CS3234 - Tutorial 4, Solutions

1.

$\phi = \forall x \forall y\ Q(g(x,y), g(y,y), z)$

Formula ϕ has the set of functions $\mathcal{F} = \{g\}$, the set of predicates $\mathcal{P} = \{Q\}$, and the set of variables $\{x,y,z\}$.

We define the model $\mathcal{M}$ and the environment ℓ such that $\mathcal{M} \models_\ell \phi$.

- $A = \{a,b\}$
- $g^{\mathcal{M}} : A^2 \to A$, with g being the constant function equal to a:

$$g(x,y) = a \ \text{ for all } x,y \in A$$

- $Q^{\mathcal{M}} \subseteq A^3$, with $Q^{\mathcal{M}} = \{(a,a,a)\}$
- The environment $\ell : \{x,y,z\} \to A$, as:

$$\ell(x) = \ell(y) = a \text{ and } \ell(z) = a$$

So, $\mathcal{M} \models_\ell \phi$ since using the definition of $g^{\mathcal{M}}$, $Q^{\mathcal{M}}$, and the evaluation of the free variable z in the environment ℓ we have:

$$\phi = \forall x \forall y\ Q^{\mathcal{M}}(g^{\mathcal{M}}(x,y), g^{\mathcal{M}}(y,y), a) = \forall x \forall y\ Q^{\mathcal{M}}(a,a,a) = \forall x \forall y\ \mathrm{T}$$

which is evaluated to T in the model $\mathcal{M}$ and the environment ℓ.

Let $\mathcal{M}'$ be the previously defined model $\mathcal{M}$ and let us define the environment ℓ' :

$$\ell : \{x,y,z\} \to A \text{ as } \ell'(x) = \ell'(y) = a \text{ and } \ell'(z) = b$$

So, $\mathcal{M} \not\models_{\ell'} \phi$ because

$$\phi = \forall x \forall y\ Q^{\mathcal{M}}(g^{\mathcal{M}}(x,y), g^{\mathcal{M}}(y,y), b) = \forall x \forall y\ Q^{\mathcal{M}}(a,a,b) = \forall x \forall y\ \mathrm{F}$$

is evaluated to F in the model $\mathcal{M}' = \mathcal{M}$ and the environment ℓ'

2.

$\forall x P(x) \lor \forall x Q(x) \models \forall(P(x) \lor Q(x))$

Let us consider a model $\mathcal{M}$ such that $\mathcal{M} \models \forall x P(x) \lor \forall x Q(x)$.
We want to show that $\mathcal{M} \models \forall(P(x) \lor Q(x))$.

We have the following proof:

$\mathcal{M} \models \forall x P(x) \lor \forall x Q(x)$
iff by definition of $\mathcal{M} \models \phi \lor \psi$
$\mathcal{M} \models \forall x P(x)$ *or* $\mathcal{M} \models \forall x Q(x)$
iff by definition of $\mathcal{M} \models \forall \phi$
$\big($*for all* $\ell : var \to A$, *for all* $a \in A$, $\mathcal{M} \models_{\ell[x \to a]} P(x)$ $\big)$ *or*
$\big($*for all* $\ell : var \to A$, *for all* $a \in A$, $\mathcal{M} \models_{\ell[x \to a]} Q(x)$ $\big)$
then
for all $\ell : var \to A$, *for all* $a \in A$, $\big(\mathcal{M} \models_{\ell[x \to a]} P(x)$ *or* $\mathcal{M} \models_{\ell[x \to a]} Q(x)$ $\big)$
iff by definition of $\mathcal{M} \models_{\ell} \phi \lor \psi$
for all $\ell : var \to A$, *for all* $a \in A$, $\mathcal{M} \models_{\ell[x \to a]} P(x) \lor Q(x)$
iff by definition of $\mathcal{M} \models_{\ell} \forall \phi$
for all $\ell : var \to A$, $\mathcal{M} \models_{\ell} \forall x(P(x) \lor Q(x))$
iff by definition of $\mathcal{M} \models \phi$
$\mathcal{M} \models \forall x(P(x) \lor Q(x))$

So, in any model $\mathcal{M}$ where $\forall x P(x) \lor \forall x Q(x)$ evaluates to true (that is $\mathcal{M} \models \forall x P(x) \lor \forall x Q(x)$) **then** also the formula $\forall x(P(x) \lor Q(x))$ evaluates to true ($\mathcal{M} \models \forall x(P(x) \lor Q(x))$).

Hence, by the definition of semantic entailment, we conclude that:

$$\forall x P(x) \lor \forall x Q(x) \models \forall(P(x) \lor Q(x))$$

3.

(a) $(\forall x \forall y (S(x,y) \to S(y,x))) \to (\forall \neg S(x,x))$

The left-hand side of the formula resembles the symmetry property of equality ($x = y \to y = x$). Based on this observation, we may have the following model, $\mathcal{M}$:

- $A = \mathbb{N}$ (the set of natural numbers)
- no function symbol ($\mathcal{F} = \emptyset$)
- $S^{\mathcal{M}} = =_{\mathbb{N}}$ (equality relation between natural numbers)

Using this model, $\forall x \forall y (x = y \to y = x)$ evaluates to true and $\forall x \neg (x = x)$ evaluates to false. Thus, the formula consisting the implication of these two is F.

So, $\mathcal{M} \nvDash (\forall x \forall y (S(x,y) \to S(y,x))) \to (\forall x \neg S(x,x))$

(b) $\exists y \; ((\forall x \, P(x)) \; \to \; P(y)$

1	$\forall x \, P(x)$	assumption
2	$P(y)$	$\forall \, x$ e 1
3	$(\forall x \, P(x)) \; \to P(y)$	$\to$i 1-2
4	$\exists y \; ((\forall x \, P(x)) \; \to P(y))$	$\exists y$ i 3

(c) $(\forall x\ (P(x) \to \exists y\ Q(y))) \ \to\ (\forall x\ \exists y\ (P(x) \to Q(y)))$

1	$\forall x\ (P(x) \to \exists y Q(y))$	assumption
2	x_0	
3	$P(x_0) \to \exists y Q(y)$	$\forall x$ e 1
4	$\neg P(x_0) \lor \exists y Q(y)$	(R)
5	$\neg P(x_0)$	assumption
6	$\neg P(x_0) \lor Q(y)$	$\lor$i1 5
7	$P(x_0) \to Q(y)$	(R)
8	$\exists y\ (P(x_0) \to Q(y))$	$\exists y$ i 7
9	$\exists y Q(y)$	assumption
10	$y_0, Q(y_0)$	assumption
11	$\neg P(x_0) \lor Q(y_0)$	$\lor$i2 10
12	$P(x_0) \to Q(y_0)$	(R)
13	$\exists y\ (P(x_0) \to Q(y))$	$\exists y$ i 12
14	$\exists y\ (P(x_0) \to Q(y))$	$\exists y$ e 9,10-13
15	$\exists y\ (P(x_0) \to Q(y))$	$\lor$e 4,5-8,9-14
16	$\forall x\ \exists y\ (P(x) \to Q(y))$	$\forall x$ i 2-15
17	$(\forall x\ (P(x) \to \exists y\ Q(y))) \ \to\ (\forall x\ \exists y\ (P(x) \to Q(y)))$	$\to$i 1-16

(d) $(\forall x\, \exists y\, (P(x) \to Q(y))) \to (\forall x\, (P(x) \to \exists y Q(y)))$

1	$\forall x\, \exists y\, (P(x) \to Q(y))$	assumption
2	x_0	
3	$\exists y\, (P(x_0) \to Q(y))$	$\forall x$ e 1
4	$y_0,\ P(x_0) \to Q(y_0)$	assumption
5	$\neg P(x_0) \vee Q(y_0)$	(R)
6	$\neg P(x_0)$	assumption
7	$\neg P(x_0) \vee \exists y Q(y)$	$\vee$i1 6
8	$P(x_0) \to \exists y Q(y)$	(R)
9	$Q(y_0)$	assumption
10	$\exists y Q(y)$	$\exists y$ i 9
11	$\neg P(x_0) \vee \exists y Q(y)$	$\vee$i2 10
12	$P(x_0) \to \exists y Q(y)$	(R)
13	$P(x_0) \to \exists y Q(y)$	$\vee$e 5,6-8,9-12
14	$P(x_0) \to \exists y Q(y)$	$\exists y$ e 3,4-13
15	$\forall x\, (P(x) \to \exists y Q(y))$	$\forall x$ i 2-14
16	$(\forall x\, \exists y\, (P(x) \to Q(y))) \to (\forall x\, (P(x) \to \exists y Q(y)))$	$\to$i 1-15

(e) $\forall x \forall y\ (S(x,y) \to (\exists z(S(x,z) \land S(z,y))))$

Let's define a model $\mathcal{M}$ in the following way:

- $A = \{a,b,c\}$
- no function symbol ($\mathcal{F} = \emptyset$)
- $S^{\mathcal{M}} = \{(a,b)\}$

Let's take an environment $\ell : var \to A$, with $\ell(x) = a$ and $\ell(y) = b$.

In these settings, $S(x,y)$ (the left-hand-side of the implication in the formula) is true only when $x = a$ and $y = b$.

In this case (when $S(x,y)$ is true), there is no z such that $S(x,z) = S(a,z)$ and $S(z,y) = S(z,b)$ are both true.

(If $S(x,z) = S(a,z)$ is true, then $z = b$; but $S(b,a)$ is false, so $S(x,z) \land S(z,y)$ is false.)

So, the right-hand-side of the implication in the formula is false when the left-hand-side is true. Hence the implication is false.

From these, we deduce that $\mathcal{M} \not\models_{\ell} \forall x \forall y(S(x,y) \to (\exists z(S(x,z) \land S(z,y))))$.

(f) $(\forall x \forall y(S(x,y) \to (x = y))) \to (\forall z \neg S(z,z))$

Let's define the following model, $\mathcal{M}$:

- $A = \mathbb{N}$ (the set of natural numbers)
- no function symbol ($\mathcal{F} = \emptyset$)
- $S^{\mathcal{M}} = =_{\mathbb{N}}$ (equality relation between natural numbers)

Then, $\forall x \forall y((x = y) \to (x = y)))$ evaluates to true, because the equality is symmetric and $\forall z \neg (z = z)$ evaluates to false, because the equality is also reflexive. So the implication of these two evaluates to false.

Hence, $\mathcal{M} \not\models (\forall x \forall y(S(x,y) \to (x = y))) \to (\forall z \neg S(z,z))$

(g) $(\forall x \exists y(S(x,y) \wedge ((S(x,y) \wedge S(y,z)) \rightarrow (x = y)))) \rightarrow (\neg \exists z \forall w(S(z,w)))$

The left-hand side of the formula resembles the antisymmetry property of the $\leq$ ordering on natural numbers, $(x \leq y \wedge y \leq x) \rightarrow x = y$.

We could choose the following model $\mathcal{M}$:

- $A = \mathbb{N}$ (the set of natural numbers)
- no function symbol ($\mathcal{F} = \emptyset$)
- $S^{\mathcal{M}} = \leq_{\mathbb{N}}$ (less-than or equal relation between natural numbers)

Thus, we have $\forall x \exists y(S(x,y) \wedge ((S(x,y) \wedge S(y,z)) \rightarrow (x = y)))$ evaluates to true (from axiom of foundation and antisymmetry for inequality of natural numbers) while $\neg \exists z \forall w(S(z,w))$ evaluates to false (because in fact there is a natural number smaller than any other, that is we may choose $z = 0$). Hence the implication of these two evaluates to false in this model.

Hence, $\mathcal{M} \not\models (\forall x \exists y(S(x,y) \wedge ((S(x,y) \wedge S(y,z)) \rightarrow (x = y)))) \rightarrow (\neg \exists z \forall w(S(z,w)))$