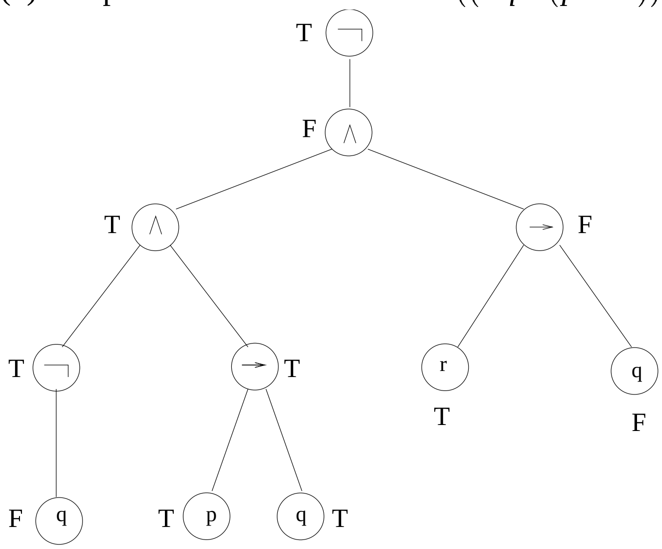
CS3234 - Tutorial 2, Solutions

1.

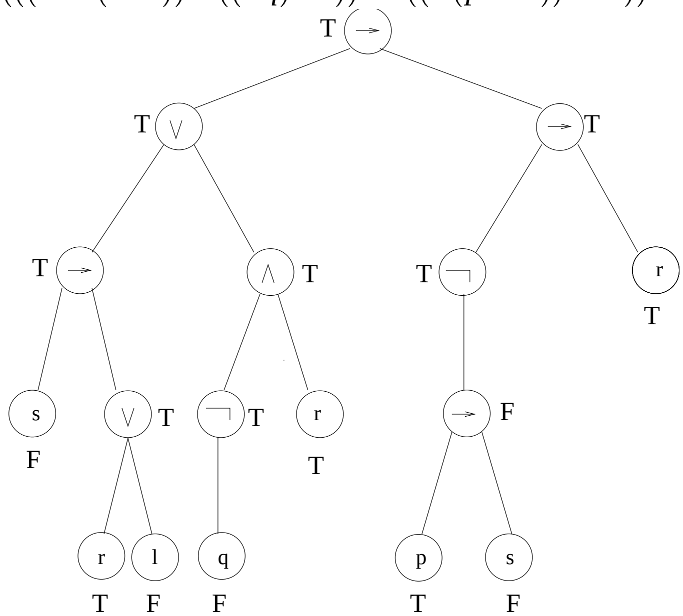
1. (1) The parse tree for the formula $\neg((\neg q \wedge (p \rightarrow r)) \wedge (r \rightarrow q))$ is:



(2) All sub-formulas are:
p, q, r, $(\neg q)$, $(p \rightarrow r)$, $(r \rightarrow q)$, $((\neg q) \wedge (p \rightarrow r))$,
$(((\neg q) \wedge (p \rightarrow r)) \wedge (r \rightarrow q))$, initial formula

(3) If p, r are T and q is F, the truth value of the formula is T.

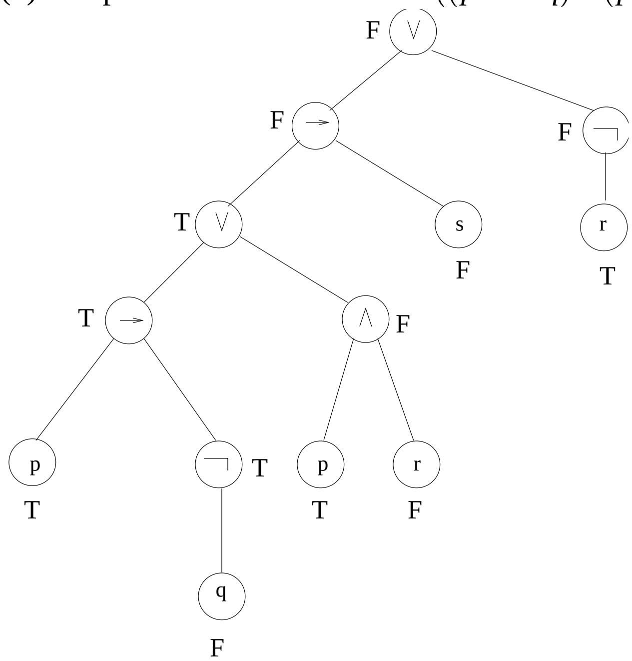
2. (1) The parse tree for the formula
$(((s \to (r \vee l)) \vee ((\neg q) \wedge r)) \to ((\neg(p \to s)) \to r))$ is:



(2) All sub-formulas are:
$r,\ l,\ q,\ p,\ s,\ (r \vee l),\ (\neg q),\ (p \to s),\ (s \to (r \vee l)),\ ((\neg q) \wedge r),\ \neg(p \to s),$
$((s \to (r \vee l)) \vee ((\neg q) \wedge r)),\ (\neg(p \to s) \to r)$, initial formula

(3) If p, r are T and q, s, l are F, the truth value of the formula is T.

3. (1) The parse tree for the formula $((p \to \neg q) \vee (p \wedge r) \to s) \vee \neg r$ is:



(2) All sub-formulas are:
q, p, s, r, $(\neg q)$, $(p \to (\neg q))$, $(p \wedge r)$, $((p \to (\neg q)) \vee (p \wedge r))$, $(((p \to (\neg q)) \vee (p \wedge r)) \to s)$, $(\neg r)$, initial formula

(3) If p, r are T and q, s are F, the truth value of the formula is F.

2.

Truth table for formula 1. $\neg((\neg q \wedge (p \to r)) \wedge (r \to q))$ is:

p	q	r	$\neg q$	$p \to r$	$r \to q$	$\neg q \wedge (p \to r)$	f_1	$\neg f_1$
F	F	F	T	T	T	T	T	F
F	F	T	T	T	F	T	F	T
F	T	F	F	T	T	F	F	T
F	T	T	F	T	T	F	F	T
T	F	F	T	F	T	F	F	T
T	F	T	T	T	F	T	F	T
T	T	F	F	F	T	F	F	T
T	T	T	F	T	T	F	F	T

We denote by f_1 the formula: $(\neg q \wedge (p \to r)) \wedge (r \to q)$.

The next formulas 2. and 3. have $card(\{T,F\})^{card(\{p,q,r,s,l\})} = 2^5 = 32$ rows, and $card(\{T,F\})^{card(\{p,q,s,r\})} = 2^4 = 16$ rows respectively.

3.

if $\phi_1, \ldots, \phi_n \vdash \psi$ then $\phi_1, \ldots, \phi_n \models \psi$

By the definition of the sequent, $\phi_1, \ldots, \phi_n \vdash \psi$ means that exists a proof using the natural deduction rules, with $\phi_1, \ldots, \phi_n$ as premises and ψ as conclusion.

In order to prove $\phi_1, \ldots, \phi_n \models \psi$ we reason by **induction on the length of the proof for** $\phi_1, \ldots, \phi_n \vdash \psi$:

$M(k)$: “For all sequents $\phi_1, \ldots, \phi_n \vdash \psi$ $(n \geq 0)$ which have a proof of length k, it is the case that $\phi_1, \ldots, \phi_n \models \psi$.“

Inductive step:
Induction hypothesis: Suppose we have $M(k')$ true for all $k' \leq k$.
To prove: We want to prove $M(k)$ true.

Assume that the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has a proof with the length k:

1.	ϕ_1	premise
	$\vdots$	
n.	ϕ_n	premise
	$\vdots$	
k.	ψ	justification

The justification from line k. in the proof is one of the natural deduction rules. We analyze one by one each natural deduction rule that could appear as justification in line k.

$$\wedge\text{e1}\ \frac{\psi \wedge \phi}{\psi}$$

Then, the proof for the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has a line $k_1,\ k_1 < k$, where $\psi \wedge \phi$ appears as the result of some natural deduction rule R and some lines, previous to k_1.

1.	ϕ_1	premise
	$\vdots$	
n.	ϕ_n	premise
	$\vdots$	
k_1.	$\psi \wedge \phi$	R
	$\vdots$	
k.	ψ	$\wedge$e1 k_1

By sequent definition, $\phi_1, \ldots, \phi_n \vdash \psi \wedge \phi$ is a sequent with a proof in k_1 steps ($k_1 < k$).

By **induction hypothesis** for $M(k_1)$, we deduce that there is a semantic entailment between $\phi_1, \ldots, \phi_n$ and $\psi \wedge \phi$. That is:

$$\phi_1, \ldots, \phi_n \models \psi \wedge \phi$$

By definition of semantic entailment, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also $\psi \wedge \phi$ to true ($\mathfrak{I}(\psi \wedge \phi) = T$).

Using the truth table for $\wedge$, we know that when $\psi \wedge \phi$ is true, both ψ and ϕ have to be true. That is any interpretation $\mathfrak{I}$ that evaluates $\psi \wedge \phi$ to true ($\mathfrak{I}(\psi \wedge \phi) = T$) will evaluate also ψ to true ($\mathfrak{I}(\psi) = T$).

Hence, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also ψ to true ($\mathfrak{I}(\psi) = T$).

By definition of semantic entailment, we conclude that:

$$\phi_1, \ldots, \phi_n \models \psi$$

$\vee$i1 $\dfrac{\psi_1}{\psi_1 \vee \psi_2}$ where ψ has the form $\psi_1 \vee \psi_2$.

Then, the proof for the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has a line $k_1,\ k_1 < k$, where ψ_1 appears as the result of some natural deduction rule R and some lines, previous to k_1.

1.	ϕ_1	premise
	$\vdots$	
n.	ϕ_n	premise
	$\vdots$	
k_1.	ψ_1	R
	$\vdots$	
k.	ψ	$\vee$i1 k_1

By sequent definition, $\phi_1, \ldots, \phi_n \vdash \psi \wedge \phi$ is a sequent with a proof in k_1 steps ($k_1 < k$).

By **induction hypothesis** for $M(k_1)$, we deduce that there is a semantic entailment between $\phi_1, \ldots, \phi_n$ and ψ_1. That is:

$$\phi_1, \ldots, \phi_n \models \psi_1$$

By definition of semantic entailment, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also ψ_1 to true ($\mathfrak{I}(\psi_1) = T$).

Using the truth table for $\vee$, we know that if ψ_1 is true, then $\psi_1 \vee \psi_2$ is true. That is any interpretation $\mathfrak{I}$ that evaluates ψ_1 to true ($\mathfrak{I}(\psi_1) = T$) will evaluate also $\psi_1 \vee \psi_2$ to true ($\mathfrak{I}(\psi_1 \vee \psi_2) = T$).

Hence, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also $\psi_1 \vee \psi_2 = \psi$ to true ($\mathfrak{I}(\psi) = T$).

By definition of semantic entailment, we conclude that:

$$\phi_1, \ldots, \phi_n \models \psi$$

$\vee$i2 analogous to $\vee$i1.

$\rightarrow$i $\quad \dfrac{\boxed{\begin{array}{c}\psi_1 \\ \vdots \\ \psi_2\end{array}}}{\psi_1 \rightarrow \psi_2}$ where ψ has the form $\psi_1 \rightarrow \psi_2$.

Then, the proof for the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has lines k_1 and k_2, $k_1, k_2 < k$, where ψ_1 and ψ_2 respectively appear as the result of some natural deduction rules $R1$ and $R2$ and some lines, previous to them.

1.	ϕ_1	premise
	$\vdots$	
n.	ϕ_n	premise
	$\vdots$	
k_1.	ψ_1	R_1
	$\vdots$	
k_2.	ψ_2	R_2
	$\vdots$	
k.	$\psi_1 \rightarrow \psi_2$	$(\rightarrow$i$)\ k_1 - k_2$

Note that the natural deduction rule includes boxes when some lines are under some assumptions, which are the first lines in the surrounding box.

By sequent definition, $\phi_1, \ldots, \phi_n, \psi_1 \vdash \psi_2$ is a sequent with a proof in k_2 steps ($k_2 < k$).

By **induction hypothesis** for $M(k_2)$, we deduce that there is a semantic entailment between $\phi_1, \ldots, \phi_n, \psi_1$ and ψ_2. That is:

$$\phi_1, \ldots, \phi_n, \psi_1 \models \psi_2$$

By definition of semantic entailment, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n, \psi_1$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n,\ \mathfrak{I}(\psi_1) = T$), will evaluate also ψ_2 to true ($\mathfrak{I}(\psi_2) = T$).

Using the truth table for $\rightarrow$, we know that if ψ_1 is false then $\psi_1 \rightarrow \psi_2$ is true, and if both ψ_1, ψ_2 are true, then $\psi_1 \rightarrow \psi_2$ is true. Any interpretation $\mathfrak{I}$ evaluates ψ_1 to true or false. So, when $\mathfrak{I}(\psi_1) = F$ then $\mathfrak{I}(\psi_1 \rightarrow \psi_2) = T$ and when $\mathfrak{I}(\psi_1) = T$, then $\mathfrak{I}(\psi_1 \rightarrow \psi_2) = T$ only if $\mathfrak{I}(\psi_2) = T$.

Hence, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate ψ_1 to true, or false. In each case, considering the justification above, $\mathfrak{I}(\psi_1 \rightarrow \psi_2) = T$. That is in fact $\mathfrak{I}(\psi) = T$.

By definition of semantic entailment, we conclude that:

$$\phi_1, \ldots, \phi_n \models \psi$$

$$\rightarrow\mathrm{e}\ \frac{\phi \qquad \phi \rightarrow \psi}{\psi}$$

Then, the proof for the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has lines k_1 and k_2 ($k_1, k_2 < k$), where ϕ appears as the result of some natural deduction rule $R1$ and some lines, previous to k_1, and respectively $\phi \rightarrow \psi$ appears applying some natural deduction rule $R2$, over some lines previous to k_2.

1.	ϕ_1	premise
	$\vdots$	
n.	ϕ_n	premise
	$\vdots$	
k_1.	ϕ	$R1$
	$\vdots$	
k_2.	$\phi \rightarrow \psi$	$R2$
	$\vdots$	
k.	ψ	$\rightarrow$e k_1, k_2

By sequent definition, $\phi_1, \ldots, \phi_n \vdash \phi$ is a sequent with a proof in k_1 steps ($k_1 < k$), and $\phi_1, \ldots, \phi_n \vdash \phi \to \psi$ is a sequent with a proof in k_2 steps ($k_2 < k$).

By **induction hypothesis** for $M(k_1)$, and $M(k_2)$, we deduce semantic entailments $\phi_1, \ldots, \phi_n \models \phi$ and $\phi_1, \ldots, \phi_n \models \phi \to \psi$.

By definition of semantic entailment, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate both ϕ and $\phi \to \psi$ to true ($\mathfrak{I}(\phi) = T$ and $\mathfrak{I}(\phi \to \psi) = T$).

Using the truth table for $\to$, we know that if ϕ is true and $\phi \to \psi$ is true, then ψ is true. That is any interpretation $\mathfrak{I}$ that evaluates ϕ and $\phi \to \psi$ to true ($\mathfrak{I}(\phi) = T$ and $\mathfrak{I}(\phi \to \psi) = T$) will evaluate also ψ to true ($\mathfrak{I}(\psi) = T$).

Hence, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also ψ to true ($\mathfrak{I}(\psi) = T$).

By definition of semantic entailment, we conclude that:

$$\phi_1, \ldots, \phi_n \models \psi$$

$$\neg\mathrm{i}\ \frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}$$

where ψ has the form $\neg\phi$.

Then, the proof for the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has lines k_1 and k_2, $k_1, k_2 < k$, where ϕ and $\bot$ respectively appear as the result of some natural deduction rules $R1$ and $R2$ and some lines, previous to them.

$$
\begin{array}{lll}
1. & \phi_1 & \text{premise} \\
 & \vdots & \\
n. & \phi_n & \text{premise} \\
 & \vdots & \\
k_1. & \boxed{\begin{array}{c} \phi \\ \vdots \\ \bot \end{array}} & \begin{array}{l} R1 \\ \\ R2 \end{array} \\
k_2. & & \\
 & \vdots & \\
k. & \neg\phi & (\neg\mathrm{i})\ k_1 - k_2
\end{array}
$$

Note that the natural deduction rule includes boxes when some lines are under some assumptions, which are the first lines in the surrounding box.

By sequent definition, $\phi_1, \ldots, \phi_n, \phi \vdash \bot$ is a sequent with a proof in k_2 steps ($k_2 < k$).

By **induction hypothesis** for $M(k_2)$, we deduce that there is a semantic entailment between $\phi_1, \ldots, \phi_n, \phi$ and $\bot$. That is:

$$\phi_1, \ldots, \phi_n, \phi \models \bot$$

By definition of semantic entailment, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n, \phi$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n,\ \mathfrak{I}(\phi) = T$), will evaluate also $\bot$ to true ($\mathfrak{I}(\bot) = T$).

But $\bot$ never evaluates to true, hence there is no interpretation $\mathfrak{I}$ that evaluates to true all $\phi_1, \ldots, \phi_n$ and ϕ. So, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$) will evaluate ϕ to false ($\mathfrak{I}(\phi) = F$).

Using the truth table for $\neg$, we know that if ϕ is false then $\neg\phi$ is true. So, any interpretation $\mathfrak{I}$ that evaluates ϕ to false, will evaluate $\neg\phi$ to true.

Hence, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), evaluates also $\neg\phi$ to true ($\mathfrak{I}(\neg\phi) = T$). That is in fact $\mathfrak{I}(\psi) = T$.

By definition of semantic entailment, we conclude that:

$$\phi_1, \ldots, \phi_n \models \psi$$

$$\neg e \ \frac{\phi \qquad \neg\phi}{\bot} \qquad \text{where } \psi \text{ is } \bot.$$

Then, the proof for the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has lines k_1 and k_2 ($k_1, k_2 < k$), where ϕ appears as the result of some natural deduction rule $R1$ and some lines, previous to k_1, and respectively $\neg\phi$ appears applying some natural deduction rule $R2$, over some lines previous to k_2.

$$\begin{array}{lll}
1. & \phi_1 & \text{premise} \\
 & \vdots & \\
n. & \phi_n & \text{premise} \\
 & \vdots & \\
k_1. & \phi & R1 \\
 & \vdots & \\
k_2. & \neg\phi & R2 \\
 & \vdots & \\
k. & \psi & \neg e\ k_1, k_2
\end{array}$$

By sequent definition, $\phi_1, \ldots, \phi_n \vdash \phi$ is a sequent with a proof in k_1 steps ($k_1 < k$), and $\phi_1, \ldots, \phi_n \vdash \neg\phi$ is a sequent with a proof in k_2 steps ($k_2 < k$).

By **induction hypothesis** for $M(k_1)$, and $M(k_2)$, we deduce semantic entailments $\phi_1, \ldots, \phi_n \models \phi$ and $\phi_1, \ldots, \phi_n \models \neg\phi$.

By definition of semantic entailment, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate both ϕ and $\neg\phi$ to true ($\mathfrak{I}(\phi) = T$ and $\mathfrak{I}(\neg\phi) = T$).

Using the truth table for $\neg$, we know that is impossible for both ϕ and $\neg\phi$ to be true in the same time.

Hence, there is no interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true. So, any interpretation that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also $\bot$ to true (there is no such interpretation $\mathfrak{I}$ so we are safe).

By definition of semantic entailment, since ψ is $\bot$, we conclude that:

$$\phi_1, \ldots, \phi_n \models \psi$$

$\perp$e $\dfrac{\perp}{\psi}$

Then, the proof for the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has a line k_1 ($k_1 < k$), where $\perp$ appears as the result of some natural deduction rule R and some lines, previous to k_1.

1.	ϕ_1	premise
	$\vdots$	
n.	ϕ_n	premise
	$\vdots$	
k_1.	$\perp$	R
	$\vdots$	
k.	ψ	$\perp$e k_1

By sequent definition, $\phi_1, \ldots, \phi_n \vdash \perp$ is a sequent with a proof in k_1 steps ($k_1 < k$).

By **induction hypothesis** for $M(k_1)$ we deduce the semantic entailment:

$$\phi_1, \ldots, \phi_n \models \perp$$

.

By definition of semantic entailment, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also $\perp$ to true ($\mathfrak{I}(\perp) = T$).

Since $\perp$ never evaluates to true, there is no interpretation $\mathfrak{I}$ that evaluates all $\phi_1, \ldots, \phi_n$ to true. So, any interpretation that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also ψ to true (there is no such interpretation $\mathfrak{I}$ so we are safe).

By definition of semantic entailment we conclude that:

$$\phi_1, \ldots, \phi_n \models \psi$$

$$\neg\neg e \ \frac{\neg\neg\psi}{\psi}$$

Then, the proof for the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ has a line k_1 ($k_1 < k$), where $\neg\neg\psi$ appears as the result of some natural deduction rule R and some lines, previous to k_1.

1.	ϕ_1	premise
	$\vdots$	
n.	ϕ_n	premise
	$\vdots$	
k_1.	$\neg\neg\psi$	R
	$\vdots$	
k.	ψ	$\neg\neg$e k_1

By sequent definition, $\phi_1, \ldots, \phi_n \vdash \neg\neg\psi$ is a sequent with a proof in k_1 steps ($k_1 < k$).

By **induction hypothesis** for $M(k_1)$ we deduce the semantic entailment:

$$\phi_1, \ldots, \phi_n \models \neg\neg\psi$$

.

By definition of semantic entailment, any interpretation $\mathfrak{I}$ that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also $\neg\neg\psi$ to true ($\mathfrak{I}(\neg\neg\psi) = T$).

Since $\neg\neg\psi$ has the same truth value as ψ under any interpretation, then any interpretation that evaluates $\phi_1, \ldots, \phi_n$ to true ($\mathfrak{I}(\phi_i) = T,\ i = 1 \ldots n$), will evaluate also ψ to true ($\mathfrak{I}(\psi) = T$).

By definition of semantic entailment we conclude that:

$$\phi_1, \ldots, \phi_n \models \psi$$

4.

$\models \phi$ with truth table

p	q	$\neg p$	$\neg p \rightarrow q$	$p \rightarrow (\neg p \rightarrow q)$
F	F	T	F	T
F	T	T	T	T
T	F	F	T	T
T	T	F	T	T

Hence $\models p \rightarrow (\neg p \rightarrow q)$

We want to prove $\vdash p \rightarrow (\neg p \rightarrow q)$ using a similar construction as in the proof for the *Completeness theorem*.

Step 1. Nothing to do for this case.

Step 2. With Proposition 1.37 (page 63 textbook) each row in the above truth table is transformed into a sequent as follows:

$$\begin{array}{rl} \neg p, \neg q \vdash p \rightarrow (\neg p \rightarrow q) & \text{(row1)} \\ \neg p, q \vdash p \rightarrow (\neg p \rightarrow q) & \text{(row2)} \\ p, \neg q \vdash p \rightarrow (\neg p \rightarrow q) & \text{(row3)} \\ p, q \vdash p \rightarrow (\neg p \rightarrow q) & \text{(row4)} \end{array}$$

Now let us construct the proof for $\vdash p \rightarrow (\neg p \rightarrow q)$ (using the proofs for the above four sequents).

1	$p \lor \neg p$	LEM
2	p	assumption
3	$q \lor \neg q$	LEM
4	q	assumption
5	$p \to (\neg p \to q)$	(row4) 2,4
6	$\neg q$	assumption
7	$p \to (\neg p \to q)$	(row3) 2,7
8	$p \to (\neg p \to q)$	$\lor$e 3,4-5,6-7
9	$\neg p$	assumption
10	$q \lor \neg q$	LEM
11	q	assumption
12	$p \to (\neg p \to q)$	(row2) 9,12
13	$\neg q$	assumption
14	$p \to (\neg p \to q)$	(row1) 9,13
15	$p \to (\neg p \to q)$	$\lor$e 10,11-12,13-14
16	$p \to (\neg p \to q)$	$\lor$e 1,2-8,9-15

By sequent definition we conclude that $\vdash p \to (\neg p \to q)$.

5.

The formula $p \to (q \lor r)$ is semantically equivalent to formulas:
1. $q \lor (\neg p \lor r)$
3. $p \land \neg r \to q$
4. $\neg q \land \neg r \to \neg p$

The semantic equivalence can be checked using truth tables. In the followings we give the truth table for all the formulas targeted in this exercise:

p	q	r	$p \to (q \vee r)$	$q \vee (\neg p \vee r)$	$q \wedge \neg r \to p$	$p \wedge \neg r \to q$	$\neg q \wedge \neg r \to \neg p$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	F	T	F	F
T	F	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	T	T	T	T	T	T	T

As we can see, in the third and fifth row, the formula $q \wedge \neg r \to p$ has a different value than the others. However, all equivalent formulas have the same truth value, in every row in the table (that is for any interpretation of the atoms).

6.

$$\phi: \quad \neg(p \to (\neg(q \wedge (\neg p \to q))))$$

A conjunctive normal form of the formula ϕ is calculated by the algorithm **CNF** that is applied over the negation normal form of the formula ϕ.

First step in obtaining the negation normal form is applying the algorithm **IMPL-FREE** which substitutes all the formulas $\phi \to \psi$ with the equivalent ones $\neg\phi \vee \psi$.

$\mathbf{IMPL-FREE}(\phi) =$
$\mathbf{IMPL-FREE}(\neg(p \to (\neg(q \wedge (\neg p \to q))))) =$
$\neg(\mathbf{IMPL-FREE}(p \to (\neg(q \wedge (\neg p \to q))))) =$
$\neg(\neg\mathbf{IMPL-FREE}(p) \vee \mathbf{IMPL-FREE}(\neg(q \wedge (\neg p \to q)))) =$
$\neg(\neg p \vee \neg(\mathbf{IMPL-FREE}(q \wedge (\neg p \to q)))) =$
$\neg(\neg p \vee \neg(\mathbf{IMPL-FREE}(q) \wedge \mathbf{IMPL-FREE}(\neg p \to q))) =$
$\neg(\neg p \vee \neg(q \wedge (\neg\mathbf{IMPL-FREE}(\neg p) \vee \mathbf{IMPL-FREE}(q)))) =$
$\neg(\neg p \vee \neg(q \wedge (\neg\neg\mathbf{IMPL-FREE}(p) \vee q))) =$
$\neg(\neg p \vee \neg(q \wedge (\neg\neg p \vee q)))$

Second step in obtaining the negation normal form is to apply the **NNF** algorithm over the equivalent formula resulted from the previous step. The **NNF** algorithm relies on the De Morgan rules, and tries to bring the negation in front of the atoms.

$$
\begin{aligned}
&\mathbf{NNF}(\mathbf{IMPL-FREE}(\phi)) = \\
&\mathbf{NNF}(\neg(\neg p \vee \neg(q \wedge (\neg\neg p \vee q)))) = \\
&\mathbf{NNF}(\neg(\neg p)) \wedge \mathbf{NNF}(\neg(\neg(q \wedge (\neg\neg p \vee q)))) = \\
&\mathbf{NNF}(p) \wedge \mathbf{NNF}(q \wedge (\neg\neg p \vee q)) = \\
&p \wedge (\mathbf{NNF}(q) \wedge \mathbf{NNF}(\neg\neg p \vee q)) = \\
&p \wedge (q \wedge (\mathbf{NNF}(\neg\neg p) \vee \mathbf{NNF}(q))) = \\
&p \wedge (q \wedge (\mathbf{NNF}(p) \vee q)) = \\
&p \wedge (q \wedge (p \vee q))
\end{aligned}
$$

Finally, a equivalent conjunctive normal form of ϕ is obtained using the **CNF** algorithm which is practically exploiting the equivalence between formulas $(\phi_1 \wedge \phi_2) \vee (\phi_3 \wedge \phi_4)$ and $(\phi_1 \vee \phi_3) \wedge (\phi_1 \vee \phi_4) \wedge (\phi_2 \vee \phi_3) \wedge (\phi_2 \vee \phi_4)$.

$$
\begin{aligned}
&\mathbf{CNF}(\mathbf{NNF}(\mathbf{IMPL-FREE}(\phi))) = \\
&\mathbf{CNF}(p \wedge (q \wedge (p \vee q))) = \\
&\mathbf{CNF}(p) \wedge \mathbf{CNF}(q \wedge (p \vee q)) = \\
&p \wedge (\mathbf{CNF}(q) \wedge \mathbf{CNF}(p \vee q)) = \\
&p \wedge (q \wedge \mathbf{DISTR}(\mathbf{CNF}(p), \mathbf{CNF}(q))) = \\
&p \wedge (q \wedge (p \vee q)) = p \wedge q \wedge (p \vee q)
\end{aligned}
$$

7.

The **HORN** algorithm aims to check if a formula is satisfiable by marking all the atoms that *have to be true*.

Note that the formula to be checked for satisfiability with **HORN** algorithm has to be a *Horn clause*.

$\mathbf{HORN}((p \wedge q \wedge w \to \bot) \wedge (t \to \bot) \wedge (r \to p) \wedge (\top \to r) \wedge (\top \to q) \wedge (r \wedge u \to w) \wedge (u \to s) \wedge (\top \to u))$:

1. mark atoms that appear in conjuncts of the form $\top \to p$: $\quad r, q, u$
2. mark atoms that appear in the right-hand-side of an implication, where all the atoms in the left-hand-side are already marked: p, w, s
3. since in the formula we have $(p \wedge q \wedge w \to \bot)$ and p, q, w are already marked (that is p, q, w have to be true), the given Horn clause is unsatisfiable (because it has a conjunct that is evaluated to $T \to \bot$)

HORN$((\top \to q) \wedge (\top \to s) \wedge (w \to \bot) \wedge (p \wedge q \wedge s \to \bot) \wedge (v \to s) \wedge (\top \to r) \wedge (r \to p))$:

1. mark atoms that appear in conjuncts of the form $\top \to p$: $\quad q, s, r$
2. mark atoms that appear in the right-hand-side of an implication, where all the atoms in the left-hand-side are already marked: p
3. since in the formula we have $(p \wedge q \wedge s \to \bot)$ and p, q, s are already marked (that is p, q, s have to be true), the given Horn clause is unsatisfiable (because it has a conjunct that is evaluated to $T \to \bot$)