Tutorial 6 questions (CS-3234) — Preparation for the mid-term exam

Solve the following questions in preparation for the mid-term exam. Ask your tutors to solve the ones you found the most difficult in the next tutorial.

1. **Terms**: Let $\mathcal{F} = \{d, f, g\}$, where d is a constant, f a function symbol with 2 arguments, and g a function symbol with 3 arguments.

   1. which of the following strings $g(d, d)$; $f(x, g(y, z), d)$; $g(x, f(y, z), d$; $g(x, h(y, z), d)$; $f(f(g(d, x), f(g(d, x), y, g(y, d)), g(d, d)), g(f(d, d, x), d), z)$ are indeed terms? If yes, draw the parse tree.
   2. The length of a term over $\mathcal{F}$ is the length of its string representation, where we count all commas and parentheses. List all terms over $\mathcal{F}$ which do not contain any variable and whose length is less than 10.
   3. The height of a term over $\mathcal{F}$ is 1 plus the length of the longest path in its parse tree. List all terms over $\mathcal{F}$ which do not contain any variable and whose height is less than 4.

2. **Formulas**:

   1. Let $F(x, y)$ mean that x is the father of y; similarly, $M(x, y)$, $H(x, y)$, $S(x, y)$, and $B(x, y)$ says that x is the mother/husband/sister/brother of y, respectively. Use constants to denote 'Ed', etc. Translate the following sentences into predicate logic:

      1. Everybody has a mother.
      2. Everybody has a father and a mother.
      3. Ed is a grandfather.
      4. Whoever has a mother has a father.
      5. All fathers are parents.
      6. All husband are spouses.
      7. No uncle is an aunt.
      8. Nobody's grandmother is anybody's father.
      9. All brother are siblings.
      10. Ed and Patsy are husband and wife.

   2. Translate the following sentences into predicate logic (using appropriate predicate & function symbols)

      (a) There are at least two saxophone players who were born in New Orleans and who play better that every sax player in New York city.

      (b) Abita Amber if the best beer which is brewed in Louisiana.

      (c) There is only one restaurant where you can get better breakfast than at the Bluebird Café.

3. **Substitutions**:

   1. Let $\phi = \exists x(P(y, z) \wedge (\forall y(\neg Q(y, x) \vee P(y, z))))$.
      (a) Which occurrences are free (resp. bound) in ϕ?

(b) Compute $\phi[w/x], \phi[w/y], \phi[f(x)/y], \phi[g(y,x)/z]$.
(c) Which of $w, f(x), g(y,z)$ are free for x (resp. for y) in ϕ?

2. Let $\psi = \neg(\forall x((\exists y P(x,y,z)) \wedge (\forall z P(x,y,z))))$.
(a) Compute $\psi[t/x], \psi[t/y], \psi[t/z]$, where $t = g(f(g(y,y)),y)$.
(b) Is t free for x (resp. for y, resp. for z) in ψ?

4. Proofs: Find natural deduction proofs for:

$$\begin{array}{rrcl} 1a & \neg\forall x\phi & \dashv\vdash & \exists x\neg\phi \\ 1b & \neg\exists x\phi & \dashv\vdash & \forall x\neg\phi \\ 4a & \forall x\forall y\phi & \dashv\vdash & \forall y\forall x\phi \\ 4a & \exists x\exists y\phi & \dashv\vdash & \exists y\exists x\phi \end{array}$$

5. Proofs: Find natural deduction proofs (for 2a,2b assuming x is not free in ψ) for:

$$\begin{array}{rrcl} 2a & \forall x\phi \wedge \psi & \dashv\vdash & \forall x(\phi \wedge \psi) \\ 2b & \forall x\phi \vee \psi & \dashv\vdash & \forall x(\phi \vee \psi) \\ 3a & \forall x\phi \wedge \forall x\psi & \dashv\vdash & \forall x(\phi \wedge \psi) \\ 3b & \exists x\phi \vee \exists x\psi & \dashv\vdash & \exists x(\phi \vee \psi) \end{array}$$

6. Proofs: Find natural deduction proofs (assuming x is not free in ψ) for:

$$\begin{array}{rrcl} 2c & \exists x\phi \wedge \psi & \dashv\vdash & \exists x(\phi \wedge \psi) \\ 2d & \exists x\phi \vee \psi & \dashv\vdash & \exists x(\phi \vee \psi) \\ 2e & \forall x(\psi \to \phi) & \dashv\vdash & \psi \to \forall x\phi \end{array}$$

7. Proofs: Find natural deduction proofs (assuming x is not free in ψ) for:

$$\begin{array}{rrcl} 2f & \exists x(\phi \to \psi) & \dashv\vdash & \forall x\phi \to \psi \\ 2g & \exists x(\psi \to \phi) & \dashv\vdash & \psi \to \exists x\phi \\ 2h & \forall x(\phi \to \psi) & \dashv\vdash & \exists x\phi \to \psi \end{array}$$

8. Proofs: Find natural deduction proofs for

1. $\forall x(\neg P(x) \wedge Q(x)) \vdash \forall x(P(x) \to Q(x))$
2. $\forall x(P(x) \vee Q(x)), \exists x\neg Q(x), \forall x(R(x) \to \neg P(x)) \vdash \exists x\neg R(x)$

9. Proofs: Find natural deduction proofs for

1. $\forall x(P(x) \wedge Q(x)), \forall y(P(x) \to R(x)) \vdash \exists x(R(x) \wedge Q(x))$
2. $\forall y Q(b,y), \forall x\forall y(Q(x,y) \to Q(s(x),s(y))) \vdash \exists z(Q(b,z) \wedge Q(z,s(s(b))))$

10. Interpretations:

1. Find the interpretation in the model

$$\mathcal{M} = (\text{natural numbers}, d^{\mathcal{M}} = 2, f^{\mathcal{M}}(k,n,m) = k * n + m, g^{\mathcal{M}}(k,n) = k + n * n)$$

and the environment

$$l = (l(x) = 5, l(y) = 7)$$

of the following terms: $f(d,x,d)$, $f(g(x,d)y, g(d,d))$, $g(f(g(d,y), f(x, g(dd), x), y), f(y, g(d,d), d))$

2. Let $\phi =_{def} \forall x \exists y \exists z (P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x))$
Which of the following models satisfies ϕ?
   (a) $\mathcal{M}$: natural numbers and $P^{\mathcal{M}} = \{(m,n) : m < n\}$
   (b) $\mathcal{M}'$: natural numbers and $P^{\mathcal{M}'} = \{(m,n) : n = 2 * m\}$
   (c) $\mathcal{M}''$: natural numbers and $P^{\mathcal{M}''} = \{(m,n) : m < n + 1\}$

11. Models: For the formulas ϕ below find models $\mathcal{M}, \mathcal{M}'$ and appropriate environments l, l' such that $\mathcal{M} \models_l \phi$ and $\mathcal{M}' \not\models_{l'} \phi$.

1. $\phi_1 =_{def} \forall x \forall y Q(g(x,y), g(y,y), z)$.
2. $\phi_2 =_{def} \forall x \neg P(x,x)$
3. $\phi_3 =_{def} \forall x \exists y S(x,y) \rightarrow \exists y \forall x S(x,y)$
4. $\phi_4 =_{def} (\exists x P(x) \wedge \exists y Q(y)) \rightarrow \exists z (P(z) \wedge Q(z))$

12. Proof & Semantic entailment Prove

1. $\forall x \forall y (S(y) \rightarrow F(x)) \vdash \exists y S(y) \rightarrow \forall x F(x)$
2. $\forall x \forall y (S(y) \rightarrow F(x)) \models \exists y S(y) \rightarrow \forall x F(x)$

13. No proofs: Find models such that the formulas on the left of '$\models$' are true, but the formula on the right is false:

1. $\forall x (P(x) \vee Q(x)) \models \forall x P(x) \vee \forall x Q(x)$
2. $\forall x (P(x) \vee R(x)), \forall x (Q(x) \vee R(x)) \models \exists x (P(x) \wedge Q(x))$
3. $\forall x P(x) \rightarrow L \models \forall x (P(x) \rightarrow L)$
   (L is a predicate symbol with no arguments)

Notice: By a soundness theorem -not proved in the class- this shows that there are no proofs for the corresponding sequents.

14. Yes or No: Check if the following formulas are theorems. If not, find a model such that it evaluates to F.

1. $\forall x \exists y (P(x) \to Q(y)) \to \exists y \forall x (P(x) \to Q(y))$
2. $\exists x (\neg P(x) \vee \neg Q(x)) \to \forall x (P(x) \vee Q(x))$

15. Resolution: Consider the following Prolog program:

```
gcd(X,0,X).
gcd(X,Y,Z) :- Y <= X, R is X rem Y, gcd(Y,R,Z).
gcd(X,Y,Z) :- Y > X, gcd(Y,X,Z).
```

and the goal

```
?- gcd(24,40,A).
```

Apply the resolution algorithm and find an answer substitution for the given goal.

16. Prolog: Write a Prolog program that computes the list of the first N Fibonacci numbers. Example goal:

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?- fib(7,L).
L=[1,1,2,3,5,8,13]
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