CS3234 - Tutorial 3, Solutions

1.

(a) $\forall x\ (P(x) \rightarrow A(m,x))$

(b) $\exists x\ (P(x) \wedge A(x,m))$

(c) $A(m,m)$

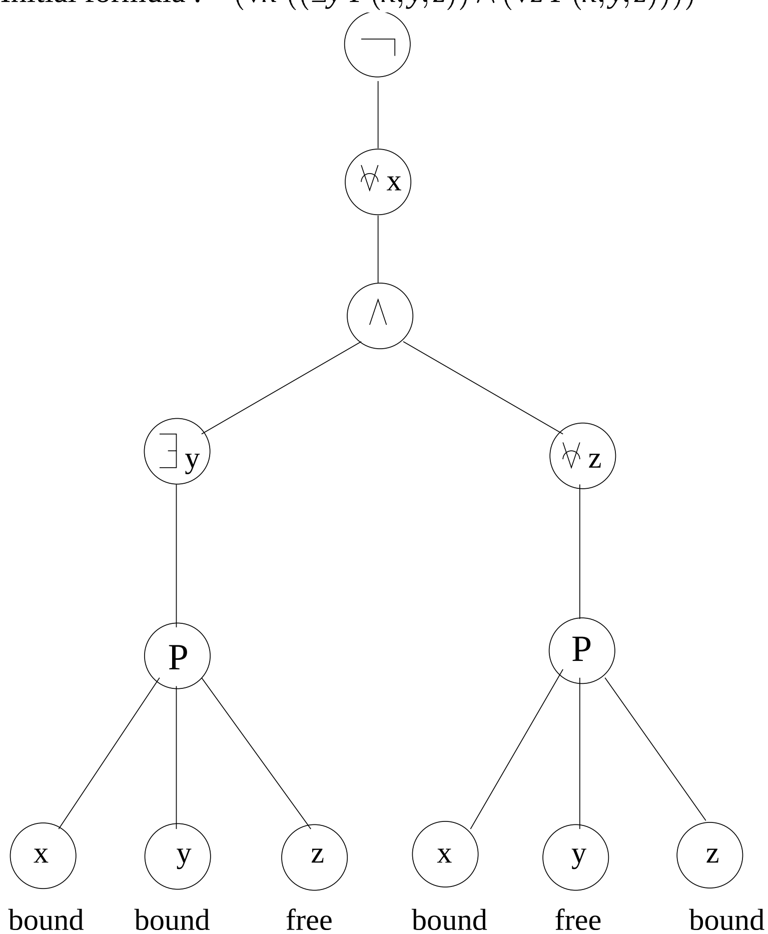
(d) $\neg\exists x\ (S(x) \wedge \forall y\ (L(y) \rightarrow B(x,y)))$

(e) $\neg\exists x\ (L(x) \wedge \forall y\ (S(y) \rightarrow B(y,x)))$

(f) $\neg(\exists x\ (L(x) \wedge \exists y\ (S(y) \wedge B(x,y))))$

2.

(a) Initial formula : $\neg(\forall x\,((\exists y\,P(x,y,z)) \wedge (\forall z\,P(x,y,z))))$



(b) All the occurrences of x are bound by the universal quantifier $\forall x$. y is bound in the left-hand side subtree by $\exists y$, and free in the right-hand side subtree. Vice versa for the occurrences of z, free in the left-hand side and bound by $\forall z$ in the right-hand side.

(c) As we mentioned at (b), the variables are y and z.

(d)

- $\phi[t/x]$, where $t = g(f(g(y,y)),y)$
  Since there are no free occurrences of x, it remains the same formula after the substitution:
  $\phi[t/x] = \phi$

- $\phi[t/y]$, where $t = g(f(g(y,y)),y)$
  The only free occurrence of y is in the right-hand side subtree, thus we have:
  $\neg(\forall x((\exists y P(x,y,z)) \wedge (\forall z P(x,y,z))))\ [t/y]\ =$
  $\neg(\forall x((\exists y P(x,y,z)) \wedge (\forall z P(x,g(f(g(y,y)),y),z))))$

- $\phi[t/z]$, where $t = g(f(g(y,y)),y)$
  The only free occurrence of z is in the left-hand side subtree. Since t contains the variable y, which is bound by $\exists y$ in the left-hand side subtree, we need to rename y in order to avoid a false bounding. We have:
  $\neg(\forall x((\exists y P(x,y,z)) \wedge (\forall z P(x,y,z))))\ [t/z]\ =$
  renaming y to u
  $\neg(\forall x((\exists u P(x,u,z)) \wedge (\forall z P(x,u,z))))\ [t/z]\ =$
  $\neg(\forall x((\exists u P(x,u,g(f(g(y,y)),y))) \wedge (\forall z P(x,u,z))))$

- Is t free for x in ϕ?
  YES, because there are no free occurences of x in the initial formula.

- Is t free for y in ϕ?
  YES, because y appears free on the right-hand side subtree and the term $t(= g(f(g(y,y)),y))$ doesn't contain any of the variables x or z, which are bound on that subtree.

- Is t free for z in ϕ?
  NO, because z appears free on the left-hand side subtree and the term $t(= g(f(g(y,y)),y))$ contains the variable y, which is bound on that subtree.

3.

(a) $\forall x\ (P(x) \wedge Q(x)) \vdash \forall x\ P(x) \wedge \forall x\ Q(x)$

1	$\forall x\ (P(x) \wedge Q(x))$	premise
2	x_0	
3	$P(x_0) \wedge Q(x_0)$	$\forall x$ e 1
4	$P(x_0)$	$\wedge$e1 3
5	$\forall x\ P(x)$	$\forall x$ i 2-4
6	x_1	
7	$P(x_1) \wedge Q(x_1)$	$\forall x$ e 1
8	$Q(x_1)$	$\wedge$e2 6
9	$\forall x\ Q(x)$	$\forall x$ i 6-8
10	$\forall x\ P(x) \wedge \forall x\ Q(x)$	$\wedge$i 5,9

(b) $\exists x\ P(x) \vee \exists x\ Q(x) \vdash \exists x\ (P(x) \vee Q(x))$

1	$\exists x\ P(x) \vee \exists x\ Q(x)$	premise
2	$\exists x\ P(x)$	assumption
3	$x_0,\ P(x_0)$	assumption
4	$P(x_0) \vee Q(x_0)$	$\vee$i1 3
5	$\exists x\ (P(x) \vee Q(x))$	$\exists x$ i 4
6	$\exists x\ (P(x) \vee Q(x))$	$\exists x$ e 2,3-5
7	$\exists x\ Q(x)$	assumption
8	$x_1,\ Q(x_1)$	assumption
9	$P(x_1) \vee Q(x_1)$	$\vee$i2 7
10	$\exists x\ (P(x) \vee Q(x))$	$\exists x$ i 9
11	$\exists x\ (P(x) \vee Q(x))$	$\exists x$ e 7,8-10
12	$\exists x\ (P(x) \vee Q(x))$	$\vee$e 1,2-5,6-9

(c) $\forall x \forall y\, P(x,y) \vdash \forall u \forall v\, P(u,v)$

1	$\forall x \forall y\, P(x,y)$	premise
2	$u_0,\ \forall y\, P(u_0,y)$	$\forall x$ e 1
3	$v_0,\ P(u_0,v_0)$	$\forall y$ e 2
4	$\forall v\, P(u_0,v)$	$\forall v$ i 3
5	$\forall u\, \forall v\, P(u,v)$	$\forall u$ i 2-4

(d) $\exists x\, \forall y\, P(x,y) \vdash \forall y\, \exists x\, P(x,y)$

1	$\exists x\, \forall y\, P(x,y)$	premise
2	y_0	
3	$x_0,\ \forall y\, P(x_0,y)$	assumption
4	$P(x_0,y_0)$	$\forall y$ e 3
5	$\exists x\, P(x,y_0)$	$\exists x$ i 4
6	$\exists x\, P(x,y_0)$	$\exists x$ e 1,3-5
7	$\forall y\, \exists x\, P(x,y)$	$\forall y$ i 2-6

(e) $P(a) \vdash \forall x\, (\, x = a \rightarrow P(x)\,)$

1	$P(a)$	premise
2	x_0	
3	$x_0 = a$	assumption
4	$P(x_0)$	$=$ e 3,1
5	$x_0 = a \rightarrow P(x_0)$	$\rightarrow$i 3-4
6	$\forall x\, (\, x = a \rightarrow P(x)\,)$	$\forall x$ i 2-5

(f) $\forall x\, P(x) \to S \vdash \exists y\, (P(y) \to S)$, ($S$ is a predicate with 0 arguments)

we denote by (R) the proof for $p \to q \dashv\vdash \neg p \vee q$

1	$\forall x\, P(x) \to S$	premise
2	$\neg(\forall x\, P(x)) \vee S$	(R)
3	$\neg(\forall x\, P(x))$	assumption
4	$\exists x\, \neg P(x)$	$\neg\forall$
5	$x_0,\ \neg P(x_0)$	assumption
6	$\neg P(x_0) \vee S$	$\vee$i1 5
7	$P(x_0) \to S$	(R)
8	$\exists y\, (P(y) \to S)$	$\exists y$ i 7
9	$\exists y\, (P(y) \to S)$	$\exists x$ e 4,5-8
10	S	assumption
11	$\neg P(y) \vee S$	$\vee$i2 10
12	$P(y) \to S$	(R)
13	$\exists y\, (P(y) \to S)$	$\exists y$ i 12
14	$\exists y\, (P(y) \to S)$	$\vee$e 2,3-9,10-13