CS1231S: Discrete Structures

Tutorial #10: Counting and Probability II and Graphs I

(Week 12: 3 – 7 November 2025)

I. Discussion Questions

D1. Suppose a random sample of 2 lightbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent of the number of defective bulbs that occur on a given trial, where $X = 0,1,2$. Find $E[X]$.

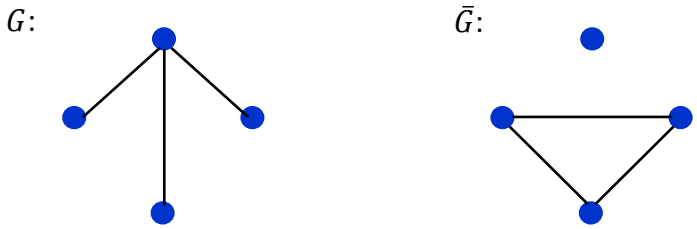
D2. How many **injective functions** are there from a set A with m elements to a set B with n elements, where $m \leq n$?

D3. How many **surjective functions** are there from a 5-element set A to a 3-element set B?

II. Definitions

Definition 1. If G is a simple graph, the ***complement*** of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

Definition 2. A ***self-complementary*** graph is isomorphic with its complement.

Definition 3. A simple circuit (cycle) of length three is called a ***triangle***.

III. Tutorial Questions

1. Find the term independent of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^9$.

2. Let's revisit Tutorial #9 Question 4:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n - k + 1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?

3. [AY2020/21 Semester 2 Exam Question]
   On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

   (i) What is the probability of rolling a 6? Write your answer as a single fraction.

   (ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction.

4. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

5. [AY2021/22 Semester 2 Exam Question]
   A rare disease broke out in a city with a prevalence of 0.1%, that is, it affects 1 out of every 1000 persons. A quick test kit has been developed that has a sensitivity of 85%, which is the probability that a person with the rare disease is tested positive. Among those who took the test, 10% of the time it came out positive. Write your answers correct to 3 significant figures.

   (a) Divoc has shown symptoms of the disease. Should he be tested positive, what is the probability that he actually has the disease?

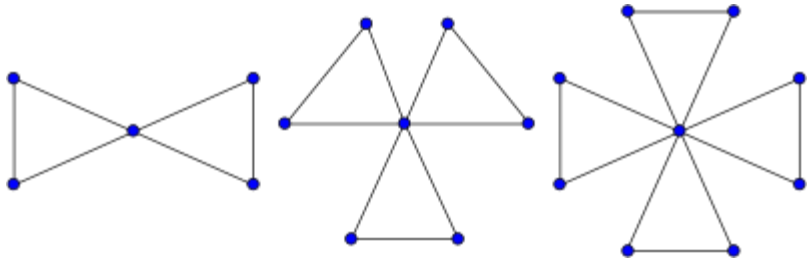
   (b) What is the probability of a false positive result, that is, a person does not have the disease but is tested positive?

6. [AY2015/16 Semester 1 Exam Question]
Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($P(S)$ denotes the powerset of S.)

Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

Can you generalize your answer to any set A with n elements?

7. The **friendship graph** (or **Dutch windmill graph** or ***n*-fan**) F_n is a planar, undirected graph with $2n + 1$ vertices and $3n$ edges. (Ref: https://en.wikipedia.org/wiki/Friendship_graph)

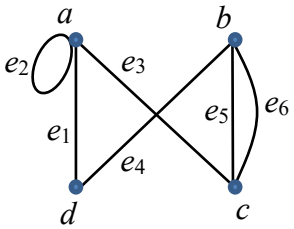


F_2 F_3 F_4

Is the friendship graph Eulerian? Is it Hamiltonian?

8. [Past year's exam question]
How many non-isomorphic undirected graphs with two vertices and two edges are there? Draw them.

9. Given the graph shown below:



(a) Give the adjacency matrix A for the graph, with vertices in the order a, b, c, d.

(b) Compute A^0, A^2 and A^3.

(c) How many walks of length 2 are there from a to b? From c to itself? List out all the walks.

(d) How many walks of length 3 are there from a to c? List out all the walks.

10. A lady hosted a party of n $(n \geq 2)$ people (including herself). At the party, various friends met and some of them shook hands with each other. The thoughtful host made sure that she shook hands with everyone at the party.

    Prove that there are at least two people who have shaken hands the same number of times.

11. [AY2021/22 Semester 1 Exam]
    A set of vertices, D, in an undirected simple graph is said to be a ***dominating set*** if every vertex not in D is adjacent to at least one vertex in D. A ***minimal dominating set*** is a dominating set such that none of its proper subsets are dominating sets.

    (a) Draw two non-isomorphic simple graphs with four vertices that have minimal dominating sets of size one. Highlight the vertices in the minimal dominating set in your graphs.

    (b) Draw two non-isomorphic simple graphs with four vertices that have minimal dominating sets of size three. Highlight the vertices in the minimal dominating set in your graphs.

12. Let G be any simple graph with 6 vertices. Prove that G or its complementary graph $\bar{G}$ contains a triangle. (This is similar to the problem of showing that in any group of 6 people, there must be either 3 mutual friends or 3 mutual strangers. But for this question, please use the graph formulation.)