Predicate Logic (1)

- ★ The need for a richer language
- ★ Predicate logic as a formal language
  - terms —variables, functions
  - formulas —predicates, quantifiers
  - free and bound variables
  - substitution
- ★ Proof theory of predicate logic
  - Natural deduction rules

The Need for a Richer Language

Propositional Logic:

- Study of declarative sentences, statements about the world which can be given a truth value
- Dealt very well with sentence components like: ***not***, ***and***, ***or***, ***if*** $\cdots$ ***then*** $\cdots$
- **Limitations:** cannot deal with modifiers like ***there exists***, ***all***, ***among***, ***only***.

Example: ***"Every student is younger than some instructor."***

- We could identify the entire phrase with the propositional symbol p.
- However, the phrase has a finer logical structure. It is a statement about the following properties:
  - being a student
  - being an instructor
  - being younger than somebody else

Predicates, Variables and Quantifiers

Properties are expressed by predicates. S, I, Y are ***predicates***.

$S(andy)$: Andy is a ***student***.
$I(paul)$: Paul is an ***instructor***.
$Y(andy, paul)$: Andy is ***younger than*** Paul.

Variables are placeholders for concrete values.

$S(x)$: x is a student.
$I(x)$: x is an instructor.
$Y(x, y)$: x is younger than y.

Quantifiers make possible encoding the phrase:

"Every student is younger than some instructor."

Two quantifiers: $\forall$ —***forall***, and $\exists$ —***exists***.

Encoding of the above sentence:

$$\forall x(S(x) \to (\exists y(I(y) \land Y(x, y))))$$

More Examples

"No books are gaseous. Dictionaries are books. Therefore, no dictionary is gaseous."

We denote: $B(x)$: x is a book
$G(x)$: x is gaseous
$D(x)$: x is a dictionary

$$\neg\exists x\,(B(x) \wedge G(x)), \forall x\,(D(x) \rightarrow B(x))$$
$$\vdash$$
$$\neg\exists x\,(D(x) \wedge G(x))$$

"Every child is younger than his mother"

We denote: $C(x)$: x is a child
$M(x,y)$: x is y's mother

$$\forall x \forall y\,(C(x) \wedge M(x,y) \rightarrow Y(x,y))$$

Denote $m(x)$: mother of x

$$\forall x\,(C(x) \rightarrow Y(x, m(x)))$$

Using the function m to encode the "mother of" relationship is more appropriate, since every person has a unique mother.

More Examples (2)

"Andy and Paul have the same maternal grandmother"

$$\forall x \forall y \forall u \forall v (M(x,y) \wedge M(y,a) \wedge M(u,v) \wedge M(v,p) \rightarrow x = u)$$

We have introduced a new, special predicate: ***equality***.

Alternative representation:

$$m(m(a)) = m(m(p))$$

Consider the relationship $B(x,y)$: x is the brother of y. This relationship must be encoded as a predicate, since a person may have more than one brother.

Predicate Logic as a Formal Language

Two sorts of "things" in a predicate formula:

- Objects such as a (Andy) and p (Paul). Function symbols also refer to objects. These are modeled by ***terms***.
- Expressions that can be given truth values. These are modeled by ***formulas***.

A predicate vocabulary consists of 3 sets:

- ***Predicate symbols*** $\mathcal{P}$;
- ***Function symbols*** $\mathcal{F}$;
- ***Constants*** C.

Each predicate and function symbol comes with a fixed ***arity*** (i.e. number of arguments)

Elements of the formal language of predicate logic:

- Terms
- Formulas
- Free and bound variables
- Substitution

Terms

Definition: ***Terms*** are defined as follows:

- Any variable is a term;
- Any constant in $\mathcal{C}$ is a term;
- If t_1,...,t_n are terms and $f \in \mathcal{F}$ has arity n, then $f(t_1, \ldots, t_n)$ is a term;
- Nothing else is a term.

Backus-Naur definition: $t \ ::= \ x \,|\, c \,|\, f(t, \ldots, t)$ where x represents variables, c represents constants in $\mathcal{C}$, and f represents function in $\mathcal{F}$ with arity n.

Remarks:

- The first building blocks of terms are constants and variables.
- More complex terms are built from function symbols using previously buit terms.
- The notion of terms is independent on the sets $\mathcal{C}$ and $\mathcal{F}$.

Formulas

Definition: We define the set of ***formulas*** over $(\mathcal{F}, \mathcal{P})$ inductively, using the already defined set of terms over $\mathcal{F}$.

- If P is a predicate with $n \geq 1$ arguments, and $t_1,\ldots,t_n$ are terms over $\mathcal{F}$, then $P(t_1,\ldots,t_n)$ is a formula.
- If Φ is a formula, then so is $\neg\Phi$.
- If Φ and Ψ are formulas, then so are $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \rightarrow \Psi$.
- If Φ is a formula and x is a variable, then $\forall x \Phi$ and $\exists x \Phi$ are formulas.
- Nothing else is a formula.

BNF definition:

$$\Phi ::= P(t_1,\ldots t_n) \mid (\neg\Phi) \mid (\Phi \wedge \Phi) \mid (\Phi \vee \Phi) \mid (\Phi \rightarrow \Phi) \mid (\forall x \Phi) \mid (\exists x \Phi)$$

where P is a predicate of arity n, t_i are terms, $i \in \{1,\ldots,n\}$, x is a variable.

Convention: We retain the usual binding priorities of the connectives $\neg$,$\wedge$, $\vee$, $\rightarrow$. We add that $\forall x$ and $\exists x$ bind like $\neg$.

Example

Consider translating the sentence:

"Every son of my father is my brother"

Two alternatives:

- ***"Father of"*** relationship encoded as a predicate.

  $S(x,y)$: x is the son of y.

  $F(x,y)$: x is the father of y.

  $B(x,y)$: x is the brother of y.

  m: constant, denoting "myself".

  Translation: $\forall x \forall y\, (F(x,m) \wedge S(y,x) \rightarrow B(y,m))$

- ***"Father of"*** relationship encoded as a function.

  $f(x)$: father of x.

  Translation: $\forall x\, (S(x, f(m)) \rightarrow B(x,m))$

Free and Bound Variables

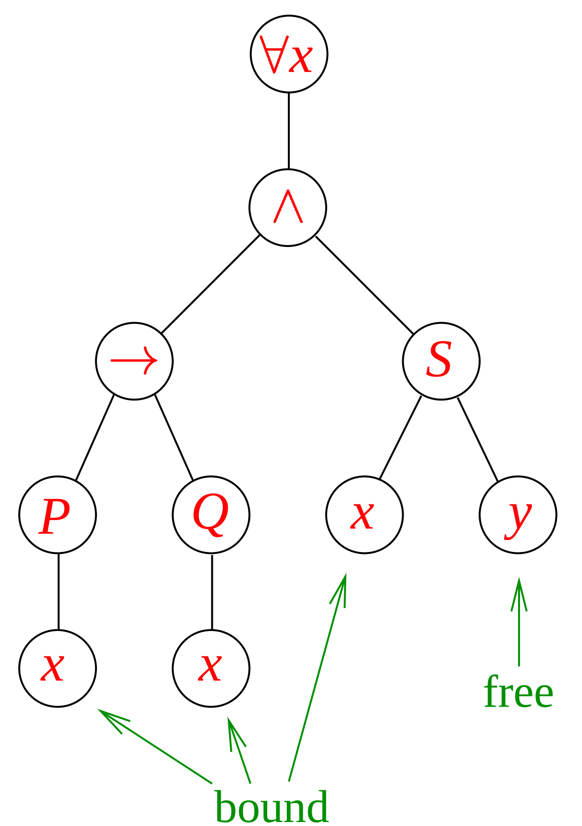
Definition: Let Φ be a formula in predicate logic. An occurrence of x in Φ is ***free in*** Φ if it is a leaf node in the parse tree of Φ such that there is no path upwards from that node x to a node $\forall x$ or $\exists x$. Otherwise, that occurrence x is called ***bound***. For $\forall x\,\Phi$, we say that Φ —minus any of its sub-formulas $\exists x\,\Psi$, or $\forall x\,\Psi$ —is the scope of $\forall x$, respectively $\exists x$.

Formula:

$$\forall x\,\underbrace{((P(x) \rightarrow Q(x)) \wedge S(x,y))}_{\text{Scope of } \forall x}$$

x is bound.

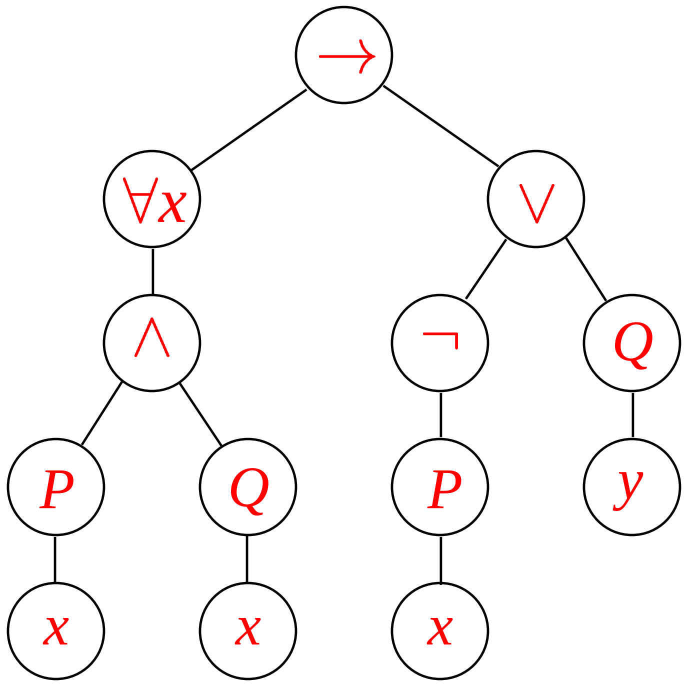
y is free.



Examples of Free and Bound Variables

Formula: $(\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y))$

Parse tree:



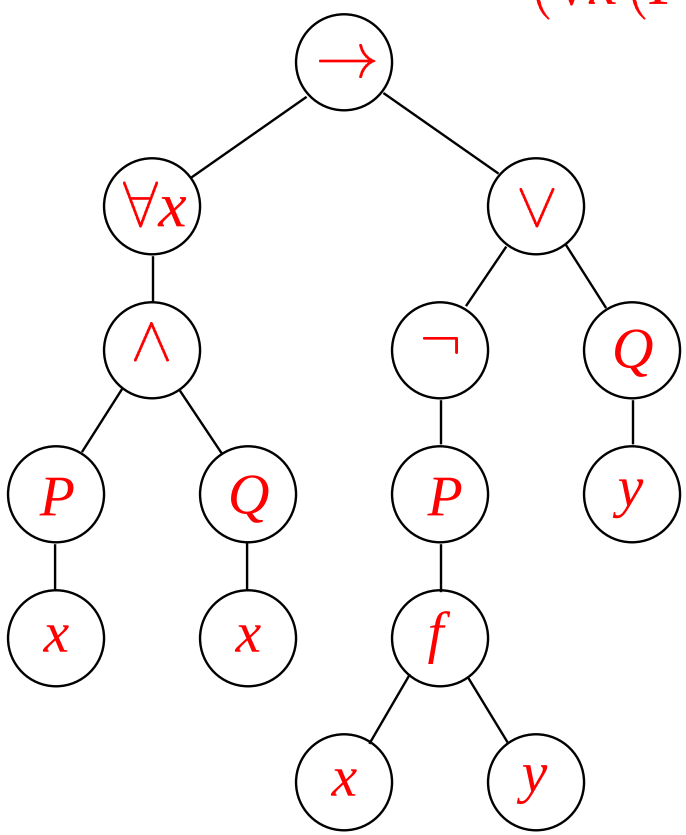
Substitution

Variables are placeholders, so we must have means of ***replacing*** them with more concrete information.

Definition: Given a variable x, a term t, and a formula Φ, we define $\Phi[t/x]$ to be the formula obtained by replacing each free occurrence of variable x in Φ with t.

$((\forall x\,(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y)))[f(x,y)/x]$ is

$$(\forall x\,(P(x) \wedge Q(x))) \rightarrow (\neg P(f(x,y)) \vee Q(y))$$



Substitution (2)

Definition: Given a term t, a variable x, and a formula Φ, we say that t ***is free for*** x ***in*** Φ if no free x leaf in Φ occurs in the scope of $\forall y$ or $\exists y$, for every variable y occurring in t.

Remark: If t is not free for x in Φ, then the substitution $\Phi[t/x]$ has unwanted effects.

Example:

$$(S(x) \land (\forall y\, (P(x) \to Q(y))))[y/x] \text{ is } S(y) \land (\forall y\, (P(y) \to Q(y)))$$

Avoid this by renaming $\forall y$ into $\forall z$.

$$(S(x) \land (\forall z\, (P(x) \to Q(z))))[y/x] \text{ is } S(y) \land (\forall z\, (P(y) \to Q(z)))$$

Proof Theory of Predicate Logic

- Natural deduction rules for propositional logic are still valid
- Natural deduction rules for predicate logic:
  - proof rules from propositional logic;
  - proof rules for equality;
  - proof rules for universal quantification;
  - proof rules for existential quantification.
- Quantifier equivalences

Proof Rules for Equality

$$\frac{}{t = t} \; =\mathrm{i} \qquad\qquad \frac{t_1 = t_2 \quad \Phi[t_1/x]}{\Phi[t_2/x]} \; =\mathrm{e}$$

Convention: When we write a substitution in the form $\Phi[t/x]$, we implicitly assume that t is free for x in Φ.

Proof example:

$$x+1 = 1+x, (x+1>1) \to (x+1>0) \vdash (1+x>1) \to (1+x>0)$$

1	$x+1 = 1+x$	premise
2	$(x+1>1) \to (x+1>0)$	premise
3	$(1+x>1) \to (1+x>0)$	$=$e 1,2

Proof Rules for Universal Quantification

$$\frac{\forall x\, \Phi}{\Phi[t/x]}\ \forall x\,\mathrm{e}$$

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad \Phi[x_0/x] \end{array}}}{\forall x\, \Phi}\ \forall x\,\mathrm{i}$$

Proof examples:

$\forall x\,(P(x) \to Q(x)), \forall x\, P(x) \vdash \forall x\, Q(x)$

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\forall x\, P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	$\forall x$ e 1
4		$P(x_0)$	$\forall x$ e 2
5		$Q(x_0)$	$\to$e 3,4
6		$\forall x\, Q(x)$	$\forall x$ i 3-5

$P(t), \forall x\,(P(x) \to \neg Q(x)) \vdash \neg Q(t)$

1	$P(t)$	premise
2	$\forall x\,(P(x) \to \neg Q(x))$	premise
3	$p(t) \to \neg Q(t)$	$\forall x$ e 2
4	$\neg Q(t)$	$\to$ e 3,1

Proof Rules for Existential Quantification

$$\frac{\Phi[t/x]}{\exists x\,\Phi}\ \exists x\,\mathrm{i}
\qquad\qquad
\frac{\exists x\,\Phi \qquad \boxed{\begin{array}{ll} x_0 & \Phi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists x\,\mathrm{e}$$

Side condition: x_0 doesn't occur in χ

Proof examples:

$$\forall x\,\Phi \vdash \exists x\,\Phi$$

1	$\forall x\,\Phi$	premise
2	$\Phi[x/x]$	$\forall x$ e 1
3	$\exists x\,\Phi$	$\exists x$ i 2

$$\forall x\,(P(x) \to Q(x)), \exists x\,P(x) \vdash \exists x\,Q(x)$$

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\exists\,P(x)$	premise
3	x_0	$P(x_0)$	assumption
4		$P(x_0) \to Q(x_0)$	$\forall x$ e 1
5		$Q(x_0)$	$\to$ e 4,3
6		$\exists x\,Q(x)$	$\exists x$ i 5
7		$\exists x\,Q(x)$	$\exists x$ e 2,3–6

Another Example

$$\exists x\, P(x), \forall x \forall y\, (P(x) \to Q(y)) \vdash \forall y\, Q(y)$$

1		$\exists x\, P(x)$	premise
2		$\forall x \forall y\, (P(x) \to Q(y))$	premise
3	y_0		
4	x_0	$P(x_0)$	assumption
5		$\forall y\, (P(x_0) \to Q(y))$	$\forall x$ e 2
6		$P(x_0) \to Q(y_0)$	$\forall y$ e 2
7		$Q(y_0)$	$\to$ 6,4
8		$Q(y_0)$	$\exists x$ e 1,4-7
9		$\forall y\, Q(y)$	$\forall y$ i 3–8