CS1231S: Discrete Structures

Tutorial #9: Counting and Probability I

(Week 11: 27 – 31 October 2025)

Answers

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1. A pack of cards consists of 52 cards with 4 suits: spades (♠), hearts (♥), diamonds (♦) and clubs (♣). Each suit has 13 cards: 2, 3, 4, 5, 6, 7, 8, 9, Ten, Jack, Queen, King and Ace.

You draw a sequence of 5 cards from a pack of cards. How many sequences have at least one picture card (picture cards are Jack, Queen and King) if

(a) you draw the cards with replacement?

(b) you draw the cards without replacement?

Answers:

(a) Number of 5-card sequences = 52^5.
Number of 5-card sequences with no picture cards = 40^5.
Therefore, number of 5-card sequences with at least one picture card = $52^5 - 40^5 = \mathbf{277,804,032}$.

(b) Number of 5-card sequences = $P(52,5)$.
Number of 5-card sequences with no picture cards = $P(40,5)$.
Therefore, number of 5-card sequences with at least one picture card = $P(52,5) - P(40,5) = \mathbf{232,914,240}$.

2. [Adapted from AY2023/24 Semester 1 Exam Question]
You have written a five-digit cheque number on a blank piece of paper and passed the sheet to the bank cashier. While the cashier can recognize the number, they cannot be certain it is the correct one because the sheet lacks a clear orientation. For example, 09168 could be misread as 89160, and vice versa. How many different five-digit numbers could lead to this type of confusion for the cashier? **Assume that the handwriting matches the font we have used in this tutorial.**

Answer:

1. There are five digits that can be read both ways: 0, 1, 6, 8, 9.
2. By multiplication rule, there are in total 5^5 5-digit cheque numbers that can be read both ways.
3. There are some numbers that are ambigrams, such as 19861. It can be read the same way if we rotate it by 180 degrees. Such numbers will not lead to any confusion for the cashier.
4. Let us count the number of ambigrams. We must fix the middle digit to be either 0, 1 or 8. Thus it can be chosen in three ways. Once we choose the first digit, there is only one way to choose the last digit. Once we choose the second digit, there is only one way to choose the second-last digit. Thus, the total number of ambigrams are $5 \times 5 \times 3 \times 1 \times 1 = 75$.
5. Therefore, by Difference Rule, the total number of five-digit numbers that can be read both ways and that are not ambigrams is $5^5 - 75 = \mathbf{3050}$.

3. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 or 3 in the correct position?

   An integer k is in the correct position if it is at the k^{th} position in the permutation. For example, the permutation 3, 2, 4, 1, 5 has integers 2 and 5 in their correct positions, and the permutation 12, 1, 3, 9, 10, 8, 7, 6, 2, 4, 11, 5 has integers 3, 7, and 11 in their correct positions. Integers that are in their correct positions are underlined for illustration.

   Answer:

   1. Let $|P_k|$ be the number of permutations with integer k in its correct position.
   2. $|P_1| = |P_2| = |P_3| = (n-1)!$
   3. $|P_1 \cap P_2| = |P_2 \cap P_3| = |P_1 \cap P_3| == (n-2)!$
   4. $|P_1 \cap P_2 \cap P_3| = (n-3)!$
   5. By the inclusion/exclusion rule (theorem 9.3.3),
      $|P_1 \cup P_2 \cup P_3| = 3(n-1)! - 3(n-2)! + (n-3)! = \mathbf{(3n^2 - 12n + 13)(n-3)!}$

4. Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   (For this tutorial, use sum of a sequence to solve this problem. In the next tutorial, we will revisit this problem using a different approach.)

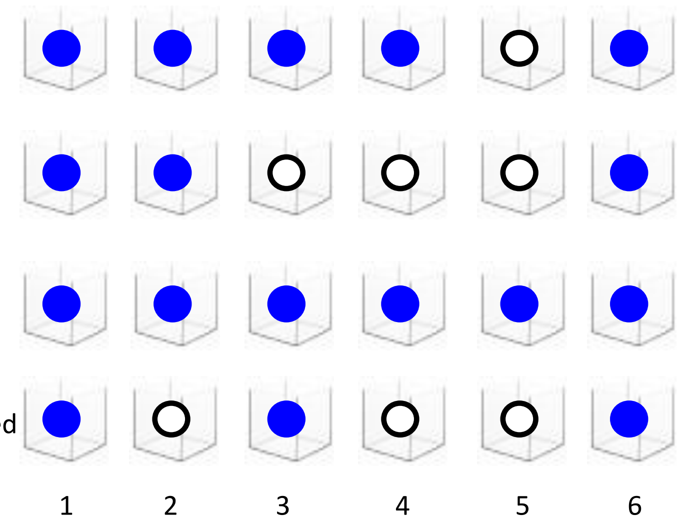
   Some examples for $n = 6$ are shown below for your reference.

Valid:

Valid:

Invalid:
(no white ball)

Invalid:
(white balls not numbered consecutively)

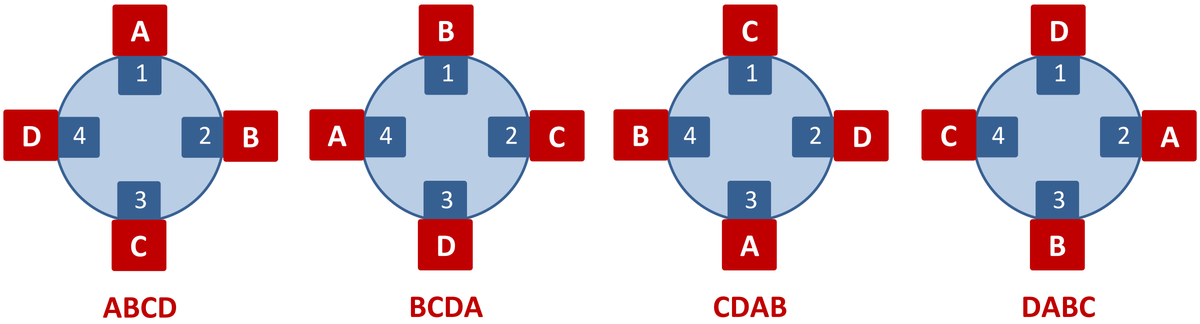


   Answer:

   1. For k $(1 \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways.
   2. Therefore, total number of ways is $\sum_{k=1}^{n}(n - k + 1) = \sum_{k=1}^{n} k = \mathbf{\frac{n(n+1)}{2}}$.

5. We have learned that the number of permutations of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chris and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

What if we seat them around a circular table? Examine the figure below.



The four seating arrangements (clockwise from top) *ABCD*, *BCDA*, *CDAB* and *DABC* are just a single permutation, as in each arrangement the persons on the left and on the right of each guest are still the same persons. Hence, these four arrangements are considered as one permutation.

This is known as *circular permutation*. The number of linear permutations of 4 persons is four times its number of circular permutations. Hence, there are $\frac{4!}{4}$ or 3! ways of circular permutations for 4 persons. In general, the number of circular permutations of n objects is $(n-1)!$

(a) In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

(b) In how many ways can 6 people sit around a circular table, but Eric would not sit next to Freddy?

(c) In how many ways can $n-1$ people sit around a circular table with n chairs?

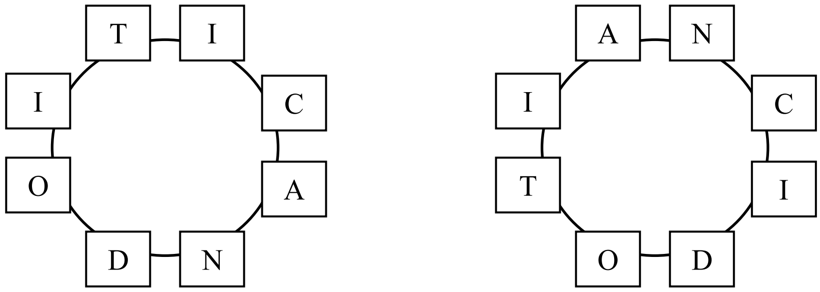
Answers:

a. 8 boys can be seated in a circle in 7! ways. There are 8 spaces between the boys, which can be occupied by 4 girls in P(8,4) ways. Hence, total number of ways = 7! × P(8,4) = 5040 × 1680 = **8467200**.

b. There are 5! = 120 ways for 6 people to sit around a circular table. There are 2 × 4! = 48 ways for Eric and Freddy to sit together. Therefore, the answer is 120 – 48 = **72** ways.

c. Treat the empty chair as just another person, therefore there are $\boldsymbol{(n-1)!}$ ways to seat $n-1$ people around a table with n chairs.

6. [AY2019/20 Semester 1 Exam Question]
You want to lay the letter tiles of these four words "I", "CAN", "DO", "IT" in a circular arrangement. The letters in the groups "CAN", "DO" and "IT" must be kept together in each group, but the letters within each group may be arranged in any order within that group. Also, no two similar letters should be placed next to each other. In how many ways can this be done? The diagram below shows two possible arrangements.



Answer:
There are two cases. (1) When "I" and "IT" are opposite of each other; then there are 2 × 3! ×2! ×2! = 48 ways. (2) When "I" and "IT" are next to each other; then the only valid arrangement between them is "ITI", and "CAN" and "DO" can be swapped, hence 2 × 3! ×2! = 24. Therefore, total = 48 + 24 = **72.**

7. [CS1231S Past Year's Exam Question]
You wish to select five persons from seven men and six women to form a committee that includes at least three men.

(a) In how many ways can you form the committee?

(b) If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men? Give your answer correct to 4 significant figures.

Answers:

(a) Let E be the event that the committee includes at least three men, that is, it has (three men and two women), or (four men and one woman) or (five men).
Therefore, $|E| = \binom{7}{3}\binom{6}{2} + \binom{7}{4}\binom{6}{1} + \binom{7}{5} = 35 \times 15 + 35 \times 6 + 21 = \mathbf{756}$.

(b) Let S be the sample space. Then $|S| = \binom{13}{5} = 1287$.
Therefore, $P(E) = \frac{|E|}{|S|} = \frac{756}{1287} = \mathbf{58.74\%}$ (or $\mathbf{0.5874}$).

8. [AY2021/22 Semester 1 Exam Question]
You have \$50,000 that you can use for investment. You are recommended 4 properties to invest in. Each investment must be in multiples of \$1000.

(a) How many different investment strategies are possible if you invest \$50,000 in total?

(b) How many different investment strategies are possible if you need not invest the entire amount of \$50,000?

Answers:

(a) Number of possible solutions to $x_1 + x_2 + x_3 + x_4 = 50$ such that $x_i \geq 0$ for $i = 1,2,3,4$ is $\binom{53}{50} = \mathbf{23426}$.

(b) Number of possible solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ such that $x_i \geq 0$ for $i = 1,2,3,4,5$ is $\binom{54}{50} = \mathbf{316251}$.

9. [AY2016/17 Semester 1 Exam Question]
Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.

Answer:

1. Divide the unit square into 25 equal smaller squares of side 1/5 each.
2. Then at least one of these small squares would contain at least three points. (Otherwise, if every square contains two points or less, the total number of points is no more than $2 \times 25 = 50$, which contradicts our assumptions that there are 51 points – Generalised PHP.)
3. Now, the circle circumvented around the small square with the three points inside also contains these three points and has radius

$$\sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^2} = \sqrt{\frac{2}{100}} = \sqrt{\frac{1}{50}} < \sqrt{\frac{1}{49}} = \frac{1}{7}$$

A common answer given by students is to divide the unit square into 16 smaller squares of side ¼ each. This is wrong. Although the area of each small square, 0.0625, is smaller than the area of a circle of radius 1/7 which is 0.0641, the small square is not entirely enclosed by the circle, because 1/8 < 1/7.

10. [AY2021/22 Semester 1 Exam Question]
Show that given any 5 distinct non-negative integers, two of them have a difference that is divisible by 4.

Answer:

1. Let $\sim_4$ denote the congruence-mod 4 relation on $\mathbb{Z}_{\geq 0}$ and let the 4 pigeonholes be the equivalence classes $[0], [1], [2]$ and $[3]$.
2. Given any 5 distinct non-negative integers, two of them will be in $[i]$, where $i \in \{0,1,2,3\}$ by the Pigeonhole Principle.
3. Let these 2 numbers be x and y. Hence, $x = 4k + i$ and $y = 4l + i$, for some integers k and l.
4. Then $x - y = (4k + i) - (4l + i) = 4(k - l)$.
5. Since $(k - l) \in \mathbb{Z}$ (by closure of integers under $-$), x and y have a difference that is divisible by 4.

11. This is the famous chess master problem to illustrate the use of the Pigeonhole Principle. Try it out yourself before googling for the answer.

A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire herself, she decides not to play more than 12 games during any one week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.

Answer:

1. There are 77 days in 11 weeks. Define P_i for $1 \leq i \leq 77$ as the total number of games the chess master has played from day 1 up to and including day i.
2. Hence $1 \leq P_1 < P_2 < \cdots < P_{77} \leq 132$ as the chess master plays at least one game per day and at most $11 \times 12 = 132$ games altogether. Note that all the P_i's are distinct.
3. Define $Q_i = P_i + 21$. Note that all the Q_i's are distinct too. Note also that the largest possible value for Q_i is $132 + 21 = 153$.
4. There are 154 numbers in the P_i's and Q_i's, but each P_i or Q_i can take a value in the range from 1 to 153 inclusive. So, there are 154 pigeons and 153 pigeonholes.
5. By PHP, two of the numbers must be equal. Hence, $P_j = Q_i = P_i + 21$ for some i, j.
6. Therefore, the chess master has played exactly 21 games in the consecutive block from day $i + 1$ to day j.