17: Dynamic Programming

CS1101S: Programming Methodology

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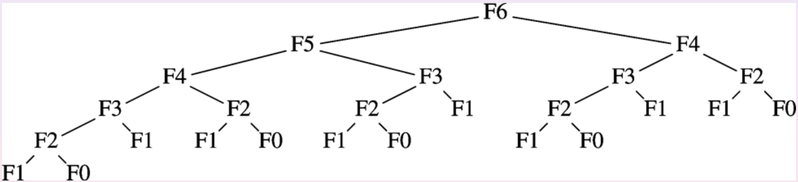
Inefficient Algorithm

```
function fib(n) {
  if (n <= 1) {
    return 1;
  } else {
    return fib(n - 1) + fib(n - 2);
} }
```

Trace of Recursion



Memoization

```
var fibs = [];
function fib(n) {
  if (fibs[n]!==undefined) {
     return fibs[n];
  } else if (n <= 1) {
     return 1;
  } else {
     var new_fib = fib(n - 1) + fib(n - 2);
     fibs[n] = new_fib;
     return new_fib;
} }
```

A Simple Loop for Fibonacci Numbers

```
function fib(n) {
  if (n <= 1) {
    return 1;
  } else {
    var last = 1, nextToLast = 1; answer = 1;
    var i = 2;
    while (i <= n) {
      answer = last + nextToLast;
      nextToLast = last;
      last = answer;
      i = i + 1;
    }
    return answer;
} }
```

Egg Dropping Puzzle

Given

n eggs, building with k floors

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Wanted

Smallest number of egg dropping experiments required to find out in all cases, which floors an egg can be safely dropped from

Assumptions

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- If an egg breaks when dropped, then it would break if dropped from a higher floor.
- If an egg survives a fall then it would survive a shorter fall.
- A first-floor drop may break eggs, and eggs may survive a drop from the highest floor.

Special Case: One Egg

Number of eggs $= 1$, number of floors $= 21$

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Number of eggs = 1, number of floors = 21

We need at most 21 experiments

Special Case: Two Eggs

Animated scenario

click here

Observations

Sub-tasks

At each point in time, we have a number of eggs n available and a number of floors k to check

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The height of the floors does not matter. At each point in time we need to check a certain number of contiguous floors, say from 10 to 14.

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Height does not matter

Checking 10 to 14 is the same as checking 20 to 24.

A simple algorithm

```
function eggDrop(n, k) {
    if (k =< 1 || n === 1) {
        return k;
    } else {
        var min = large_constant;
        var x = 1;
        var res = undefined;
        while (x <= k) {
            res = max(eggDrop(n-1, x-1),
                      eggDrop(n, k-x));
            if (res < min) min = res;
            x = x + 1;
        }
        return min + 1;
} }
```

Solution Idea

Observation

We compute eggDrop(i,j) over and over again.

Remember results in a table

Allocate a 2-D table eggFloor that remembers the results; after computing s = eggDrop(i,j), remember s in a table.

```
eggDrop[i][j] = s;
```

Optimal Binary Search Tree

Given

- a set of words $\{w_1, \ldots, w_n\}$
- probabilities of each word's occurrence $\{p_1, \ldots, p_n\}$

Wanted

Binary tree that includes all words and has the lowest expected cost:

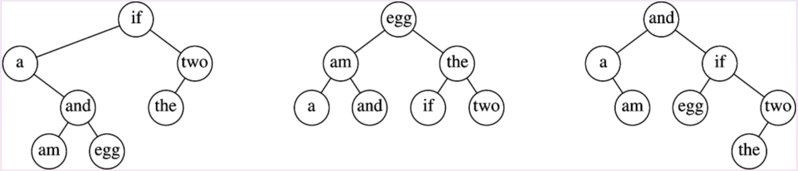
$$\mathit{expected\ cost} = \sum_{i=1}^{n} d_i p_i$$

where d_i is the depth of word i in the tree

Sample Input

Word	Probability
a	0.22
am	0.18
and	0.20
egg	0.05
if	0.25
the	0.02
two	0.08

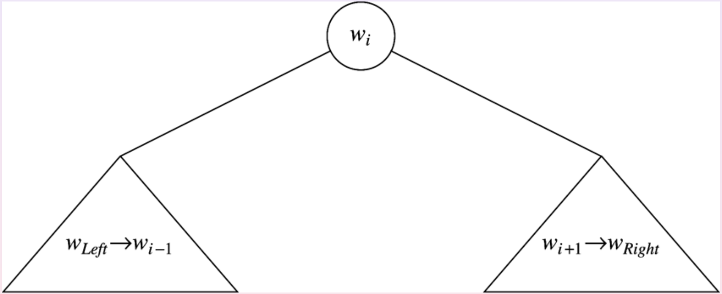
Three Possible Binary Search Trees



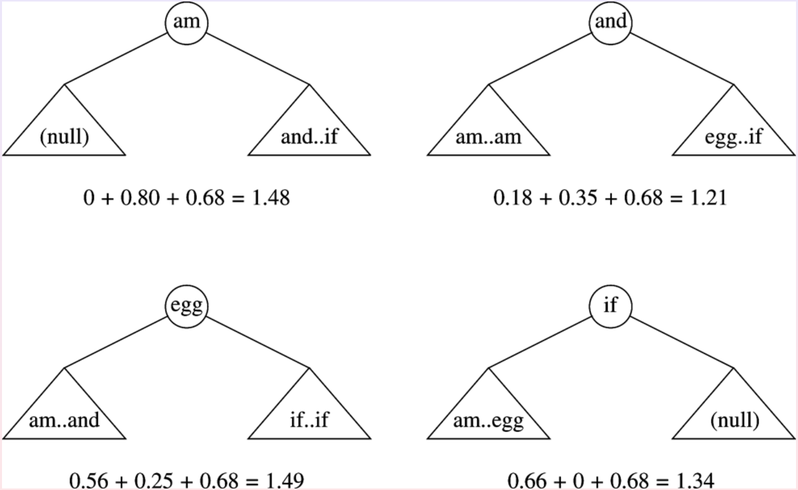
Comparison of the Three Trees

Input		Tree #1		Tree #2		Tree #3	
Word	Probability	Access Cost		Access Cost		Access Cost	
w_i	p_i	Once	Sequence	Once	Sequence	Once	Sequence
a	0.22	2	0.44	3	0.66	2	0.44
am	0.18	4	0.72	2	0.36	3	0.54
and	0.20	3	0.60	3	0.60	1	0.20
egg	0.05	4	0.20	1	0.05	3	0.15
if	0.25	1	0.25	3	0.75	2	0.50
the	0.02	3	0.06	2	0.04	4	0.08
two	0.08	2	0.16	3	0.24	3	0.24
Totals	1.00		2.43		2.70		2.15

Structure of Optimal Binary Search Tree



Example


am
(null)
and..if
0 + 0.80 + 0.68 = 1.48
and
am..am
egg..if
0.18 + 0.35 + 0.68 = 1.21
egg
am..and
if..if
0.56 + 0.25 + 0.68 = 1.49
if
am..egg
(null)
0.66 + 0 + 0.68 = 1.34

Idea

Proceed in order of growing tree size

For each range of words, compute optimal tree

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Memoization

For each range, store optimal tree for later retrieval

Computation of Optimal Binary Search Tree

	Left=1	Left=2	Left=3	Left=4	Left=5	Left=6	Left=7
Iteration=1	a..a .22 a	am..am .18 am	and..and .20 and	egg..egg .05 egg	if..if .25 if	the..the .02 the	two..two .08 two
Iteration=2	a..am .58 a	am..and .56 and	and..egg .30 and	egg..if .35 if	if..the .29 if	the..two .12 two	
Iteration=3	a..and 1.02 am	am..egg .66 and	and..if .80 if	egg..the .39 if	if..two .47 if		
Iteration=4	a..egg 1.17 am	am..if 1.21 and	and..the .84 if	egg..two .57 if			
Iteration=5	a..if 1.83 and	am..the 1.27 and	and..two 1.02 if				
Iteration=6	a..the 1.89 and	am..two 1.53 and					
Iteration=7	a..two 2.15 and						

Run Time

For each cell of table

Consider all possible roots

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Overall runtime

$$O(N^3)$$