CS1231S: Discrete Structures

Tutorial #2: Logic of Quantified Statements (Predicate Logic)

(Week 4: 1– 5 September 2025)

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1. Discussion Questions

These are meant for you to discuss on Canvas. No answers will be provided.

D1 An elementary definition in number theory is the following on divisibility:

For integers d and n, $d|n$ if and only if $n = kd$ for some integer k.

(Here, $d|n$ means "d divides n", and d is called a *divisor* or *factor*.)

(a) State the above definition symbolically.

(b) According to the above definition, does 2 divide $2\sqrt{2}$?

D2. Explain why English can be ambiguous at times using the following sentence:

"Every boy loves a girl."

Interpret the above sentence in two ways and write the quantified statement for each of the interpretations.

D3. The following table shows when the quantified statements are true and when they are false.

Statement	True when…	False when…
$\forall x\ P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\ P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Complete the table below for mixed quantifiers.

Statement	True when…	False when…
$\forall x \forall y\ P(x,y)$		
$\forall x \exists y\ P(x,y)$		
$\exists x \forall y\ P(x,y)$		
$\exists x \exists y\ P(x,y)$		

D4. Let the domain of discourse D be the set of all students at NUS, and let $M(s)$ be "s is a Math major", $C(s)$ be "s is a Computer Science major" and $E(s)$ be "s is an Engineering major". Express each of the following statements using quantifiers, variables and the predicates $M(s)$, $C(s)$ and $E(s)$. Part (a) has been done for you.

(a) Every Computer Science major is an Engineering major.

Answer: $\forall s \in D \left(C(s) \rightarrow E(s)\right)$.

Discuss: Why is the following answer wrong?

Wrong answer: $\forall s \in D \left(C(s) \land E(s)\right)$.

Can you give an example to show the difference between these two answers?

(b) No Computer Science major are Engineering majors.

(c) Some Computer Science major are not Math majors.

(d) If a student is not a Math major, then the student is either a Computer Science major or an Engineering major, but not both.

2. Common Mistakes

- Using commas (,) in place of appropriate connectives, for example, $\forall x\ P(x), Q(x)$ where it should be $\forall x \left(P(x) \land Q(x)\right)$. Note that the comma does not represent conjunction, disjunction, or any logical connective.
- Treating predicates as functions returning some value. Eg: given the following predicates
  - $Loves(x, y)$: x loves y
  - $Reindeer(x)$: x is a reindeer

  Some students wrote $Loves\big(x, Reindeer(y)\big)$ in part of their answers. Since $Reindeer(y)$ is a predicate, its value is either **true** or **false**. So, the above is akin to writing $Loves(x, true)$ or $Loves(x, false)$ which does not make sense! The correct way is to use the appropriate connectives, eg: $Reindeer(y) \land Loves(x, y)$.

3. Additional Notes

Note that "logic of quantified statements" (chapter 3) is commonly known as "predicate logic", as opposed to "propositional logic" in chapter 2.

We picked up some frequently asked questions and created this *Additional Notes* section to include some materials not covered in lecture that might be of interest to you.

Equivalent expressions: The following quantified statements are equivalent. We use the shorter notation on the left.

- $\forall x \in D, P(x) \equiv \forall x\ \big((x \in D) \rightarrow P(x)\big)$
- $\exists x \in D, P(x) \equiv\ \exists x\ \big((x \in D) \wedge P(x)\big)$

Sometimes, the comma is omitted, eg: $\forall x \in D\ P(x)$.

Well-formed formula (wff)

- **true** and **false** are wffs.
- A proposition variable (eg: x, p) is a wff.
- A predicate name followed by a list of variables (eg: $P(x)$, $Q(x, y)$), which is called an *atomic formula*, is a wff.
- If A, B and C are wffs, then so are $\sim A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.
- If x is a proposition variable and A is a wff, then so are $\forall x\ A$ and $\exists x\ A$.

Bound variables, Scope of quantifiers, and the use of Parentheses

- When a quantifier is used on a variable x in a predicate statement, we say that variable x is *bound*. If no quantifier is used on a variable, we say that the variable x is *free*.

  Examples: In $\forall x\ \exists y\ P(x, y)$, both x and y are bound. In $\forall x\ P(x, y)$, x is bound but y is free.
- The scope of a quantifier is the range in the formula where the quantifier "engages in". It is put right after the quantifier, often in parentheses (eg: $\forall x\ \big(P(x)\big)$).

  Sometimes, when parentheses are not present (eg: $\forall x\ P(x)$), the scope is understood to be the smallest wff following the quantification.
- For example, in $\exists x\ P(x, y)$, the variable x is bound while y is free. In $\forall x\big(\exists y\ P(x, y) \vee Q(x, y)\big)$, x and the red y in $P(x, y)$ are bound, but the blue y in $Q(x, y)$ is free, because the scope of $\exists y$ is only $P(x, y)$ (the smallest wff), whereas the scope of $\forall x$ is $\big(\exists y\ P(x, y) \vee Q(x, y)\big)$. If you want to change the blue y to red y, you need to add parentheses: $\forall x\ \Big(\exists y\ \big(P(x, y) \vee Q(x, y)\big)\Big)$. Then, the outermost pair of parentheses may be removed, i.e.: $\forall x\ \exists y\ \big(P(x, y) \vee Q(x, y)\big)$.

4. Tutorial Questions

1. For each of the following statements, write its **converse**, **inverse** and **contrapositive**. Indicate which among the statement, its converse, its inverse, and its contrapositive are true and which are false. Give a counterexample for each that is false. Proof not required if it is true. The predicate $Even(n)$ means that n is an even integer.

   a. $\forall n \in \mathbb{Z}\ (6|n \rightarrow 2|n \wedge 3|n)$. (Note: "$d|n$" is as defined in D1.)

   b. $\forall x\ (x \in \mathbb{Q} \rightarrow x \in \mathbb{Z})$.

   c. $\forall p, q \in \mathbb{Z}\ \big(Even(p) \wedge Even(q) \rightarrow Even(p+q)\big)$.

2. A long time ago, you already knew the following:

   a. There is no biggest number, i.e. no matter how big a number is, there is always another number that is bigger.

   b. Given any two distinct numbers, you can always find another number between them.

   Formulate these two ideas symbolically, using quantified statements. What are the domains of the numbers that make the above statements true?

3. Later in this course you will learn "relations". For a binary relation R on a set A, there are three important properties:

   a. R is **reflexive** if and only if "xRx for any x in A".

   b. R is **symmetric** if and only if "for every x and y in A, if xRy then yRx".

   c. R is **transitive** if and only if "for all x, y and z in A, if xRy and yRz, then xRz".

   For each property, rewrite the condition in "…" symbolically, using quantified statements with logical connectives instead of words such as "for all, and, if then".

4. Recall the definition of rational numbers (Lecture 1 slide 37):

   $$r \text{ is rational} \Leftrightarrow \exists a, b \in \mathbb{Z} \text{ s.t. } r = \frac{a}{b} \text{ and } b \neq 0.$$

   Prove or disprove the following statements:

   a. Integers are closed under division.

   b. Rational numbers are closed under addition.

   c. Rational number are closed under division.

5. Given $A = \{1,3,5,7,11,13\}$ and $B = \{0,2,4,6\}$, for each of the statements (a) to (j), explain whether the statement is true or false.

   a. $\forall x \in B\ \forall y \in B\ (x - y \in B)$

   b. $\forall x \in A\ \forall y \in A\ \big((x < y) \wedge (y < 10) \rightarrow y - x \in B\big)$

   c. $\forall x \in A\ \forall y \in B\ (x = y + 1)$

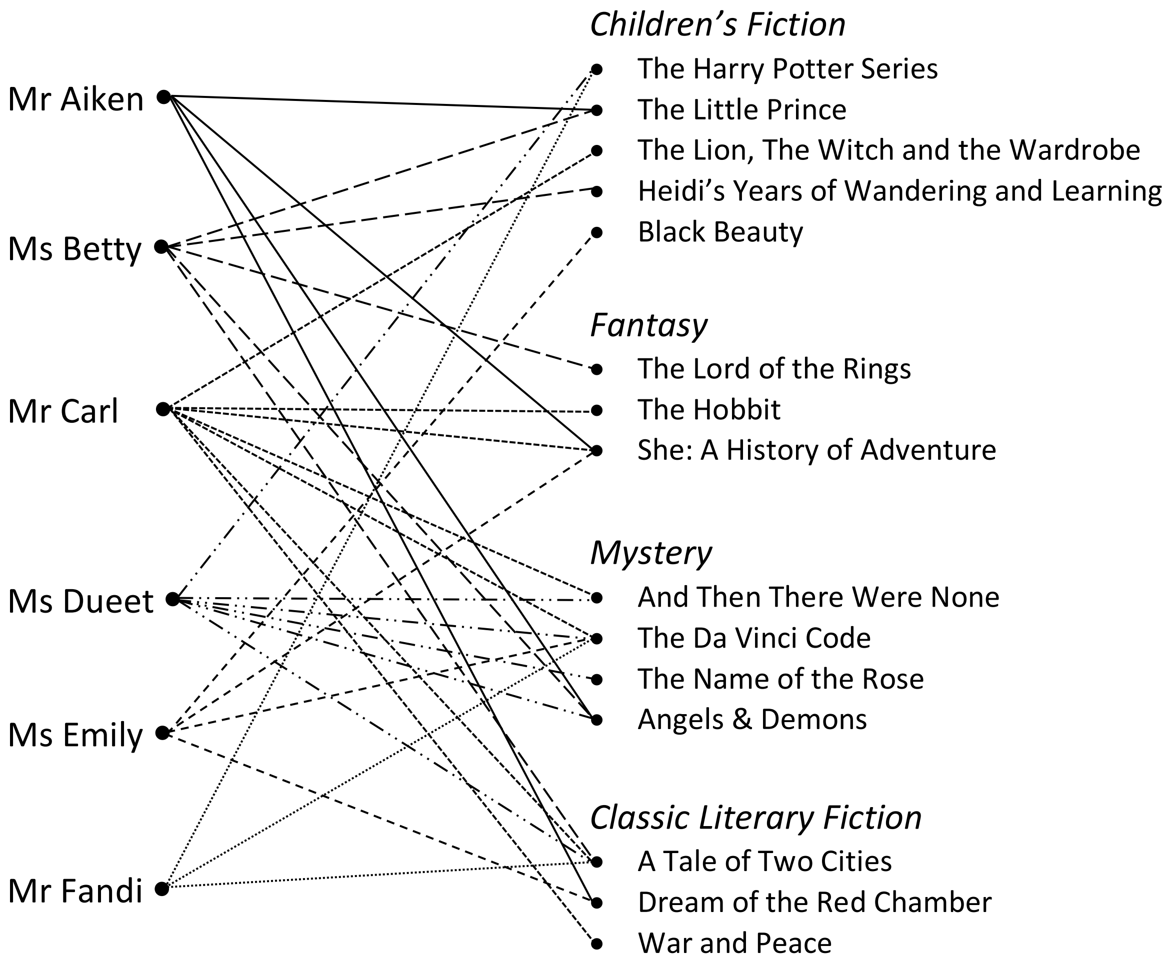
   d. $\exists x \in A\ \exists y \in B\ (x = y + 1)$

   e. $\exists y \in B\ \exists x \in A\ (x = y + 1)$

   f. $\forall x \in A\ \forall y \in B\ (x \neq y + 1)$

g. $\exists x \in A\ \exists y \in B\ (x \neq y + 1)$
h. $\forall x \in A\ \exists y \in B\ (x > y)$
i. $\forall x \in A\ \exists y \in B\ (x \leq y)$
j. $\exists y \in B\ \forall x \in A\ (x > y)$

6. Refer to the figure below, which shows six readers and four book genres (Children's Fiction, Fantasy, Mystery and Classic Literary Fiction) with selected book titles of each genre. A line is drawn between a reader and a book title if, and only if, that reader reads that book. For example, Mr Aiken reads "The Little Prince", but Ms Dueet does not read "Black Beauty".



For each of the following, indicate whether the statement is true or false and explain why. (You are not required to write the quantified statements.)

a. Some title is read by all the female readers.
b. Every reader reads some title in every genre.
c. Some reader reads all titles of some genre.
d. There is some genre for which some reader does not read any of its titles.

7. This question illustrates that one can "prove" anything, including nonsense, using bad logic.

   a. The following is a proof for: $\forall x \in \mathbb{R}\ (x^2 \geq 0)$. What is wrong with this "proof"?

      "There are 3 cases to consider: $x < 0, x = 0$ and $x > 0$. If $x < 0$, for example, $x = -3$, then $x^2 = 9 \geq 0$; if $x = 0$, then $x^2 = 0$; if $x > 0$, say $x = 4$, then $x^2 = 16 \geq 0$. Therefore, in all cases, $x^2 \geq 0$."

   b. Use the same logic in (a) to prove: $\forall x \in \mathbb{R}\ (x^3 = x)$.

   c. The following is another proof for: $\forall x \in \mathbb{R}\ (x^2 \geq 0)$. What is wrong with this "proof"?

      "Prove by contradiction. Suppose $x^2 < 0$ for all real numbers x. Let $x = 3$, then $x^2 = 9 > 0$ which is a contradiction. Therefore, $\forall x \in \mathbb{R}\ (x^2 \geq 0)$."

   d. Use the same logic in (c) to prove: $\forall x \in \mathbb{R}\ (x^3 = x)$.

8. The following is a partial proof of the claim:

$$\forall x \in \mathbb{R}\ \big((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\big).$$

1. Let r be an arbitrarily chosen real number.
2. Suppose $r^2 > r$.
2.1. Then $r^2 - r > 0$, or $r(r-1) > 0$. (by basic algebra)
2.2. So, both r and $r - 1$ are positive, or both are negative. (by Appendix A, T25)
2.3. …
3. Therefore, $\forall x \in \mathbb{R}\ \big((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\big)$.

Note: Some students drew diagrams (eg: graphs) and used them as proofs. In this module, do not use diagrams for proofs, unless otherwise instructed. If you use diagrams, you need to explain the diagrams.

   a. In step 2, we explore the case $r^2 > r$. Do we need to include the case $r^2 \leq r$? Why?

   b. Complete the proof.

   c. Step 3 is an application of **universal generalization**. Explain what it means.

9. Let V be the set of all visitors to Universal Studios Singapore on a certain day, $T(v)$ be "v took the Transformers ride", $G(v)$ be "v took the Battlestar Galactica ride", $E(v)$ be "v visited the Ancient Egypt", and $W(v)$ be "v watched the Water World show".

   Express each of the following statements using quantifiers, variables, and the predicates $T(v)$, $G(v)$, $E(v)$ and $W(v)$. The statements are not related to one another. Part (a) has been done for you.

   a. Every visitor watched the Water World show.

Answer for (a): $\forall v \in V\ (W(v))$.

   b. Every visitor who took the Battlestar Galactica ride also took the Transformers ride.

   c. There is a visitor who took both the Transformers ride and the Battlestar Galactica ride.

   d. No visitor who visited the Ancient Egypt watched the Water World show.

   e. Some visitors who took the Transformers ride also visited the Ancient Egypt but some (who took the Transformers ride) did not (visit the Ancient Egypt).

10. Given the following argument:

   1. If an object is above all the triangles, then it is above all the blue objects.
   2. If an object is not above all the gray objects, then it is not a square.
   3. Every black object is a square.
   4. Every object that is above all the gray objects is above all the triangles.

   $\therefore$ If an object is black, then it is above all the blue object.

   a. Reorder the premises in the argument to show that the conclusion follows as a valid consequence from the premises, by applying universal transitivity (Lecture 3 slide 93). (Hint: It may be helpful to rewrite the statements in if-then form and replace some statements by their contrapositives.)

      You may use self-explanatory predicate names such as $Triangle(x)$, $Square(x)$, etc.

   b. Rewrite your answer in part (a) using predicates and quantified statements.

11. [Past year's midterm test question]

   Prove that if n is a product of two positive integers a and b, then $a \leq n^{1/2}$ or $b \leq n^{1/2}$.