NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 1: AY2025/26)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY SIX (26)** questions in **TWO (2)** parts and comprises **TWELVE (12)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEETS** provided.
5. Do **not** write your name on the ANSWER SHEETS.
6. The maximum mark of this assessment is 100.

Question	Max. mark
Part A: Q1 – 23	46
Part B: Q24	20
Part B: Q25	20
Part B: Q26	14
Total	**100**

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions [Total: 23×2 = 46 marks]

Each multiple choice question (MCQ) is worth **TWO marks**. Shade your answer on the corresponding bubble on the Answer Sheets. You are to shade only one bubble for each question.

1. Which of the following topics is not covered in CS1231S?

   A. Mathematical induction.

   B. Predicate logic.

   C. Graph theory.

   D. Calculus.

   E. None of the above, that is, all the topics in (A),(B),(C) and (D) are covered in CS1231S.

2. Given the following argument on statement variables p and q:

   $p \rightarrow q$
   $\sim p \rightarrow q$
   $\therefore p$

   Which of the following statements is true?

   A. The argument is a contradiction.

   B. The argument is invalid.

   C. The argument is valid and sound.

   D. The argument is valid but unsound.

   E. None of options (A), (B), (C), (D) are correct.

3. Given the following three statements on variables p, q, r and s.

   $(p \vee q) \rightarrow r$

   $\sim r$

   $s \vee p$

   Assuming that all three statements above are true, which of the following is true?

   A. $\sim s$

   B. $q \wedge s$

   C. $\sim p \wedge s$

   D. $s \rightarrow q$

   E. None of options (A), (B), (C), (D) are correct.

4. $P(x)$ and $Q(x)$ are predicates. Let $A = \{x : P(x)\}$ and $B = \{x : Q(x)\}$. Which of the following states that $A \subseteq B$?

   A. $\forall x\, (Q(x) \vee {\sim}P(x))$
   B. $\forall x\, (Q(x) \rightarrow P(x))$
   C. $\exists x\, (P(x) \wedge Q(x))$
   D. $\exists x\, ({\sim}P(x) \vee Q(x))$
   E. None of options (A), (B), (C), (D) are correct.

5. Let predicate $P(x, y, z)$ mean $x + y = z$. Given these two statements:

   (i) $\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ \forall z \in \mathbb{N}\ P(x, y, z)$
   (ii) $\forall x \in \mathbb{N}\ \forall z \in \mathbb{N}\ \exists y \in \mathbb{N}\ P(x, y, z)$

   Which of the following is correct?

   A. Both (i) and (ii) are false.
   B. (i) is true and (ii) is false.
   C. (i) is false and (ii) is true.
   D. Both (i) and (ii) are true.

For questions 6 – 8, an empty relation R is a relation with no elements, that is, $R = \emptyset$.

6. Let R be a binary relation on A. Given these statements:

   (i) If R is not transitive, then R is non-empty.
   (ii) If R is non-empty, symmetric and transitive, then R is reflexive.
   (iii) If R is symmetric and asymmetric, then R is empty.

   Which of the following is correct?

   A. Only (i) is true.
   B. Only (ii) is true.
   C. Only (iii) is true.
   D. All (i), (ii), (iii) are true.
   E. None of options (A), (B), (C), (D) are correct.

7. Let A be any set and R an empty binary relation on A. Which of the following statements is true?

   A. R may not be reflexive.

   B. R may not be symmetric.

   C. R may not be anti-symmetric.

   D. R may not be transitive.

   E. None of options (A), (B), (C), (D) are correct.

8. Let $A = \{a, b, c\}$. Given these two statements:

   (i) There exist distinct non-empty binary relations R and S on A such that R and S are symmetric, and $R \cap S$ is non-empty and antisymmetric.

   (ii) There exist distinct non-empty binary relations R and S on A such that R and S are transitive, and $R \cup S$ is non-empty and not transitive.

   Which of the following is correct?

   A. Both (i) and (ii) are false.

   B. (i) is true and (ii) is false.

   C. (i) is false and (ii) is true.

   D. Both (i) and (ii) are true.

9. Let $G = \{V, E\}$ be a graph. Define a binary relation R on V such that xRy if and only if there exists a walk from x to y, where $x, y \in V$.

   Given these statements:

   (i) If G is undirected, then R is an equivalence relation.

   (ii) If G is directed, then R is a partial order.

   (iii) If G is directed and for every $(x, y) \in E$, $(y, x) \notin E$, then R is a partial order.

   Which of the following is correct?

   A. Only (i) is true.

   B. Only (ii) is true.

   C. Only (iii) is true.

   D. All (i), (ii), (iii) are true.

   E. None of options (A), (B), (C), (D) are correct.

10. A strictly increasing function $f : X \to Y$ is defined as follows:

$$\forall a, b \in X\ \big((a < b) \Rightarrow f(a) < f(b)\big)$$

Given the following sets:

$$F = \{f : \{0,1\} \to \{0,1,2,3\} \mid f \text{ is a strictly increasing function}\}.$$
$$G = \{g : \{0,1\} \to \{0,1,2\} \mid g \text{ is an injective function}\}.$$

Which of the following statements is true regarding the above sets?

A. $|F| = (|G| + 1)$.
B. $|F| = |G|$.
C. $(|F| + 1) < |G|$.
D. $(|F| + 1) = |G|$.
E. None of options (A), (B), (C), (D) are correct.

11. Let f be a function defined on $\mathcal{B} \times \mathcal{B}$, where $\mathcal{B} = \{true, false\}$, as follows:

$$f\big((p,q)\big) = \big(p, (\sim p \wedge q) \vee (p \wedge \sim q)\big)$$

Given the following statements:

(i) $f = f^{-1}$
(ii) $f^{-1} \circ f = f$
(iii) $f \circ f = id_{\mathcal{B}\times\mathcal{B}}$

Which of the above statements are true?

A. Only (i).
B. Only (ii).
C. Only (i) and (iii).
D. All of (i),(ii) and (iii).
E. None of options (A), (B), (C), (D) are correct.

12. Given the function f in question 11 above, and function g on $\mathcal{B} \times \mathcal{B}$ as defined below:

$$g\big((p,q)\big) = \big((\sim p \wedge q) \vee (p \wedge \sim q), q\big)$$

What is the order of $(g \circ f)$?

A. 1
B. 2
C. 3
D. 4
E. None of options (A), (B), (C), (D) are correct.

13. An equivalence relation $\sim$ on the set $\mathbb{Z} \times \mathbb{Z}^+$ is defined as follows:

$$(a, b) \sim (c, d) \text{ if and only if } ad = bc.$$

Let $Q = (\mathbb{Z} \times \mathbb{Z}^+)/\sim$ be the quotient set under the relation. Given the following statements:

(i) There exists a bijection $f_1: Q \to \mathbb{Z}$.
(ii) There exists a bijection $f_2: Q \to \mathbb{Q}$.
(iii) There exists a bijection $f_3: Q \to (\mathbb{Z} \times \mathbb{Z}^+)$.

Which of the above statements are true?

A. Only (i).
B. Only (ii).
C. Only (i) and (iii).
D. All of (i),(ii) and (iii).
E. None of options (A), (B), (C), (D) are correct.

14. A set S of strings over the alphabet $\{a, b\}$ is recursively defined as follows:

(1)	$a \in S$.	(base clause)
(2)	If $x \in S$, then $xa \in S$.	(recursion clause)
(3)	If $x \in S$, then $bxb \in S$.	(recursion clause)
(4)	Membership for S can always be demonstrated by finitely many successive applications of the clauses above.	(minimality clause)

How many elements in S have length of at most 4?

A. 4
B. 7
C. 8
D. 16
E. None of options (A), (B), (C), (D) are correct.

15. The recurrence relation for Fibonacci sequence is given as follows:

$$F_0 = 0, F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n > 1.$$

Let G_n denote the number of elements in S with length n, where S is as defined in question 14 above. Which of the following is correct?

A. $G_n = F_n$ for $n \geq 1$.
B. $G_n = F_{n-1}$ for $n \geq 1$.
C. $G_n = F_{n+1}$ for $n \geq 1$.
D. $G_1 = G_2 = 1$, and $G_n = F_n + F_{n-1}$ for $n > 2$.
E. None of options (A), (B), (C), (D) are correct.

16. A sequence of sets $S_0, S_1, S_2, \cdots$ is recursively defined as follows:

    (1) $S_0 = \{1,2,3\}$. (base clause)

    (2) For every integer $n \geq 0, S_{n+1} = (S_n \cup \{x + 3 : x \in S_n\}) \setminus \{2,3\}$. (recursion clause)

    (3) Membership in any S_k, $k \geq 0$, can always be demonstrated by finitely many successive applications of the clauses above. (minimality clause)

    What is $\bigcup_{n \geq 0} S_n$?

    A. S_0
    B. $S_0 \cup \{3k : k \in \mathbb{Z}^+\}$
    C. $\mathbb{Z}^+$
    D. $\mathbb{Z}^+ \setminus \{2,3\}$
    E. None of options (A), (B), (C), (D) are correct.

17. Consider the following two statements:

    (i) If set A is countably infinite and set B is uncountable, then $A \cup B$ is uncountable.

    (ii) If set A is countably infinite and set B is uncountable, then $A \cap B$ is countable.

    Which of the following is correct?

    A. Both (i) and (ii) are false.
    B. (i) is true and (ii) is false.
    C. (i) is false and (ii) is true.
    D. Both (i) and (ii) are true.

18. Consider the following two statements:

    (i) Any partition of $\mathbb{N}$ is countable.

    (ii) The set of all partitions of $\mathbb{N}$ is countable.

    Which of the following is correct?

    A. Both (i) and (ii) are false.
    B. (i) is true and (ii) is false.
    C. (i) is false and (ii) is true.
    D. Both (i) and (ii) are true.

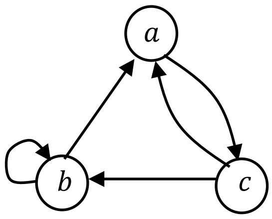
19. Assuming that the left and right children are not distinguished, there are 2 non-isomorphic, unlabelled, rooted binary trees with 3 vertices, and there are 3 non-isomorphic, unlabelled, rooted binary trees with 4 vertices.

    How many non-isomorphic, unlabelled, rooted binary trees with 5 vertices are there?

    A. 4
    B. 5
    C. 6
    D. 7
    E. None of options (A), (B), (C), (D) are correct.

20. Which of the following has the minimum number of **walks of length 4** among all pairs of vertices in the directed graph given below?

    A. Vertex a to vertex c
    B. Vertex b to vertex b
    C. Vertex c to vertex a
    D. Vertex c to vertex b
    E. None of options (A), (B), (C), (D) are correct.



21. A certain binary tree T has the following preorder and inorder traversals:

    Preorder: G B E D F C A
    Inorder: B G F D C E A

    What is the postorder traversal of T?

    A. A B C D E F G
    B. F C D A B E G
    C. B D C A F E G
    D. B F C D A E G
    E. None of options (A), (B), (C), (D) are correct.

22. At most how many distinct binary trees have the following preorder and postorder traversals?

    Preorder: A B C D E

    Postorder: C E D B A

    A. 2
    B. 4
    C. 8
    D. 16
    E. None of options (A), (B), (C), (D) are correct.

23. An ice-cream parlor offers 10 flavours for its sundaes. If the customers say the secret code "Aiken Dueet!", they may choose exactly 3 flavours for their sundae, otherwise, they would be allowed to choose exactly 2 flavours. Suppose 20% of the customers are CS1231S students and hence they know the secret code, what is the expected number of distinct sundaes per customer?

    A. 33
    B. 60
    C. 105
    D. 165
    E. None of options (A), (B), (C), (D) are correct.

Part B: There are 3 questions in this part [Total: 54 marks]

24. Counting and Probability [Total: 20 marks]

Working is not required for this question.

(a) You have 12 balls, 3 of which are red, 4 are blue and 5 are green. Balls of the same colour are identical. How many distinct arrangements are possible if you lay the 12 balls in a straight row? Write your answer as a single number. [2 marks]

(b) A committee of 5 is to be formed from 6 men and 5 women such that there are at least 3 women. How many possible committees are there? Write your answer as a single number. [2 marks]

(c) A password consists of 6 letters chosen from the 26 uppercase letters of the English alphabet (A, B, C, …, Z). The vowels are A, E, I, O and U. If letters in the password are not allowed to be repeated, how many passwords contain exactly 2 vowels? Write your answer as a single number. [2 marks]

(d) How many ways are there to seat n couples ($2n$ people), $n > 1$, around a circular table if the men and women must alternate? All people are identifiable, that is, they are distinct. [2 marks]

(e) How many ways are there to seat n couples ($2n$ people), $n > 1$, around a circular table if every woman has to be seated next to her husband? All people are identifiable, that is, they are distinct. [2 marks]

(f) A drawer contains 8 red, 7 blue, 6 green, 5 black and 4 white socks. What is the minimum number of socks you must draw without looking to guarantee that you have either (i) 4 socks of the same colour, or (ii) 3 socks of one colour and 3 socks of another colour (that is, two distinct colours each appearing at least 3 times)?
Write your answer as a single number. [2 marks]

(g) You roll 3 fair dice. The dice are distinguishable. In how many ways can the sum of the 3 dice be 10? Write your answer as a single number. [3 marks]

(h) (Total: 5 marks) A mail service labels an incoming email as Spam (S) or Ham (H = not spam). Historical data shows that $P(S) = 0.2$ and $P(H) = 0.8$. For a given email, let's define two features:

- O: the word "offer" appears at least once;
- U: the word "urgent" appears at least once.

From past emails, we know that $P(O|S) = 0.5, P(U|S) = 0.35, P(O|H) = 0.08$ and $P(U|H) = 0.04$. Assume that O and U are independent given the class (spam vs ham).

(i) Determine $P(S|O)$. Write your answer correct to <u>2 decimal places</u>. [2 marks]

(ii) Determine $P(S|U)$. Write your answer correct to <u>2 decimal places</u>. [2 marks]

(iii) Which word, "offer" or "urgent", is a stronger spam signal? (Your answer for this part is graded only if you have answered both parts (i) and (ii).) [1 mark]

25. Graphs and Trees [Total: 20 marks]

Definitions:

A **cycle graph** (or circular graph) is an undirected connected graph that has at least 3 vertices and consists of a single cycle where every vertex is on the cycle. The cycle graph with n vertices is denoted as C_n.

A **bipartite graph** $G = (U, V, E)$ is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to a vertex in V, and there are no edges within U or within V.
A **bipartite matching** is a subset $F \subseteq E$ such that no two edges in F share a common vertex.
A **maximum matching** is a matching that contains the largest possible number of edges.
A **maximal matching** is a matching that cannot be extended by adding another edge.
A **perfect matching** is a matching that covers every vertex of the graph.

Hall's Marriage Theorem
A bipartite graph $G = (U, V, E)$ has a matching that covers all vertices in U if and only if for every subset $S \subseteq U$,

$$|N(S)| \geq |S|$$

where $N(S)$ is the set of neighbours of S in V. The neighbour of a vertex is any vertex that is directly connected to it by an edge. A vertex is not considered its own neighbour unless it has a loop.

$K_{m,n}$ denotes a complete bipartite graph where $m = |U|$ and $n = |V|$. You may assume that $m, n \geq 2$.

(a) Is $K_{3,4}$ a planar graph? Answer "yes" or "no". [2 marks]

(b) What is the necessary and sufficient condition for $K_{m,n}$ to be an Eulerian graph? [2 marks]

(c) What is the necessary and sufficient condition for $K_{m,n}$ to be a Hamiltonian graph? [2 marks]

(d) Which of the following cycle graphs are bipartite graphs? List out all of them. [2 marks]

$$C_{12}, C_{57}, C_{123}, C_{1231}, C_{8888}, C_{12345}$$

(e) A matching that satisfies the condition in the Hall's Marriage Theorem may not be a perfect matching of the bipartite graph $G = (U, V, E)$. What is the additional condition that makes it a perfect matching? [2 marks]

(f) Draw a bipartite graph where $2 \leq |U|, |V| \leq 4$ with the property that there is a maximal matching which is not a maximum matching. Label the vertices of your graph and write out the maximal matching and the maximum matching found for this question. [4 marks]

(g) Let $G_1 = (U, V, E)$ be a bipartite graph. Suppose $U = \{a, b, c, d\}, V = \{w, x, y, z\}$ and $E = \{(a, x), (b, w), (b, y), (c, y), (d, y), (d, z)\}$, is there a perfect matching for G_1? If there is, write out the set of edges in the perfect matching; if there isn't, give a counterexample based on Hall's Marriage Theorem. [3 marks]

(h) Let $G_2 = (U, V, E)$ be a bipartite graph. Suppose $U = \{a, b, c, d\}, V = \{w, x, y, z\}$ and $E = \{(a, w), (a, x), (b, w), (b, x), (c, x), (d, y), (d, z)\}$, is there a perfect matching for G_2? If there is, write out the set of edges in the perfect matching; if there isn't, give a counterexample based on Hall's Marriage Theorem. [3 marks]

26. Functions [Total: 14 marks]

(a) [4×2 marks] For each of the following functions, answer

- **A** if the function is neither injective nor surjective;
- **B** if the function is injective but it is not surjective;
- **C** if the function is not injective but it is surjective;
- **D** if the function is both injective and surjective.

(i) Let $f : \mathbb{N} \to \mathbb{Z}^+$, $f(n) = n!$

(ii) Let $G = (V, E)$ be a simple undirected graph where $|V| = n > 0$. Define a function g from the set of all simple undirected graphs on n vertices to the set $\left\{0,1,2,\ldots,\frac{n(n-1)}{2}\right\}$ (that is, the set of integers from 0 to $\frac{n(n-1)}{2}$ inclusive), where $g(G) = |E|$.

(iii) Let $h : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$, $h(A) = |A|$.
(Note: $\mathcal{P}(X)$ is the powerset of X.)

(iv) Let $k : \mathbb{R} \to \mathbb{R}$, $k(x) = \begin{cases} x+1, & x < 0; \\ 2x, & x \geq 0 \end{cases}$

(b) [3×2 marks]
Let $\mathbb{Z}_5$ denote the set of equivalence classes of integers under congruence modulo 5. That is,

$$\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}, \qquad [a] = \{n \in \mathbb{Z} : n \equiv a \pmod 5\}$$

Define $z : \mathbb{Z}_5 \to \mathbb{Z}_5$ given by $z(x) = 2x + 1$ (mod 5).

Given a function $f(x)$, we define the function $\boldsymbol{f^{(n)}(x)}$ to be the result of n applications of f to x, where $n \in \mathbb{Z}^+$. For example, $f^{(3)}(x) = f\left(f\left(f(x)\right)\right)$. We also define the ***order*** of an input x with respect to f to be the smallest positive integer m such that $f^{(m)}(x) = x$.

(i) What is $z^{-1}([3])$?

(ii) What is $z^{(2)}(x)$?

(iii) What is the order of [3] with respect to the function z?

=== END OF PAPER ===