CS1231S Assignment #2

AY2026/27 Semester 2

Deadline: Monday, 2 November 2026, 1:00pm

IMPORTANT: Please read the instructions below carefully.

This is a graded assignment worth 7% of your final grade. There are **four questions** with a total score of 21 marks. Please work on it by yourself, not in a group or discussion/in collaboration with anybody. Anyone found committing plagiarism (submitting other's work as your own, submitting an AI answer, or sending your answers to others, or other forms of academic dishonesty, will be penalised with a straight zero for the assignment, and reported to the school. Please see SoC website "Preventing Plagiarism" https://www.comp.nus.edu.sg/cug/plagiarism/.

You must submit your assignment to **Canvas > Assignments > Assignment 2 submission** before the deadline.

Your answers may be typed or handwritten. Make sure that it is legible (for example, don't use very light pencil or ink if it is handwritten, or font size smaller than 11 if it is typed) or marks may be deducted.

You are to submit a SINGLE pdf file, where each page is A4 size. Do not submit files in other formats. If you submit multiple times, only the last submitted file will be graded.

Late submission will NOT be accepted. We have set the closing time of submission to slightly after 1pm to give you a few minutes of grace, but in your mind, you should treat **1pm** as the deadline. If you think you might forget, please submit earlier. Also, avoid submitting in the last few minutes, as the system may get sluggish due to overload and you will miss the deadline, and we will not extend the deadline for you.

Note the following:

- An Answer Sheet is provided on Canvas. Download it and write/type your answers into it and submit it as a pdf file. Name your pdf file with your Student Number (which begins with 'A' and not with 'e' which is your NUSNET-id).
- Note that if you submit multiple times, Canvas will append some number to your filename; that's fine.
- You are not to write beyond the boxes provided in the Answer Sheet or add additional pages.
- As this is an assignment given well ahead of time, we expect you to work on it early. You should submit polished work, not answers that are untidy or appear to have been done in a hurry, for example, with scribbling and cancellation all over the places. Marks may be deducted for untidy work.
- It is always good to be clear and leave no gap so that the marker does not need to make guesswork on your answer. When in doubt, the marker would usually not award the mark.
- Do not use any methods that have not been covered in class. When using a theorem or result that has appeared in class (lectures or tutorials), please quote the theorem number/name, the lecture and slide number, or the tutorial number and question number in that tutorial, failing which marks may be deducted. Remember to use numbering and give justification for important steps in your proof, or marks may be deducted.

To combine all pages into a single pdf document for submission, you may find the following scanning apps helpful if you intend to scan your handwritten answers:

for Android: https://fossbytes.com/best-android-scanner-apps/

for iphone: https://www.switchingtomac.com/tutorials/ios-tutorials/the-best-ios-scanner-apps-to-scan-documents-images/

If you need any clarification about this assignment, please do NOT email us or post on telegram, but post on QnA "Assignments" topic so that everybody can read the answers to the queries.

Question 1. (4 marks)

Define a function $f\colon \mathbb{Z} \to \mathbb{Z}$ by setting, for each $x \in \mathbb{Z}$,

$$f(x) = \begin{cases} \dfrac{x}{2}, & \text{if } x \text{ is even;} \\ 3x + 1, & \text{if } x \text{ is odd.} \end{cases}$$

For (a) and (b), answer "yes" with a proof, or "no" with a counterexample. Mark will not be awarded with a mere "yes" or "no" answer without a proof or counterexample.

(a) Is f injective? [1 mark]

(b) Is f surjective? [1 mark]

(c) An integer x is called *a fixed point* of f if $f(x) = x$. Find all fixed points of f. You do not need to show your working. [1 mark]

(d) An integer x is called a *periodic point of period 2* if $f(f(x)) = x$, $f(x) \neq x$. Find all integers that are periodic points of period 2. You do not need to show your working. [1 mark]

Answers:

(a) No.

1. Note $f(1) = 3 \times 1 + 1 = 4 = \frac{8}{2} = f(8)$.
2. As $1 \neq 8$, we conclude that f is not injective.

(b) Yes.

1. Let $y \in \mathbb{Z}$.
2. Then $2y$ is an even integer.
3. So $f(2y) = \frac{2y}{2} = y$.

(c) The only fixed point is $x = 0$ as $f(0) = 0$.

Proof that 0 is the only fixed point (not required from students; for reference).

1. Case 1: x is even
   1.1. Since $f(x) = x$, we have $\frac{x}{2} = x$, hence $x = 0$.
2. Case 2: x is odd
   2.1. Since $f(x) = x$, we have $3x + 1 = x$, hence $x = -\frac{1}{2}$. However, $-\frac{1}{2} \notin \mathbb{Z}$.
   2.2. Hence there is no odd integer that is a fixed point of f.

(d) The integers that are periodic points of period 2 are -1 and -2.
$f(f(-1)) = f(-2) = -1$ and $f(f(-2)) = f(-1) = -2$. Thus $-1 \to -2 \to -1$.

Proof that -1 and -2 are the only 2 periodic points of period 2 (not required from students; for reference)

Suppose $x \in \mathbb{Z}$ satisifes $f(f(x)) = x$.

1. Case 1: x is even, then $f(x) = \frac{x}{2}$
   - 1.1. Subcase $\frac{x}{2}$ is even
     - 1.1.1. Then $f\big(f(x)\big) = \frac{x}{4}$
     - 1.1.2. Since $f\big(f(x)\big) = x$, $\frac{x}{4} = x \Rightarrow x = 0$.
     - 1.1.3. However, $f(0) = 0$, so $x = 0$ is not a periodic point of period 2.
   - 1.2. Subcase $\frac{x}{2}$ is odd
     - 1.2.1. Then $f\big(f(x)\big) = 3\left\langle\frac{x}{2}\right\rangle + 1$.
     - 1.2.2. Since $f\big(f(x)\big) = x$, $\frac{3x}{2} + 1 = x \Rightarrow x = -2$.
     - 1.2.3. This satisfies the required conditions, and we have already verified that -2 is a periodic point of period 2.
2. Case 2: x is odd
   - 2.1. Since x is odd, $f(x) = 3x + 1 = x$.
   - 2.2. Observe that $3x + 1$ is even whenever x is odd.
   - 2.3. Therefore, $f\big(f(x)\big) = \frac{3x+1}{2}$.
   - 2.4 Since $f\big(f(x)\big) = x$, we have $\frac{3x+1}{2} = x \Rightarrow x = -1$.
   - 2.5. This satisfies the required conditions, and we have already verified that -1 is a periodic point of period 2.

Question 2. (5 marks)

Let $A = \{a, b\}$ and S be the set of all strings over A. Define the function $f : S \to S$ as follows:

> $\forall s \in S$, $f(s)$ is a new string that replaces the leftmost occurrence of a in s with b.
> If a does not occur in s, then $f(s) = s$.

Another function $g : S \to \mathbb{N}$ is defined as follows:

> $\forall s \in S$, $g(s) = max(0,$ (the number of a's in s) $-$ (the number of b's in s))

where $max(x, y)$ is the maximum of x and y.

Examples: $f(\varepsilon) = \varepsilon; f(a) = b; f(bbb) = bbb; f(bbaab) = bbbab$, $g(\varepsilon) = 0; g(abaa) = 2; g(bb) = 0; g(babaa) = 1; g(abbab) = 0$.

For (a), (c) and (d), answer "yes" with a proof, or "no" with a counterexample. Mark will not be awarded with a mere "yes" or "no" answer without a proof or counterexample.

(a) Is f injective? [1 mark]

(b) What is $(f \circ f)(abaa)$? Show your working. [1 mark]

(c) Is $f \circ f$ injective? [1 mark]

(d) Is g surjective? [2 marks]

Answers:

(a) **No.** $f(a) = b = f(b)$ but $a \neq b$.

(b) $(f \circ f)(abaa) = f\big(f(abaa)\big) = f(bbaa) = \boldsymbol{bbba}$.

(c) **No.** $(f \circ f)(aab) = f\big(f(aab)\big) = f(bab) = bbb = (f \circ f)(bbb)$ but $aab \neq bbb$.

(d) **Yes.**

Proof (by construction) (other constructions possible)

1. Take any $n \in \mathbb{N}$.
2. Let $s = aa \cdots a$ (exactly n a's). (Note: n is possibly 0. If so, $s = \varepsilon$.)
3. Hence, $s \in S$ since its members are all a's.
4. Now, $g(s) = n - 0 = n$ by the definition of g.
5. Therefore, g is surjective.

The trick is to construct a suitable $s \in S$ that works. Other constructions are possible, eg. a string containing $(n + k)$ a's followed by k b's.

Question 3. (6 marks)

Recall Theorem 4.4.1 (The Quotient-Remainder Theorem) from Lecture 5 and Tutorial 8:

> "For any integer n, and any positive integer d, there exist unique integers q and r such that $n = dq + r$, where $0 \leq r < d$."

(a) Use the First Principle of Mathematical Induction (Weak Mathematical Induction) to prove the existence of q and r for positive n. Part of the proof is given in the Answer Sheet.

(b) Using the result of (a), prove the statement for $n \in \mathbb{Z}$. Part of the proof is given in the Answer Sheet.

You may quote closure of integers under subtraction. There is no need to provide justification for basic algebra from now on.

The proof for uniqueness is given below:

1. Suppose there are two pairs of integers (q_1, r_1) and (q_2, r_2) satisfying $n = dq_1 + r_1 = dq_2 + r_2$ where $0 \leq r_1, r_2 < d$.
2. Rearranging, we have $d(q_1 - q_2) = r_2 - r_1$
3. Since $0 \leq r_1, r_2 < d$, we have $-d < r_2 - r_1 < d$
4. Hence, $-d < d(q_1 - q_2) < d$ (by substitution 2 into 3)
5. Dividing by d, we have $-1 < (q_1 - q_2) < 1$ (note that $d > 0$)
6. Since $q_1 - q_2$ is an integer, it follows that $q_1 - q_2 = 0$ or $q_1 = q_2$
7. Substituting into the original equality $dq_1 + r_1 = dq_2 + r_2$, we have $r_1 = r_2$.
8. Therefore, the quotient and remainder are unique.

Answer:

Proof of existence for positive n:

1. Let $P(n) \equiv (\forall d \in \mathbb{Z}^+ \; \exists q, r \in \mathbb{Z}$ such that $n = dq + r$, where $0 \leq r < d)$, for any $n \in \mathbb{Z}^+$.
2. Basis step: $n = 1$
   - 2.1. Case 1: If $d = 1$, then $n = 1 = dq + r = 1(1) + 0$.
   - 2.2. Case 2: If $d > 1$, then $n = 1 = dq + r = d(0) + 1$.
   - 2.3. In both cases, $P(1)$ is true.
3. Inductive step: Assume $P(k)$ is true for any $k \in \mathbb{Z}^+$, i.e. $\forall d \in \mathbb{Z}^+ \; \exists q, r \in \mathbb{Z}$ such that $k = dq + r$, where $0 \leq r < d$.
   - 3.1. Then $k + 1 = (dq + r) + 1$ (by the inductive hypothesis).
   - 3.2. Case 1: $r < d - 1$
     - 3.2.1. Let $r' = r + 1$.
     - 3.2.2. Hence $k + 1 = (dq + r) + 1 = dq + (r + 1) = dq + r'$.
     - 3.2.3. Note that $r' = r + 1 < d$. ← this is important, to ensure that $k + 1$ can be expressed in the desired form
     - 3.2.4. Hence $P(k + 1)$ is true.
   - 3.3. Case 2: $r = d - 1$
     - 3.3.1. Then $k + 1 = (dq + d - 1) + 1 = dq + d = d(q + 1) + 0$.
     - 3.3.2. Note that $q + 1$ is an integer since q is an integer (closure of integers under +).
     - 3.3.3. Hence $P(k + 1)$ is true.
   - 3.4. In both cases, $P(k + 1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+ \; P(n)$ is true by mathematical induction.

Proof of existence for integer n:

1. $0 = d(0) + 0$. So $P(0)$ is true. It suffices to show that $P(n)$ is true for all negative integers n.
2. Suppose $m \in \mathbb{Z}^-$, then $m = -n$ for some $n \in \mathbb{Z}^+$ such that $n = dq + r$, where $q, r \in \mathbb{Z}$ and $0 \leq r < d$ (from part (a)).
   2.1. Now, $m = -n = -(dq + r) = d(-q) - r$ by basic algebra.
   2.2. Case 1: $r = 0$
      2.2.1. Then $m = d(-q) + 0$ where $-q \in \mathbb{Z}$ (by closure of integers under $\times$) and $0 \leq 0 < d$.
   2.3. Case 2: $0 < r < d$
      2.3.1. Then $m = d(-q) - r = d(-q - 1) + (d - r)$ where $-q - 1 \in \mathbb{Z}$ (by closure of integers under $-$) and $0 < (d - r) < d$.
3. Therefore, in all cases $P(n)$ is true for $n \in \mathbb{Z}^-$.
4. With part (a), $P(n)$ is true for $n \in \mathbb{Z}$.

Question 4. (6 marks)



The ADB – Aiken Dueet Balloon – shop sells balloons with five colours. Balloons of the same colour are indistinguishable.

The following questions are independent of one another. Show your workings.

(a) If the ADB shop has at least 18 balloons of each colour, how many combinations of 18 balloons can you have? [1 mark]

(b) If the ADB shop has only 5 red balloons but at least 18 of each of the other colours, how many combinations of 18 balloons can you have? [2 marks]

(c) If the ADB shop has only 5 red balloons and 5 green balloons but at least 18 of each of the other colours, how many combinations of 18 balloons can you have? [3 marks]

Answers:

The number of multisets of size r that can be selected from a set of n elements is $\binom{r+n-1}{r}$.

(a) Here, $r = 18, n = 5$, so $\binom{r+n-1}{r} = \binom{22}{18} = \mathbf{7315}$. [1 mark]

(b) Let $R_{\geq 6}$ denote the set of selections containing at least 6 red balloons, $R_{\leq 5}$ the set of selections containing at most 5 red balloons, and T the set of selections with no restriction.

$T = R_{\leq 5} \cup R_{\geq 6}$ and $R_{\leq 5} \cap R_{\geq 6} = \emptyset$.

$|R_{\geq 6}| = \binom{12+5-1}{12} = \binom{16}{12} = 1820.$

Therefore $|R_{\leq 5}| = |\mathrm{T}| - |R_{\geq 6}| = 7315 - 1820 = \mathbf{5495}$. [2 marks]

(c) Let $G_{\geq 6}$ denote the set of selections containing at least 6 green balloons, and $G_{\leq 5}$ the set of selections containing at most 5 green balloons.

$|G_{\geq 6}| = \binom{12+5-1}{12} = \binom{16}{12} = 1820.$

$|R_{\geq 6} \cap G_{\geq 6}| = \binom{6+5-1}{6} = \binom{10}{6} = 210.$

$|R_{\geq 6} \cup G_{\geq 6}| = |R_{\geq 6}| + |G_{\geq 6}| - |R_{\geq 6} \cap G_{\geq 6}| = 1820 + 1820 - 210 = 3430.$

$|R_{\leq 5} \cap G_{\leq 5}| = |T| - |\overline{R_{\leq 5} \cap G_{\leq 5}}| = |T| - |R_{\geq 6} \cup G_{\geq 6}| = 7315 - 3430 = \mathbf{3885}$. [3 marks]

=== End of paper ===