Lecture #13: Trees
Summary

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10.5 Trees

- Definitions: circuit-free, tree, trivial tree, forest
- Characterizing trees: terminal vertex (leaf), internal vertex

10.6 Rooted Trees

- Definitions: rooted tree, root, level, height, child, parent, sibling, ancestor, descendant
- Definitions: binary tree, full binary tree, subtree
- Binary tree traversal: breadth-first-search (BFD), depth-first-search (DFS)

10.7 Spanning Trees and Shortest Paths

- Definitions: spanning tree, weighted graph, minimum spanning tree (MST)
- Kruskal's algorithm, Prim's algorithm
- Dijkstra's shortest path algorithm (non-examinable)

Summary

Definition: Tree

(The graph is assumed to be undirected here.)

A **graph** is said to be **circuit-free** if and only if it has no circuits.

A simple graph is called a **tree** if and only if it is circuit-free and connected.

A **trivial tree** is a tree that consists of a single vertex.

A simple graph is called a **forest** if and only if it is circuit-free and not connected.

Definitions: Terminal vertex (leaf) and internal vertex

Let T be a tree. If T has only one or two vertices, then each is called a **terminal vertex** (or **leaf**). If T has at least three vertices, then a vertex of degree 1 in T is called a **terminal vertex** (or **leaf**), and a vertex of degree greater than 1 in T is called an **internal vertex**.

Summary

10.5 Trees

Lemma 10.5.1

Any non-trivial tree has at least one vertex of degree 1.

Theorem 10.5.2

Any tree with n vertices ($n > 0$) has $n - 1$ edges.

Lemma 10.5.3

If G is any connected graph, C is any circuit in G, and one of the edges of C is removed from G, then the graph that remains is still connected.

Theorem 10.5.4

If G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

Definitions: Rooted Tree, Level, Height

A **rooted tree** is a tree in which there is one vertex that is distinguished from the others and is called the **root**.

The **level** of a vertex is the number of edges along the unique path between it and the root.

The **height** of a rooted tree is the maximum level of any vertex of the tree.

Definitions: Child, Parent, Sibling, Ancestor, Descendant

Given the root or any internal vertex *v* of a rooted tree, the **children** of *v* are all those vertices that are adjacent to *v* and are one level farther away from the root than *v*.

If *w* is a child of *v*, then *v* is called the **parent** of *w*, and two distinct vertices that are both children of the same parent are called **siblings**.

Given two distinct vertices *v* and *w*, if *v* lies on the unique path between *w* and the root, then *v* is an **ancestor** of *w*, and *w* is a **descendant** of *v*.

Definitions: Binary Tree, Full Binary Tree

A **binary tree** is a rooted tree in which every parent has at most two children. Each child is designated either a **left child** or a **right child** (but not both), and every parent has at most one left child and one right child.

A **full binary tree** is a binary tree in which each parent has exactly two children.

Definitions: Left Subtree, Right Subtree

Given any parent v in a binary tree T, if v has a left child, then the **left subtree** of v is the binary tree whose root is the left child of v, whose vertices consist of the left child of v and all its descendants, and whose edges consist of all those edges of T that connect the vertices of the left subtree.

The **right subtree** of v is defined analogously.

Summary

Theorem 10.6.1: Full Binary Tree Theorem

If T is a full binary tree with k internal vertices, then T has a total of $2k + 1$ vertices and has $k + 1$ terminal vertices (leaves).

Theorem 10.6.2

For non-negative integers h, if T is any binary tree with height h and t terminal vertices (leaves), then

$$t \le 2^h$$

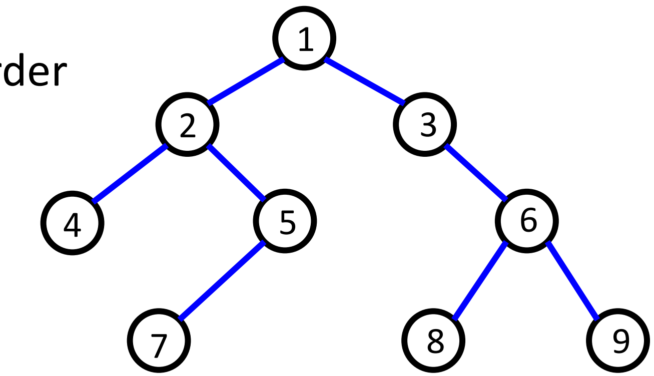
Equivalently,

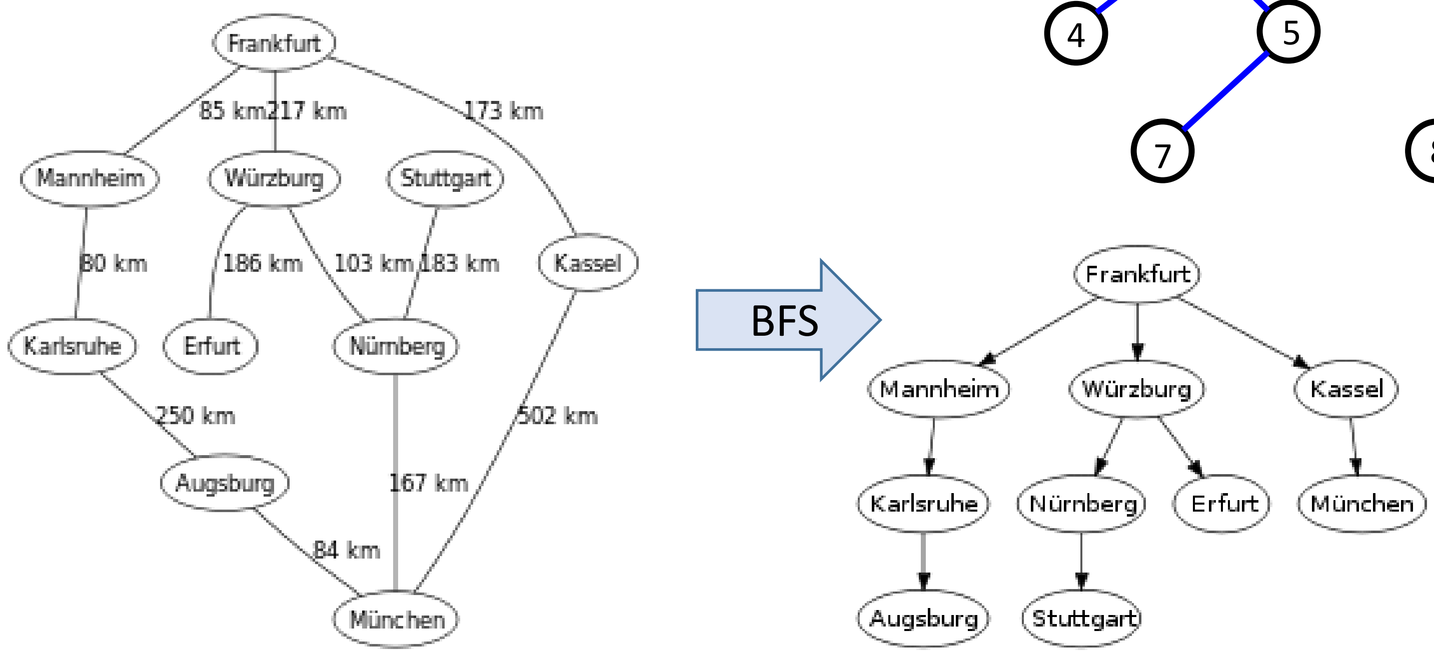
$$\log_2 t \le h$$

Breadth-First Search

In breadth-first search (by E.F. Moore), it starts at the root and visits its adjacent vertices, and then moves to the next level.

The figure shows the order of the vertices visited.





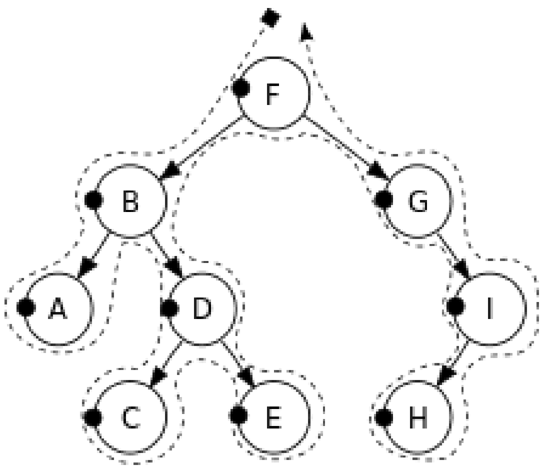
Acknowledgement: Wikipedia https://en.wikipedia.org/wiki/Breadth-first_search

Depth-First Search

There are three types of depth-first traversal:

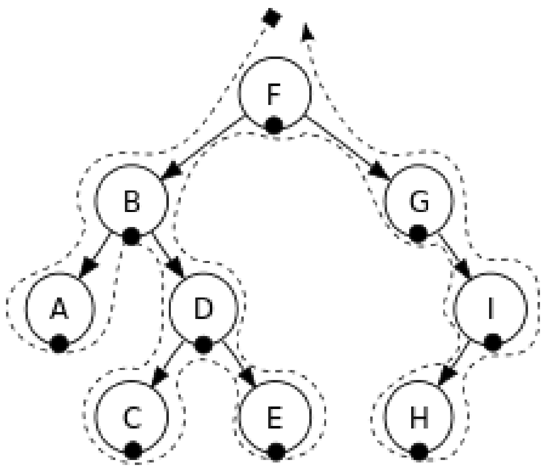
- Pre-order
  - Print the data of the root (or current vertex)
  - Traverse the left subtree by recursively calling the pre-order function
  - Traverse the right subtree by recursively calling the pre-order function
- In-order
  - Traverse the left subtree by recursively calling the in-order function
  - Print the data of the root (or current vertex)
  - Traverse the right subtree by recursively calling the in-order function
- Post-order
  - Traverse the left subtree by recursively calling the post-order function
  - Traverse the right subtree by recursively calling the post-order function
  - Print the data of the root (or current vertex)

Depth-First Search



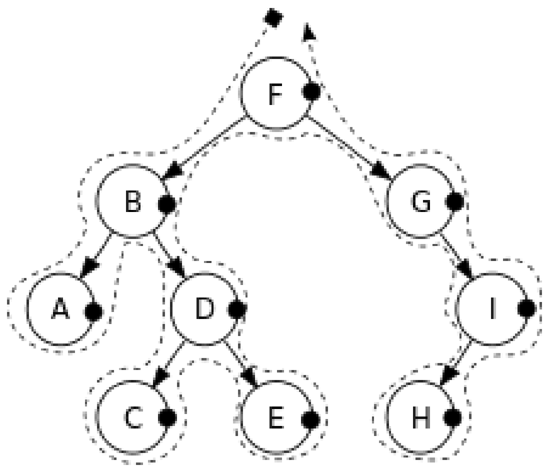
Pre-order:

F, B, A, D, C, E, G, I, H



In-order:

A, B, C, D, E, F, G, H, I



Post-order:

A, C, E, D, B, H, I, G, F

Acknowledgement: Wikipedia https://en.wikipedia.org/wiki/Tree_traversal

Summary

Definition: Spanning Tree

A **spanning tree** for a graph *G* is a subgraph of *G* that contains every vertex of *G* and is a tree.

Proposition 10.7.1

1. Every connected graph has a spanning tree.
2. Any two spanning trees for a graph have the same number of edges.

Definitions: Weighted Graph, Minimum Spanning Tree

A **weighted graph** is a graph for which each edge has an associated positive real number **weight** . The sum of the weights of all the edges is the **total weight** of the graph.

A **minimum spanning tree** for a connected weighted graph is a spanning tree that has the least possible total weight compared to all other spanning trees for the graph.

If *G* is a weighted graph and *e* is an edge of *G*, then $\boldsymbol{w(e)}$ denotes the weight of *e* and $\boldsymbol{w(G)}$ denotes the total weight of *G*.

Algorithm 10.7.1 Kruskal

Input: G [a connected weighted graph with n vertices]

Algorithm:

1. Initialize T to have all the vertices of G and no edges.
2. Let E be the set of all edges of G, and let $m = 0$.
3. While ($m < n - 1$)
   - 3a. Find an edge e in E of least weight.
   - 3b. Delete e from E.
   - 3c. If addition of e to the edge set of T does not produce a circuit, then add e to the edge set of T and set $m = m + 1$

   End while

Output: T [T is a minimum spanning tree for G]

Algorithm 10.7.2 Prim

Input: G [a connected weighted graph with n vertices]

Algorithm:

1. Pick a vertex v of G and let T be the graph with this vertex only.
2. Let V be the set of all vertices of G except v.
3. For $i = 1$ to $n - 1$
   - 3a. Find an edge e of G such that (1) e connects T to one of the vertices in V, and (2) e has the least weight of all edges connecting T to a vertex in V. Let w be the endpoint of e that is in V.
   - 3b. Add e and w to the edge and vertex sets of T, and delete w from V.

Output: T [T is a minimum spanning tree for G]

To skip for this semester.

Algorithm 10.7.3 Dijkstra

Inputs:

- G [a connected simple graph with positive weight for every edge]
- ∞ [a number greater than the sum of the weights of all the edges in G]
- $w(u, v)$ [the weight of edge $\{u, v\}$]
- a [the source vertex]
- z [the destination vertex]

Algorithm:

1. Initialize T to be the graph with vertex a and no edges. Let $V(T)$ be the set of vertices of T, and let $E(T)$ be the set of edges of T.
2. $L(a) \leftarrow 0$, and for all vertices u in G except a, $L(u) \leftarrow \infty$. [The number $L(u)$ is called the label of u.]
3. Initialize $v \leftarrow a$ and $F \leftarrow \{a\}$. [The symbol v is used to denote the vertex most recently added to T.]

To skip for this semester.

Algorithm 10.7.3 Dijkstra (continued...)

Let Adj(x) denote the set of vertices adjacent to vertex x.

4. While ($z \notin V(T)$)
   a. $F \leftarrow (F - \{v\}) \cup \{\text{vertices} \in \text{Adj}(v) \text{ and } \notin V(T)\}$
      [The set F is the set of fringe vertices.]
   b. For each vertex $u \in \text{Adj}(v)$ and $\notin V(T)$,
      if $L(v) + w(v, u) < L(u)$ then
      $L(u) \leftarrow L(v) + w(v, u)$
      $D(u) \leftarrow v$

      [The notation $D(u)$ is introduced to keep track of which vertex in T gave rise to the smaller value.]
   c. Find a vertex x in F with the smallest label.
      Add vertex x to $V(T)$, and add edge $\{D(x), x\}$ to $E(T)$.
      $v \leftarrow x$

Output: $L(z)$ [this is the length of the shortest path from a to z.]

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