NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 1: AY2024/25)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY SIX (26)** questions in **TWO (2)** parts and comprises **THIRTEEN (13)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEETS** provided.
5. Do **not** write your name on the ANSWER SHEETS.
6. The maximum mark of this assessment is 100.

Question	Max. mark
Part A: Q1 – 23	46
Part B: Q24	20
Part B: Q25	20
Part B: Q26	14
Total	**100**

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions [Total: 23×2 = 46 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer. Shade the corresponding bubble on the Answer Sheets.

1. What is the course CS1231S primarily about?

   A. Developing mathematical maturity.

   B. Understanding why mothers should learn logic to communicate with their children.

   C. About counting 1,2,3 and back to square 1 again.

   D. Studying about the relationship between Aiken and Dueet.

   E. This is a trick question, there is no such course.

2. Given the following argument:

   p
   $p \rightarrow q$
   $p \rightarrow {\sim}q$
   $\therefore r$

   Which of the following statements is true?

   A. The argument is a contradiction.

   B. The argument is invalid.

   C. The argument is valid and sound.

   D. The argument is valid but unsound.

   E. None of options (A), (B), (C), (D) are correct.

3. Given the following arguments:

   (i) $p \rightarrow q$
   $q \rightarrow p$
   $\therefore p \vee q$

   (ii) $p \rightarrow r$
   $q \rightarrow r$
   $\therefore p \vee q \rightarrow r$

   (iii) $p \rightarrow r$
   $q \rightarrow r$
   $\therefore p \wedge q \rightarrow r$

   Which of the above arguments are valid?

   A. Only (iii).

   B. Only (i) and (ii).

   C. Only (ii) and (iii).

   D. All of (i), (ii) and (iii).

   E. None of options (A), (B), (C), (D) are correct.

4. The following two statements are assumed to be true even though they divert from known facts.
   - All crows are horses.
   - Some hens are crows.

   Assuming that the sets of crows, horses and hens are non-empty, which of the following statements are true?

   (i) Some horses are hens.

   (ii) Some hens are horses.

   A. Only (i).
   B. Only (ii).
   C. Both (i) and (ii).
   D. Neither (i) nor (ii).

5. Given the following predicates on the domain $\mathbb{Z}$:

$$P(x) \equiv x \geq 1$$
$$Q(y) \equiv y = 0$$
$$R(x, y) \equiv x + y > 0$$

   Which of the following statements are true?

   (i) $\forall x, y \in \mathbb{Z}\ \Big(P(x) \rightarrow \big(Q(y) \rightarrow R(x, y)\big)\Big)$

   (ii) $\forall x, y \in \mathbb{Z}\ \big(Q(y) \wedge R(x, y) \rightarrow P(x)\big)$

   A. Only (i).
   B. Only (ii).
   C. Both (i) and (ii).
   D. Neither (i) nor (ii).

6. Let $Bool = \{true, false\}$. Define f: $\mathrm{Bool}^2 \rightarrow \mathrm{Bool}^2$ as follows:

$$f\big((p, q)\big) = (p, p \leftrightarrow q)$$

   Which of the following statements is true?

   A. f is not well-defined.
   B. f is a bijection.
   C. f is injective but not surjective.
   D. f is surjective but not injective
   E. None of options (A), (B), (C), (D) are correct.

7. Let $A = \{-2, -1, 0, 1, 2\}, B = \{0, 1, 4\}$ and $C = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. Which of the following statements are true?

   (i) $\forall x \in C \left((x \in A) \leftrightarrow (x^2 \in B)\right)$.

   (ii) $\forall x \in C \left((\forall y \in B\ xy \in B) \rightarrow (x^2 = x)\right)$.

   (iii) $\exists x \in A\ \forall y \in A \left((x \neq 0) \wedge (xy \in B)\right)$.

   A. Only (i) and (ii).

   B. Only (i) and (iii).

   C. Only (ii) and (iii).

   D. All of (i), (ii) and (iii).

   E. None of options (A), (B), (C), (D) are correct.

8. Which of the following statements are true about relations on a non-empty set?

   (i) The transitive closure of a reflexive relation is always symmetric.

   (ii) The reflexive closure of a transitive relation is always symmetric.

   (iii) It is not possible to have a symmetric closure of an anti-symmetric relation.

   A. Only (iii).

   B. Only (i) and (ii).

   C. Only (ii) and (iii).

   D. None of (i), (ii) and (iii).

   E. None of options (A), (B), (C), (D) are correct.

9. Let A, B, C be finite sets. If $R \subseteq A \times B$ and $S \subseteq B \times C$, then which of the following are true?

   (i) $R \circ S \subseteq A \times C$

   (ii) $R = \emptyset \Rightarrow S \circ R = \emptyset$

   (iii) $R \neq \emptyset \wedge S \neq \emptyset \Rightarrow S \circ R \neq \emptyset$

   A. Only (i).

   B. Only (ii).

   C. Only (ii) and (iii).

   D. All of (i), (ii) and (iii).

   E. None of options (A), (B), (C), (D) are correct.

10. Consider the congruence-mod 9 relation on $\mathbb{Z}$. Which of the following are related to 11?

(i) 110 (ii) -7 (iii) 29 (iv) 56

A. Only (i) and (iii).
B. Only (ii) and (iv).
C. Only (i), (iii) and (iv).
D. All of (i), (ii), (iii) and (iv).
E. None of options (A), (B), (C), (D) are correct.

11. Given the following three relations R_1, R_2 and R_3:

Relation R_1 defined on $\mathbb{R}$ by: $xR_1y \Leftrightarrow (x+y)^2 = x^2 + y^2$.
Relation R_2 defined on $\mathbb{R}$ by: $xR_2y \Leftrightarrow |x| + |y| = |x+y|$.
Relation R_3 defined on $\mathbb{R}^2$ by: $(a,b)R_3(c,d) \Leftrightarrow (a=c) \vee (b=d)$.

Note that $|x|$ is the absolute value of x, that is,

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Which of the following statements are true?

(i) None of the three relations are transitive.
(ii) Exactly one of the three relations is reflexive.
(iii) All three relations are symmetric.

A. Only (iii).
B. Only (i) and (iii).
C. Only (ii) and (iii).
D. All of (i), (ii) and (iii).
E. None of options (A), (B), (C), (D) are correct.

12. Define a relation $\sim$ on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}$,

$$x \sim y \Leftrightarrow 3 \mid (x + 2y)$$

Define addition and multiplication on $\mathbb{Z}/\sim$ as follows: whenever $[x], [y] \in \mathbb{Z}/\sim$,

$$[x] + [y] = [x+y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

Which of the following statements is true?

A. Both $+$ and $\cdot$ are well defined on $\mathbb{Z}/\sim$.
B. $+$ is well defined on $\mathbb{Z}/\sim$ but $\cdot$ is not well defined on $\mathbb{Z}/\sim$.
C. $\cdot$ is well defined on $\mathbb{Z}/\sim$ but $+$ is not well defined on $\mathbb{Z}/\sim$.
D. Neither $+$ nor $\cdot$ is well defined on $\mathbb{Z}/\sim$.

13. Which of the following statements are true?

    (i) Any function $f : \mathbb{Z} \to \mathbb{Z}$ is surjective.

    (ii) There exists a function $g : \mathbb{Z} \to \mathbb{R}$ where g is injective.

    (iii) There exists a function $h : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ where h is surjective.

    A. Only (i).
    B. Only (ii).
    C. Only (iii).
    D. None of (i), (ii) and (iii).
    E. None of options (A), (B), (C), (D) are correct.

14. Let R, S, T be relations on a non-empty set A and $T = R \cup S$. Given that R and S are partial orders, which of the following statements are true?

    (i) T is reflexive.

    (ii) T is anti-symmetric.

    (iii) T is transitive.

    A. Only (i).
    B. Only (i) and (ii).
    C. Only (ii) and (iii).
    D. All of (i), (ii) and (iii).
    E. None of options (A), (B), (C), (D) are correct.

15. Let $\preccurlyeq$ be a partial order on a non-empty set A. A subset D of A is called an **antichain** if and only if no two distinct elements in D are comparable, that is, $\forall a, b \in D\ \big(a \neq b \to {\sim}(a \preccurlyeq b \vee b \preccurlyeq a)\big)$. A **maximal antichain** is an antichain that is not a proper subset of any other antichain.

    Given the set $A = \{1,2,3,4,5,6\}$ and the subset relation $\subseteq$ on $\mathcal{P}(A)$, what is the size of the largest maximal antichain in the poset $(\mathcal{P}(A), \subseteq)$?

    A. 6
    B. $6!$
    C. $\binom{6}{3}$
    D. $\left(\frac{6}{2}\right)^2$
    E. None of options (A), (B), (C), (D) are correct.

16. Which of the following statements can be proved by using mathematical induction?

A. $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$ for all $a \in \mathbb{N}$.

B. $n^2 + n + 1$ is always even for all $n \in \mathbb{N}$.

C. $x^2 \geq 0$ for all $x \in \mathbb{R}$.

D. The interior angle sum of an n-sided polygon is $(n-2) \times 180°$, for all integers $n > 2$.

E. None of options (A), (B), (C), (D) are correct.

17. A set T is recursively defined as follows on the alphabet $\Sigma = \{0,1\}$:

(1) $\varepsilon \in T$. (Note: ε is the empty string.) (base clause)

(2) If $x \in T$, then $0x \in T$ and $x1 \in T$. (recursion clause)

(3) Membership for T can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

What is T?

A. All binary strings.

B. All binary strings with no 1's before 0's.

C. The binary string 1 and all binary strings with more 0's than 1's.

D. The binary string 0 and all binary strings with at least one 1.

E. None of options (A), (B), (C), (D) are correct.

18. Which of the following statements are true?

(i) If $\mathbb{Z} \in A$, then A is countably infinite.

(ii) If $\mathbb{Z} \subseteq A$, then A is countably infinite.

(iii) If $A \in \mathcal{P}(\mathbb{Z}) \setminus \{\emptyset\}$, then A is countably infinite.

A. Only (i).

B. Only (ii).

C. Only (i) and (ii).

D. Only (ii) and (iii).

E. None of options (A), (B), (C), (D) are correct.

19. Which of the following statements are true?

    (i) There exist countably infinite sets A and B where $A \neq B$ such that $A \setminus B$ is finite.

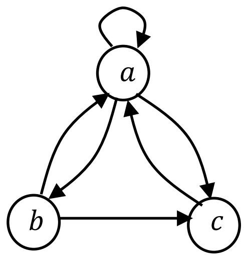
    (ii) There exist countably infinite sets A and B such that $A \setminus B$, $B \setminus A$ and $A \cap B$ are countably infinite.

    (iii) There exists a finite set A for which $\mathcal{P}(A)$ is countably infinite.

    A. Only (i).
    B. Only (i) and (ii).
    C. Only (i) and (iii).
    D. Only (ii) and (iii).
    E. None of options (A), (B), (C), (D) are correct.

20. Which of the following has the minimum number of **walks of length 4** among all pairs of vertices in the directed graph given below?

    A. Vertex a to vertex b
    B. Vertex b to vertex b
    C. Vertex c to vertex a
    D. Vertex c to vertex b
    E. None of options (A), (B), (C), (D) are correct.



21. Aiken drew 25 labelled simple graphs with 3 vertices. At least how many of these graphs are isomorphic to one another? You are to choose the tightest bound.

    A. 4
    B. 5
    C. 7
    D. 9
    E. None of the above.

22. Which of the following statements are true?

    (i) A graph is bipartite if and only if it has no odd cycle (i.e. cycle of odd length).

    (ii) A graph is bipartite if and only if the sum of degrees of all its vertices is even.

    (iii) Any complete bipartite graph $K_{2,n}$ where $n > 3$, is planar.

    A. Only (i).

    B. Only (i) and (iii).

    C. Only (ii) and (iii).

    D. All of (i), (ii) and (iii).

    E. None of options (A), (B), (C), (D) are correct.

23. One ball is drawn randomly from a bowl containing four balls numbered 1,2, 3 and 4. Define the following three events:

    - Let A be the event that a 1 or 2 is drawn, that is, $A = \{1,2\}$.
    - Let B be the event that a 1 or 3 is drawn, that is, $B = \{1,3\}$.
    - Let C be the event that a 1 or 4 is drawn, that is, $C = \{1,4\}$.

    Which of the following is true?

    A. Events A, B, C have different probabilities.

    B. Events A, B, C are not pairwise independent.

    C. Events A, B, C are pairwise independent but not mutually independent.

    D. Events A, B, C are mutually independent.

    E. None of options (A), (B), (C), (D) are correct.

Part B: There are 3 questions in this part [Total: 54 marks]

24. Counting and Probability [Total: 20 marks]

Workings are not required for this question.

(a) The lab coordinator for an AI lab at SoC found that 5 GPUs were available for 5 students to use during the week. What is the probability (correct to 4 significant figures) that at least one GPU was used by more than one student? We may assume that students randomly choose any GPU with equal probability, and none of the GPUs was unused. [2 marks]

(b) There are k ways to choose three balls from a bucket containing n balls. There are $k/2$ ways to choose two balls after removing one ball from the bucket. How many balls are there in the bucket? [2 marks]

(c) Write your answer as a single integer for each of the parts below. [2+2 = 4 marks]

Given the equation: $x_1 + x_2 + x_3 + x_4 = 12$.

(i) How many integer solutions does the equation have, assuming that $x_i \geq 0$?

(ii) How many integer solutions does the equation have, assuming that $x_i \geq 0$ and $x_4 \leq 3$?

(d) Ten couples (a couple consists of two persons) went to watch a horror movie on Halloween Day. Not all of them had the courage to stay throughout the entire movie. Four persons left the cinema hall in the middle of the movie. [2+2 = 4 marks]

(i) Fix an arbitrary couple. What is the probability of the couple watched the entire movie? Write your answer as a single fraction.

(ii) What is the expected number of couples that watched the entire movie? Write your answer as a single fraction.

(e) Aaron wants to distribute markers to his tutors before the tutorials start in week 3. He has a box that contains b blue markers and r red markers. Every time a tutor visits, he draws two markers at random, one at a time, and gives them to the tutor. [2+1+2+3 = 8 marks]

(i) What is the probability of the first marker being red?

(ii) Is the statement “the probability of the first marker being red equals to the probability of second marker being red” true or false?

(iii) What is the probability of drawing two markers of different colours?

(iv) There are altogether 25 markers in the box and Aaron remembers that there are more red markers than blue markers. How many red markers are there if the probability of drawing two markers of the same colour is equal to the probability of drawing two markers of different colours?

25. Graphs and Trees [Total: 20 marks]

Workings are not required for this question.

Definitions:

The **chromatic number** of an undirected graph G, denoted as $\chi(G)$, is the minimum number of colours required to colour the vertices of G in such a way that no two adjacent vertices have the same colour.

The **chromatic index** of an undirected graph G, denoted as $\chi'(G)$, is the minimum number of colours required to colour the edges of G in such a way that no two adjacent edges have the same colour.

A **cycle graph** (or circular graph) is an undirected connected graph that has at least 3 vertices and consists of a single cycle where every vertex is on the cycle. The cycle graph with n vertices is denoted as C_n.

Let $G = (V, E)$ be a connected undirected graph. A **vertex cut set** is a set S, where $S \subseteq V$, such that the removal of the vertices in S, along with their incident edges, disconnects G (that is, resulting in G having more than one connected component). A **minimum vertex cut set** is a vertex cut set of the smallest possible size.

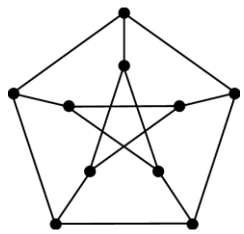
(a) What is the chromatic number of each of the following graphs? [4 marks]

(i) K_{1231}

(ii) $K_{1231,1231}$

(iii) C_{1231}

(iv) The Petersen graph as shown on the right.



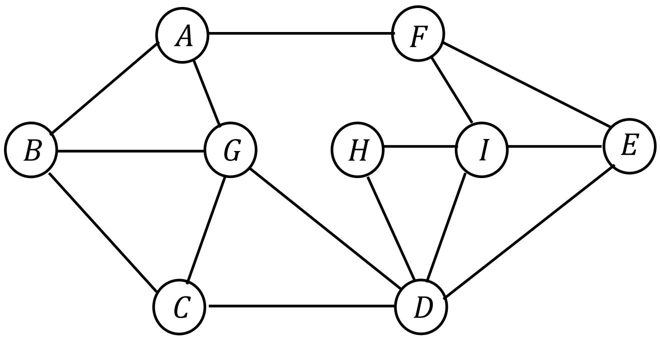
Petersen graph

(b) What is the chromatic index of each of the following graphs? [3 marks]

(i) K_{100} (ii) $K_{1231,1231}$ (iii) The Petersen graph

(c) Is the Petersen graph Eulerian? Is it Hamiltonian? [2 marks]

(d) Write out a minimum vertex cut set for the graph below. [2 marks]

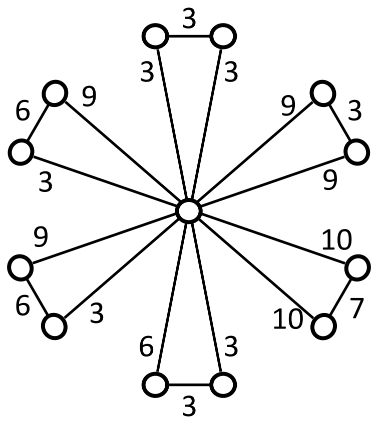


25. (continue…)

(e) An undirected weighted graph is shown below. [4 marks]

(i) What is the weight of its minimum spanning tree?

(ii) How many minimum spanning trees are there if the vertices are labelled distinctly?



(f) (i) What is the name of the graph in part (e) above? (Note that the name of the graph is required, not the type, eg: Eulerian graph, planar graph, simple graph, etc. These will not be accepted.)

(ii) Given such a graph with $2n + 1$ vertices, what is its chromatic index for $n > 1$? [2 marks]

(g) The preorder and postorder traversals of a certain binary tree are given below. Write out the inorder traversal of this tree. [3 marks]

Preorder: A M P L I T U D E

Postorder: P L M T D E U I A

26. Mathematical Induction [Total: 14 marks]

Consider a set S recursively defined by the following three clauses:

Clause (1)	$1 \in S$
Clause (2)	If $x \in S$ then $x + 2 \in S$
Clause (3)	If $x \in S$ then $3x \in S$

Membership in S can always be demonstrated by a finite application via one or more of the above three clauses. Let $Bool = \{true, false\}$.

(a) As you are aware, the mathematical induction principle for $\mathbb{Z}^+$ can be established by an inductive proof rule for any predicate $P : \mathbb{Z}^+ \to Bool$ as follows:

$$P(1)$$
$$\forall x \in \mathbb{Z}^+ \big(P(x) \Rightarrow P(x+1)\big)$$
$$\therefore\ \forall x \in \mathbb{Z}^+\ P(x)$$

Write a suitable mathematical induction principle, in the above format, for S. [2 marks]

(b) Given the following statements, indicate whether each of them is true or false. [4 marks]

(i) $\forall x \in S\ (x \geq 1)$.

(ii) $\forall x \in S\ (x \neq 8)$.

(iii) $\forall x \in S\ (\exists k \in \mathbb{Z}^+\ x = 3^k)$.

(iv) $\forall x \in S\ false$.

(c) A predicate (or property) $Q(x)$ is said to be stronger than another predicate $R(x)$ if we can prove that $Q(x) \Rightarrow R(x)$.

The strongest predicate $Q(x)$ that holds for any given well-ordered set S can be described by $\forall x\ \big(x \in S \Leftrightarrow Q(x)\big)$.

This statement is unfortunately not in a form that can be used by our induction proof rule. To establish if a given property $Q(x)$ is indeed the strongest property for some well-ordered set S, we would need express it differently, as the following statement:

$$\big(\forall x \in S\ Q(x)\big) \wedge \big(\forall P : S \to Bool\ (\ldots)\big)$$

Complete the missing part "…" in the statement above. [4 marks]

(d) Define a mathematical function $f : \mathbb{Z}^+ \to \mathbb{N}$ as follows:

$$f(x) = \begin{cases} 0, & \text{if } x \notin S; \\ n, & \text{if } x \in S \text{ and } n \text{ is the minimum number of clauses used to generate } x. \end{cases}$$

Some examples are:

$f(1) = 1$ since 1 can be generated only by the base case clause (1).

$f(2) = 0$ since $2 \notin S$.

$f(3) = 2$ since 3 can be generated using a minimum of 2 clauses via $3 = 3 \times 1$.

$f(7) = 4$ since 7 can be generated using a minimum of 4 clauses via $7 = \big((3 \times 1) + 2\big) + 2$.

You are asked to define this function mathematically, as follows,

$$f(n) = r \Leftrightarrow \text{" \ldots "}$$

Complete the missing part "…". [4 marks]

=== END OF PAPER ===