

CS1231S: Discrete Structures
Tutorial #8: Cardinality and Revision
(Week 10: 19 – 23 October 2026)

I. Discussion Questions

- D1 Is the set of perfect squares $\{0, 1, 4, 9, 16, \dots\}$ countable? Prove or disprove it.
- D2. Aiken spoke about a set being “uncountable and infinite”. Dueet commented that Aiken must have meant “uncountably infinite.” Comment on what Aiken and Dueet said.
- D3. [AY2021/22 Semester 2 Exam Multiple-Response Question].
Which of the following sets are countable?
- A. The set A of all points in the plane with rational coordinates.
 - B. The set B of all infinite sequences of integers.
 - C. The set C of all functions $f: \{0, 1\} \rightarrow \mathbb{N}$.
 - D. The set D of all functions $f: \mathbb{N} \rightarrow \{0, 1\}$.
 - E. The set E of all 2-element subsets of $\mathbb{N}$.

II. Tutorial Questions

1. In lecture example #3, we showed that $\mathbb{Z}$ is countable by defining a bijection $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even;} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The above is based on the definition $\aleph_0 = |\mathbb{Z}^+|$. Suppose we adopt the definition $\aleph_0 = |\mathbb{N}|$ instead, define a bijection $g: \mathbb{N} \rightarrow \mathbb{Z}$ using a single-line formula to show that $\mathbb{Z}$ is countable.

2. Let B be a countably infinite set and C a finite set. Show that $B \cup C$ is countable
- (a) by using the sequence argument;
 - (b) by defining a bijection $g: \mathbb{N} \rightarrow B \cup C$.

3. Recall the definition of $\bigcup_{i=m}^n A_i$ in Tutorial 3.

- (a) Consider this claim:

“Suppose $A_1, A_2, \dots$ are finite sets. Then $\bigcup_{i=1}^n A_i$ is finite for any $n \geq 2$.”

The above statement is true. However, consider the following “proof”:

“We will prove by induction on n . Since A_1 and A_2 are finite, then $A_1 \cup A_2$ is finite, so the claim is true for $n = 2$. Now suppose the claim is true for $n = k$, so $\bigcup_{i=1}^k A_i$ is finite. Let $A_{k+1} = \emptyset$. Then $\bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^k A_i) \cup A_{k+1} = \bigcup_{i=1}^k A_i$ which is finite by the induction hypothesis, so the claim is true for $n = k + 1$. Therefore, the claim is true for all $n \geq 2$.”

What is wrong with this “proof”?

- (b) Disprove the following: “Suppose $A_1, A_2, \dots$ are finite sets. Then $\bigcup_{k=1}^{\infty} A_k$ is finite.”
[The point here is: induction takes you to any finite n , but not to infinity.]

4. Suppose $A_1, A_2, A_3, \dots$ are countable sets.
 - (a) Prove, by induction, that $\bigcup_{i=1}^n A_i$ is countable for any $n \in \mathbb{Z}^+$.
 - (b) Does (a) prove that $\bigcup_{i=1}^{\infty} A_i$ is countable?
5. Let S_i be a countably infinite set for each $i \in \mathbb{Z}^+$. Prove that $\bigcup_{i \in \mathbb{Z}^+} S_i$ is countable.
[Hint: Use this theorem covered in class: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.]
6. Let B be a (not necessarily countable) infinite set and C be a finite set. Define a bijection $B \cup C \rightarrow B$.
7. Let A be a countably infinite set. Prove that $\mathcal{P}(A)$ is uncountable. ($\mathcal{P}(A)$ is the power set of A .)

8. [AY2022/23 Semester 1 Exam]

Given the following statements on any finite set A ,

- (i) If R is a reflexive relation on A , then $|A| \leq |R|$.
- (ii) If R is a symmetric relation on A , then $|A| \leq |R|$.
- (iii) If R is a transitive relation on A , then $|A| \leq |R|$.

Prove or disprove each of the statements.

9. [AY2022/23 Semester 2 Exam]

The Fibonacci sequence F_n is defined for $n \in \mathbb{N}$ as follows:

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 1.$$

Prove by Mathematical Induction the following statement:

$$\text{Even}(F_n) \Leftrightarrow \text{Even}(F_{n+3}), \forall n \in \mathbb{N}.$$

The predicate $\text{Even}(x)$ is true when the integer x is even and false otherwise. You may use the following fact in your proof:

$$\text{Fact 1: } \text{Even}(x + y) \Leftrightarrow (\text{Even}(x) \Leftrightarrow \text{Even}(y))$$

10. [AY2022/23 Semester 2 Exam]

- (a) Let $\sim$ be an equivalence relation on X and let $g : X \rightarrow Y$ be a function such that

$$g(a) = g(b) \Leftrightarrow a \sim b \quad \forall a, b \in X.$$

Prove or disprove the following statement:

The following function f is well-defined:

$$f : X/\sim \rightarrow Y \text{ given by the formula } f([x]) = g(x) \quad \forall x \in X.$$

- (b) If the function f in part (a) above is well-defined, prove or disprove whether function f is injective or not injective.

If the function f in part (a) above is not well-defined, would changing the function g in part (a) to

$$g(a) = g(b) \Leftrightarrow a \not\sim b \quad \forall a, b \in X. \text{ (Note: } \not\sim \text{ is the negation of } \sim.)$$

make the function f in part (a) well-defined? Prove or disprove.

- (c) Given the following bijections f and g on the set $A = \{1, 2, 3, 4\}$,

$$f = \{(1, 2), (2, 4), (3, 1), (4, 3)\};$$

$$g = \{(1, 4), (2, 1), (3, 2), (4, 3)\}.$$

Find the order of f , g and $(f^{-1} \circ g)$.