NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING
EXAMINATION FOR

SEMESTER 1 2008/2009

CS 3234 - LOGIC AND FORMAL SYSTEMS

November 2008 **Time Allowed: 2 Hours**

INSTRUCTIONS TO CANDIDATES

This is an open book examination. Any written and printed material may be used during the examination.

Please observe the following items:

1. Enter your matriculation number here:

2. Enter the answers to the questions in this answer book in the provided spaces.
3. This examination booklet has 13 pages, including this cover page, and contains 8 questions.
4. Answer all questions.
5. The maximal marks achievable is indicated for each question. Overall, the maximal number of marks is 50.

Good Luck !

Do not write below this line

Q#	1	2	3	4	5	6	7	8	Σ
Max	5	8	9	20	12	12	6	8	80
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Propositional Logic

Question 1: (5 marks) Prove the following sequent using the basic rules of natural deduction:

$$p, r \rightarrow \neg p \vdash \neg r$$

Question 2: (8 marks) A formula in propositional logic is called *linear* if any propositional atom occurs at most once. All linear formulas are satisfiable.
Give an algorithm in pseudo-code that finds a satisfying assignment for a given linear formula ϕ, and that runs in $O(n)$ where n is the size of ϕ.

Predicate Calculus

Question 3: (9 marks) Consider a set $\mathcal{P} = \{P\}$ containing one binary predicate symbol P, and the set $\mathcal{F} = \{null\}$ containing one nullary function symbol *null*. Consider a model $\mathcal{M}$ whose universe is the set of non-negative integers, and where $null^{\mathcal{M}}$ is the integer 0.

- (3 marks) Give an interpretation $P^{\mathcal{M}}$ for P such that $\mathcal{M}$ satisfies the formula:

$$\forall x((\neg \exists y P(y,x)) \rightarrow x = null)$$

- (3 marks) Give an interpretation $P^{\mathcal{M}}$ for P such that $\mathcal{M}$ satisfies the formula:

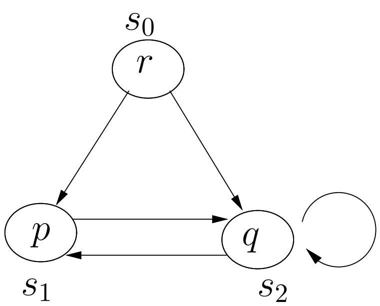
$$\forall x \exists y(P(x,y) \wedge P(y,x)) \wedge \forall x \forall y(P(x,y) \rightarrow (\forall z(P(z,x) \rightarrow y = z)))$$

- (3 marks) Give an interpretation $P^{\mathcal{M}}$ for P such that $\mathcal{M}$ satisfies the formula:

$$\begin{array}{l}\forall x \exists y P(x, y) \wedge \\ \forall x \forall y \forall z (P(x, y) \wedge P(x, z) \rightarrow y = z) \wedge \\ \forall x \forall y \forall z (P(x, y) \wedge P(y, z) \rightarrow P(z, x))\end{array}$$

Verification by Model Checking

Question 4: (20 marks) Consider the transition system $\mathcal{M}$ depicted in the following diagram:



- For the following LTL formulas ϕ, does $\mathcal{M}, s_0 \models \phi$ hold?

  If yes, give a short justification. If no, give a counterexample.

  - (2 marks) $F\ q$

  - (2 marks) $F\ G\ q$

– (2 marks) $G\ F\ q$

– (2 marks) $(p\ \vee\ X\ p)\ U\ q$

– (2 marks) $(r \vee p)\ U\ G\ q$

- For the following CTL formulas ϕ, does $\mathcal{M}, s_0 \models \phi$ hold? Give a short justification of your answer.
  - (2 marks) $EG\ AF\ p$
  - (2 marks) $AF\ E[p\ U\ q]$
  - (2 marks) $EG\ E[q\ U\ p]$
  - (2 marks) $EF\ (q \wedge \neg EX\ q)$
  - (2 marks) $AF\ (q \wedge \neg AX\ q)$

Program Verification

Question 5: (12 marks)

- (4 marks) Consider the following program `plusabs` in the core programming language:

```
if (b > 0) {
   c = a + b;
} else {
   c = a - b;
}
```

Give a proof for the following Hoare triple.

$$\vdash_{\texttt{par}} (\!|\top|\!)\ \texttt{plusabs}\ (\!|c = a + |b||\!)$$

Recall that a proper proof in the proof calculus annotates every line with the name of the rule applied to derive that line. Indicate

- (8 marks) Consider the following program `square` in the core programming language:

```
a = 0;
b = x;
while (b > 0) {
   a = a + x;
   b = b - 1;
}
```

Give a proof for the following Hoare triple.

$$\vdash_{\texttt{tot}} (\!|x > 0|\!)\ \texttt{square}\ (\!|a = x^2|\!)$$

Recall that a proper proof in the proof calculus annotates every line with the name of the rule applied to derive that line. Indicate clearly what *variant* and *invariant* you are using.

Modal Logic

Question 6: (12 marks) Consider all frames in which

$$\Diamond\phi \to \Box\phi$$

holds.

- (3 marks) Give a necessary and sufficient condition on these frames' accessibility relation R. Be precise and formal in your language, and do not use terms such as "transitive" or "linear" without defining them.

- (9 marks) Prove your claim in detail.

Question 7: (6 marks) Consider the following modified scenario of the "wise-men puzzle":
The following scenario is common knowledge among the wise men:

> There are three red hats and two white hats. The king puts a hat on each wise man so that they are not able to see their own hat, but each sees both of the other men's hats. The king asks each one in turn whether they know the colour of the hat on their head. **The second man is deaf and thus does not come to know the first man's answer.**

All three men answer "no".

- (3 marks) Now the king asks the first man again if he knows the colour of his hat. What is his answer? Why?

- (3 marks) The formula set Γ in KT45^n that describes the scenario before the men answer remains unchanged:

$$\begin{aligned}\Gamma \quad = \{ \quad & C(p_1 \vee p_2 \vee p_3), \\ & C(p_1 \rightarrow K_2 p_1), C(\neg p_1 \rightarrow K_2 \neg p_1), \\ & C(p_1 \rightarrow K_3 p_1), C(\neg p_1 \rightarrow K_3 \neg p_1), \\ & C(p_2 \rightarrow K_1 p_2), C(\neg p_2 \rightarrow K_1 \neg p_2), \\ & C(p_2 \rightarrow K_3 p_2), C(\neg p_2 \rightarrow K_3 \neg p_2), \\ & C(p_3 \rightarrow K_1 p_3), C(\neg p_2 \rightarrow K_1 \neg p_3), \\ & C(p_3 \rightarrow K_2 p_3), C(\neg p_2 \rightarrow K_2 \neg p_3)\}\end{aligned}$$

  What formula can we add to Γ to represent the first man's answer "no"?

Binary Decision Diagrams

Question 8: (8 marks)

- (4 marks) The size of a reduced OBDD depends on the chosen variable ordering. Give a minimal-size reduced OBDD for the following boolean function:

$$f(x, y, z) = (\overline{x} + \overline{y}) \cdot z$$

- (4 marks) Reduced OBDDs for *compatible* variable orderings representing a given function have identical structure. For some boolean functions, **all** OBDDs have identical structure, regardless of the variable ordering.

  Give a sufficient and necessary condition for such boolean functions, or the resulting OBDDs.

END OF QUESTIONS