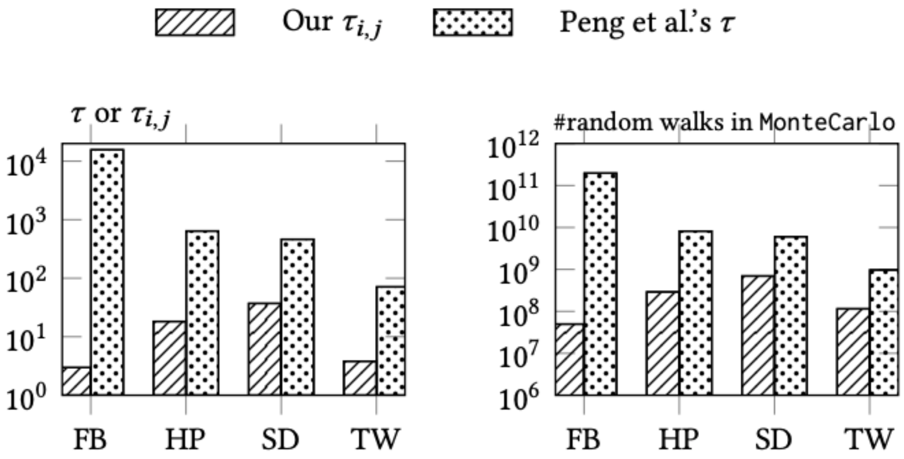
| Name | #nodes | #edges |
|---|---|---|
| Facebook [30] | 4,039 | 88,234 |
| HepPh [20] | 34,401 | 420,784 |
| Slashdot [22] | 77,360 | 469,180 |
| Twitch [35] | 168,114 | 6,797,557 |
| Orkut [50] | 3,072,441 | 117,185,082 |

# EXPERIMENTS

- $\epsilon$-approximate AESC solutions
  - spanning tree sampling
    - ST-Edge [IJCAI'16]
  - random walk sampling **with our τ**
    - MonteCarlo [KDD'21]
    - MonteCarlo-C [KDD'21]
  - our proposal
    - TGT: our $\tau$ + reverse graph traversal
    - TGT+: our $\tau$ + reverse graph traversal + random walk

# EXPERIMENTS

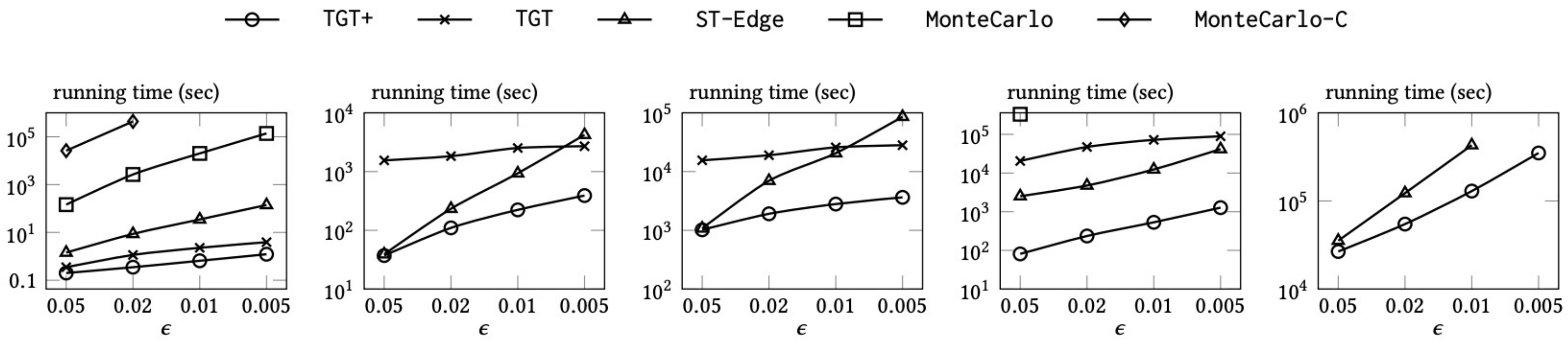
- Our $\tau$ vs. Peng et al.'s $\tau$




(a) truncated lengths (b) the number of walks

# EXPERIMENTS

- running time vs. absolute error $\epsilon$




(a) *Facebook* (b) *HepPh* (c) *Slashdot* (d) *Twitch* (e) *Orkut*

# SUMMARY

- Personalized random walk length
- TGT: deterministic graph traversal in a reverse manner
- TGT+: deterministic graph traversal + random walk sampling

# THANK YOU!

Q&A

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# BACKUP MATERIAL

- Tradeoff between running time and actual average absolute error