CS1231S: Discrete Structures

Tutorial #6: Functions

(Week 8: 6 – 10 October 2025)

I. Discussion Questions

D1. Which of the following is a function? If it is not a function, explain.

(a) Define $f: \mathbb{Z} \to \mathbb{Z}$ by $\forall z \in \mathbb{Z}, f(z) = \begin{cases} 1, & \text{if } 2 \mid z, \\ 2, & \text{if } 3 \mid z. \end{cases}$

(b) Define $f: \mathbb{Z} \to \mathbb{Z}$ by $\forall z \in \mathbb{Z}, f(z) = \begin{cases} 1, & \text{if } 2 \mid z, \\ 2, & \text{if } 2 \nmid z. \end{cases}$

(c) Define $f: \mathbb{R} \to \mathbb{Z}$ by $\forall x \in \mathbb{R}, f(x) = 2x$.

(d) Define $f: \mathbb{Z} \to \mathbb{R}$ by $\forall x \in \mathbb{Z}, f(x) = 2x$.

D2. Let function $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ be defined by setting, $\forall x. y \in \mathbb{Z}, f(x, y) = \frac{x+y}{3}$.
Find three distinct pre-images of 2.

D3. Definitions: Given any real number x,

(1) the **floor** of x, denoted $\lfloor x \rfloor$, is the unique integer n such that $n \le x < n + 1$;
(2) the **ceiling** of x, denoted $\lceil x \rceil$, is the unique integer n such that $n - 1 < x \le n$.

Let $f, g: \mathbb{Q} \to \mathbb{Q}$ be defined by setting, for each $x \in \mathbb{Q}$,

$$f(x) = \lfloor x \rfloor + 1 \quad \text{and} \quad g(x) = \lceil x \rceil.$$

What is the range of f? What is the range of g? Is $f = g$? Why?

D4. To prove that a composition of two surjections is a surjection, Aiken wrote:

1. Suppose $f: X \to Y$ and $g: Y \to Z$ are surjections.
2. Then $\forall y \in Y \; \exists x \in X$ such that $f(x) = y$ as f is surjective,
3. and $\forall z \in Z \; \exists y \in Y$ such that $g(y) = z$ as g is surjective.
4. So $(g \circ f)(x) = g\big(f(x)\big) = g(y) = z$.
5. Hence $g \circ f$ is a surjection.

Explain the mistakes in this "proof".

II. Tutorial Questions

1. Define the following relations on $\mathbb{N}$:

   $\forall x, y \in \mathbb{N}\ (x\ R_1\ y \iff x^2 = y^2)$;
   $\forall x, y \in \mathbb{N}\ (x\ R_2\ y \iff y \mid x)$;
   $\forall x, y \in \mathbb{N}\ (x\ R_3\ y \iff y = x + 1)$.

   Are the relations R_1, R_2 and R_3 functions? Prove or disprove.

2. Let $A = \{s, u\}$. Define a function $len: A^* \to \mathbb{Z}_{\geq 0}$ by setting $len(\sigma)$ to be the length of σ for each $\sigma \in A^*$.
   (a) What is $len(suu)$?
   (b) What is $len(\{\varepsilon, ss, uu, ssss\})$?
   (c) What is $len^{-1}(\{3\})$?
   (d) Does len^{-1} exist? Explain your answer.

3. Given any two bijections $f: A \to B$ and $g: B \to C$, prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

4. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here we denote by $Bool$ the set {**true**, **false**}.
   (a) $f: \mathbb{Q} \to \mathbb{Q}$;
   $x \mapsto 12x + 31$.
   (b) $g: Bool^2 \to Bool$;
   $(p, q) \mapsto p \wedge {\sim}q$.
   (c) $h: Bool^2 \to Bool^2$;
   $(p, q) \mapsto (p \wedge q, p \vee q)$.
   (d) $k: \mathbb{Z} \to \mathbb{Z}$;
   $x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$.

5. [AY2022/23 Semester 2 Exam Questions]
   The following definitions are given.

   > Given a function $f: A \to B$, we say that
   > - $g: B \to A$ is a **left inverse** of f if and only if $g(f(a)) = a$ for all $a \in A$.
   > - $h: B \to A$ is a **right inverse** of f if and only if $f(h(b)) = b$ for all $b \in B$.

   You do not need to provide proofs for the following parts.
   (a) Which of the 4 functions given in question 5 have a left inverse?
   (b) Which of the 4 functions given in question 5 have a right inverse?
   (c) Which of the following statements are true?
      (i) If a function has a left inverse, then it has a right inverse.
      (ii) If a function has a right inverse, then it has a left inverse.

6. We have shown in Theorem 7.3.3 that if $f: X \to Y$ and $g: Y \to Z$ are both injective, then $g \circ f$ is injective.

   Now, let $f: B \to C$. Suppose we have a function g with domain C such that $g \circ f$ is injective. Show that f is injective.

7. Let $A = \{1,2,3\}$. The ***order*** of a bijection $f: A \to A$ is defined to be the smallest $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \cdots \circ f}_{n\text{-many } f\text{'s}} = id_A.$$

   Define functions $g, h: A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

   Find the orders of $g, h, g \circ h$, and $h \circ g$.

8. [AY2023/24 Semester 2 Exam Questions]
   Recall Theorem 4.4.1 (The Quotient-Remainder Theorem):

   > "Given any integer n and positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \le r < d$."

   Definitions:

Let $n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$. Suppose $n = dq + r$ and $0 \le r < d$, we define $n \,\%\, d = r$.

Given a function $f(x)$, we define the function $\boldsymbol{f^{(n)}(x)}$ to be the result of n applications of f to x, where $n \in \mathbb{Z}^+$. For example, $f^{(3)}(x) = f\left(f\left(f(x)\right)\right)$. We also define the ***order*** of an input x with respect to f to be the smallest positive integer m such that $f^{(m)}(x) = x$.

   Define a function $g : A \to A$ by setting, for each $x \in A$, $g(x) = 3x \,\%\, 5$.

   (a) What is $g^{(3)}(21)$?

   (b) What is the order of 3 with respect to the function g?

   (c) Let $A = \{0,1,2,3,4\}$. Define the relation R on A as follows:

   $x \, R \, y$ iff the order of x is equal to the order of y with respect to the function g.

   R is an equivalence relation. Write out all the distinct equivalence classes of R using <u>set-roster notation</u>. Do not use the [ ] notation.

9. Let $f: A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$. Justify your answers for the following:

(a) Is it always the case that $X \subseteq \boldsymbol{f}^{-1}(\boldsymbol{f}(X))$? Is it always the case that $\boldsymbol{f}^{-1}(\boldsymbol{f}(X)) \subseteq X$?

(b) Is it always the case that $Y \subseteq \boldsymbol{f}\left(\boldsymbol{f}^{-1}(Y)\right)$? Is it always the case that $\boldsymbol{f}\left(\boldsymbol{f}^{-1}(Y)\right) \subseteq Y$?

Note that $\boldsymbol{f}(X)$ is the **(setwise) image** of X, and $\boldsymbol{f}^{-1}(Y)$ the **(setwise) preimage** of Y under f, where $X \subseteq A$ and $Y \subseteq B$. Without the colored font to disambiguate the two kinds of functions, it should also be clear what $f(U)$ denotes, depending on whether $U \in A$ or $U \in P(A)$.

10. Consider the equivalence relation $\sim$ on $\mathbb{Q}$ defined by setting, for all $x, y \in \mathbb{Q}$,

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}.$$

Define addition and multiplication on $\mathbb{Q}/\sim$ as follows: whenever $[x], [y] \in \mathbb{Q}/\sim$,

$$[x] + [y] = [x + y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

(a) Is $+$ well defined on $\mathbb{Q}/\sim$?

(b) Is $\cdot$ well defined on $\mathbb{Q}/\sim$?

Prove that your answers are correct.