CS1231S: Discrete Structures

Tutorial #9: Counting and Probability I

(Week 11: 27 – 31 October 2025)

I. Discussion Questions

D1. A box contains three blue balls and seven white balls. One ball is drawn, its colour recorded, and it is returned to the box. Then another ball is drawn and its colour is recorded as well.

(a) What is the probability that the first ball is blue and the second is white?
(b) What is the probability that both balls drawn are white?
(c) What is the probability that the second ball drawn is blue?

D2. Calculate
(a) the probability that a randomly chosen positive three-digit integer is a multiple of 6.
(b) the probability that a randomly chosen positive four-digit integer is a multiple of 7.

D3. Write down all possible functions $\{1,2,3\} \rightarrow \{4,5\}$. How many possible functions $f: A \rightarrow B$ are there if $|A| = n$ and $|B| = k$?

D4. Assuming that all years have 365 days and all birthdays occur with equal probability. What is the smallest value for n so that in any randomly chosen group of n people, the probability that two or more persons having the same birthday is at least 50%?

Write out the equation to solve for n and write a program to compute n.
(This is the well-known *birthday problem*, whose solution is counter-intuitive but true.)

II. Tutorial Questions

1. A pack of cards consists of 52 cards with 4 suits: spades (♠), hearts (♥), diamonds (♦) and clubs (♣). Each suit has 13 cards: 2, 3, 4, 5, 6, 7, 8, 9, Ten, Jack, Queen, King and Ace.

You draw a sequence of 5 cards from a pack of cards. How many sequences have at least one picture card (picture cards are Jack, Queen and King) if

(a) you draw the cards with replacement?
(b) you draw the cards without replacement?

2. [Adapted from AY2023/24 Semester 1 Exam Question]
You have written a five-digit cheque number on a blank piece of paper and passed the sheet to the bank cashier. While the cashier can recognize the number, they cannot be certain it is the correct one because the sheet lacks a clear orientation. For example, 09168 could be misread as 89160, and vice versa. How many different five-digit numbers could lead to this type of confusion for the cashier? **Assume that the handwriting matches the font we have used in this tutorial.**

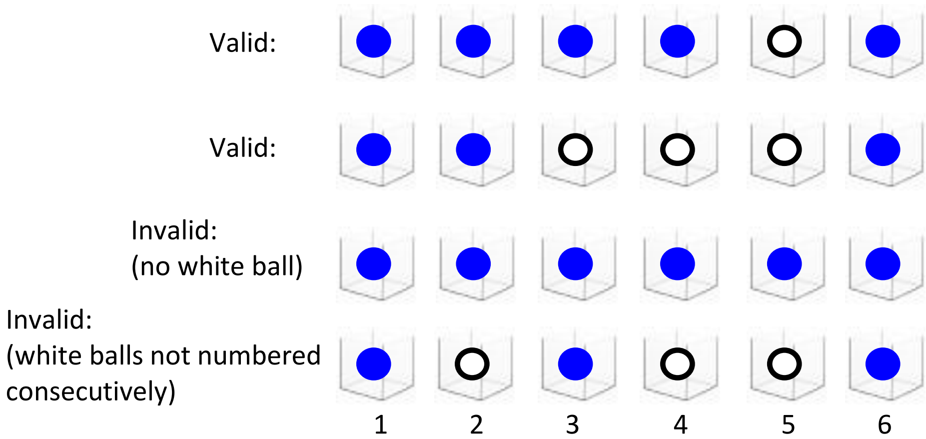
3. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 or 3 in the correct position?

   An integer k is in the correct position if it is at the k^{th} position in the permutation. For example, the permutation 3, $\underline{2}$, 4, 1, $\underline{5}$ has integers 2 and 5 in their correct positions, and the permutation 12, 1, $\underline{3}$, 9, 10, 8, $\underline{7}$, 6, 2, 4, $\underline{11}$, 5 has integers 3, 7, and 11 in their correct positions. Integers that are in their correct positions are underlined for illustration.

4. Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

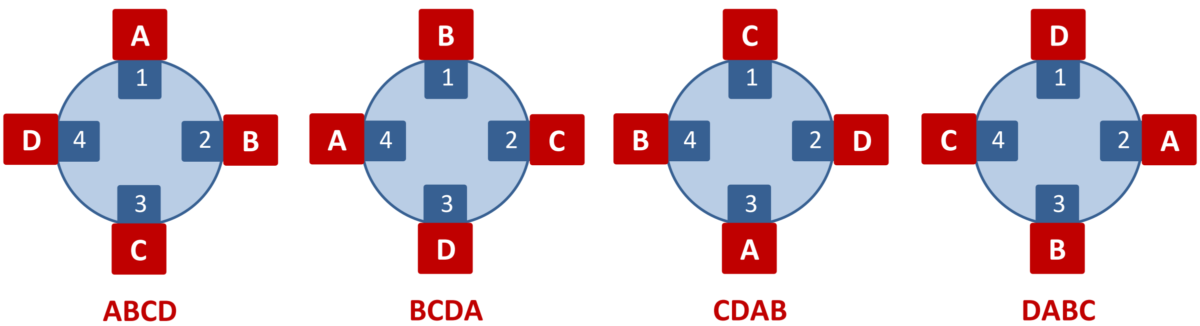
   (For this tutorial, use sum of a sequence to solve this problem. In the next tutorial, we will revisit this problem using a different approach.)

   Some examples for $n = 6$ are shown below for your reference.



5. We have learned that the number of permutations of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chris and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

   What if we seat them around a circular table? Examine the figure below.



   The four seating arrangements (clockwise from top) *ABCD*, *BCDA*, *CDAB* and *DABC* are just a single permutation, as in each arrangement the persons on the left and on the right of each guest are still the same persons. Hence, these four arrangements are considered as one permutation.

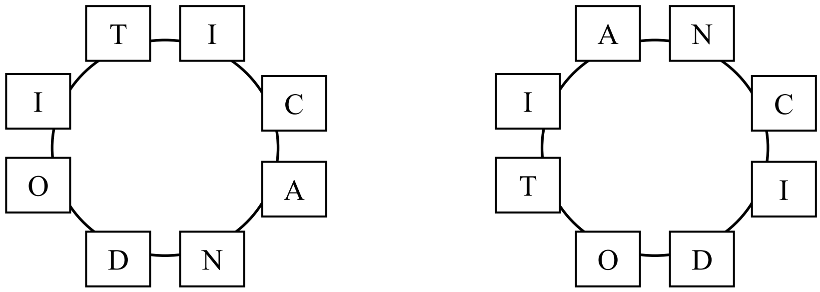
This is known as *circular permutation*. The number of linear permutations of 4 persons is four times its number of circular permutations. Hence, there are $\frac{4!}{4}$ or $3!$ ways of circular permutations for 4 persons. In general, the number of circular permutations of n objects is $(n-1)!$

(a) In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

(b) In how many ways can 6 people sit around a circular table, but Eric would not sit next to Freddy?

(c) In how many ways can $n-1$ people sit around a circular table with n chairs?

6. [AY2019/20 Semester 1 Exam Question]
You want to lay the letter tiles of these four words "I", "CAN", "DO", "IT" in a circular arrangement. The letters in the groups "CAN", "DO" and "IT" must be kept together in each group, but the letters within each group may be arranged in any order within that group. Also, no two similar letters should be placed next to each other. In how many ways can this be done? The diagram below shows two possible arrangements.



7. [CS1231S Past Year's Exam Question]
You wish to select five persons from seven men and six women to form a committee that includes at least three men.

(a) In how many ways can you form the committee?

(b) If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men? Give your answer correct to 4 significant figures.

8. [AY2021/22 Semester 1 Exam Question]
You have $50,000 that you can use for investment. You are recommended 4 properties to invest in. Each investment must be in multiples of $1000.

(a) How many different investment strategies are possible if you invest $50,000 in total?

(b) How many different investment strategies are possible if you need not invest the entire amount of $50,000?

9. [AY2016/17 Semester 1 Exam Question]
   Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.

10. [AY2021/22 Semester 1 Exam Question]
    Show that given any 5 distinct non-negative integers, two of them have a difference that is divisible by 4.

11. This is the famous chess master problem to illustrate the use of the Pigeonhole Principle. Try it out yourself before googling for the answer.

    A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire herself, she decides not to play more than 12 games during any one week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.