Tutorial 2
Logic of Quantified Statements

1 Discussion questions

Discussion questions are meant for discussion on the IVLE Forum. You may try them on your own or discuss them with your classmates. No answers will be provided by us.

D1. Using the following English sentence, explain why English can be ambiguous.

"Every boy loves a girl."

Interpret the above sentence in two ways and write the logical statement for each of them.

D2. Let the domain of discourse D be the set of all students at your school, and let $M(s)$ be "s is a math major", let $C(s)$ be "s is a computer science student", and let $E(s)$ be "s is an engineering student". Express each of the following statements using quantifiers, variables, and the predicates $M(s)$, $C(s)$, and $E(s)$.

Part (a) has been done for you.

a. Every computer science student is an engineering student.
   Answer: $\forall s \in D\ (C(s) \rightarrow E(s))$.

   Discuss: Why is the following answer wrong? Can you also give an example to show the difference between this incorrect answer and the above answer?
   Incorrect answer: $\forall s \in D\ (C(s) \wedge E(s))$.

b. No computer science students are engineering students.

c. Some computer science students are not math majors.

d. If a student is not a math major, then he or she is either a computer science student or an engineering student, but not both.

D3. The following table shows when the quantified statements are true and when they are false.

Statement	True when...	False when...
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Complete the table below for mixed quantifiers.

Statement	True when...	False when...
$\forall x \forall y P(x, y)$		
$\forall x \exists y P(x, y)$		
$\exists x \forall y P(x, y)$		
$\exists x \exists y P(x, y)$		

2 Additional Notes

We picked up some frequently asked questions from students in the past semesters and created this *Additional Notes* section to include some materials not covered in lecture that might be of interest to you.

Equivalent expressions: The following quantified statements are equivalent.

- $\forall x \in D, P(x) \equiv \forall x\, (x \in D \rightarrow P(x))$
- $\exists x \in D, P(x) \equiv \exists x\, (x \in D \wedge P(x))$

Scope of quantifiers

- When a quantifier is used on a variable x in a predicate statement, we say that x is *bound*. If no quantifier is used on a variable, we say that the variable is *free*.

  Examples: In the statement $\forall x \exists y\, P(x,y)$, both x and y are bound. In the statement $\forall x\, P(x,y)$, x is bound but y is free.
- A statement is called a *well-formed formula* (or *wff*) when all variables are properly quantified.
- The set of all variables bound by a common quantifier is the *scope* of that quantifier.

  If there are no parentheses, then the scope is the smallest *wff* following the quantification.

  Example: For $\forall x(\exists y\, P(x,y) \vee Q(x,y))$, x and y in $P(x,y)$ are bound, while y in $Q(x,y)$ is free, because the scope of $\exists y$ is $P(x,y)$. The scope of $\forall x$ is $(\exists y\, P(x,y) \vee Q(x,y))$.

3 Tutorial questions

Q1. For each of the following statements write the converse, inverse, and contrapositive. Indicate which among the statement, its converse, its inverse, and its contrapositive are true and which are false. Give a counterexample for each that is false.

a. $\forall n \in \mathbb{Z}^+$, if n is prime then n is odd or $n = 2$.

b. $\forall n \in \mathbb{Z}$, if $(6 \mid n)$, then $(2 \mid n)$ and $(3 \mid n)$.

c. $\forall a, b, c \in \mathbb{Z}$, if $a - b$ is even and $a - c$ is even, then $b - c$ is even.

Q2. Prove the statement in Question 1c:

$\forall\ a, b, c \in \mathbb{Z}$, if $a - b$ is even and $a - c$ is even, then $b - c$ is even.

Q3. Let V be the set of all visitors to Universal Studios Singapore on a certain day, $T(v)$ be "v took the Transformers ride", $G(v)$ be "v took the Battlestar Galactica ride", $E(v)$ be "v visited the Ancient Egypt", and $W(v)$ be "v watched the Water World show".

Express each of the following statements using quantifiers, variables, and the predicates $T(v)$, $G(v)$, $E(v)$ and $W(v)$. The statements are not related to one another.

a. Every visitor watched the Water World show.

b. Every visitor who took the Battlestar Galactica ride also took the Transformers ride.

c. There is a visitor who took both the Transformers ride and the Battlestar Galactica ride.

d. No visitor who visited the Ancient Egypt watched the Water World show.

e. Some visitors who took the Transformers ride also visited the Ancient Egypt but some did not (visit the Ancient Egypt).

Q4. Refer to Figure 3.3.2 (in page 120 of Susanna Epp's book and also on slide 52 of lecture slides "The Logic of Quantified Statements"). Determine whether each of the following statements is true or false.

a. $\forall$ students S, $\exists$ a dessert D such that S chose D.

b. $\forall$ students S, $\exists$ a salad T such that S chose T.

c. $\exists$ a beverage B such that $\forall$ students S, S chose B.

d. $\exists$ an item I such that $\forall$ S, S did not choose I.

e. $\exists$ a station Z such that $\forall$ students S, $\exists$ an item I such that S chose I from Z.

Q5. Some of the arguments below are valid by *universal modus ponens* or *universal modus tollens*; others are invalid and exhibit the *converse error* or the *inverse error*. State which are valid and which are invalid. Justify your answers.

a. If a person likes coffee, he likes toffee.
Jack does not like toffee.
$\therefore$ Jack does not like coffee.

b. All positive odd integers greater than 1 are primes.
4 is not a positive odd integer greater than 1.
$\therefore$ 4 is not a prime.

c. Anyone above 18 and passed the driving test gets a driving licence.
Sharon gets a driving licence.
$\therefore$ Sharon is above 18 and passed the driving test.

Q6. Indicate which of the following statements are true and which are false. Justify your answers.

a. $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ such that $a + b = 0$. (AY2015/6 Sem2 Midterm test)

b. $\exists a \in \mathbb{Z}$ such that $\forall b \in \mathbb{Z}$, $a + b = 0$. (AY2015/6 Sem2 Midterm test)

c. $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+$ such that their arithmetic mean is equal to their geometric mean.

Q7. Given the following argument:

1. If an object is above all the triangles, then it is above all the blue objects.
2. If an object is not above all the gray objects, then it is not a square.
3. Every black object is a square.
4. Every object that is above all the gray objects is above all the triangles.

$\therefore$ If an object is black, then it is above all the blue objects.

a. Reorder the premises in the argument to show that the conclusion follows as a valid consequence from the premises. It may be helpful to rewrite the statements in if-then form and replace some (if any) statements by their contrapositives.

b. Rewrite your answer in part (a) using predicates and quantified statements.

Q8. (AY2015/6 Sem2 Midterm test)
You are given the following English statements:

1. All swimmers are able to swim across the river.
2. No archers are short-sighted.
3. Patrick wears glasses.
4. Everybody is either an archer or a swimmer.

a. Rewrite each of the above sentences into formal statements, using quantifiers wherever appropriate, and well-named predicates.

You may assume that the domain of discourse is the set of people, which may be omitted in your statements. You may use the logically equivalent form in your statements. You may use 'Patrick' as an instance instead of creating a predicate called 'Patrick'.

b. There is a missing statement above. Adding that missing statement would allow you to answer this question "Is Patrick able to swim across the river?" Write down the missing statement (as a formal quantified statement) and the conclusion (in English) about Patrick.

c. Show your proof to derive the conclusion about Patrick in part (b) above.