04a—Propositional Logic II

CS 3234: Logic and Formal Systems

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Atoms

Convention

We usually use p, q, p_1, etc, instead of sentences like “The sun is shining today”.

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More formally, we fix a set A of propositional atoms.

Meaning of Atoms

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More formally

A model (valuation) for a propositional logic for the set A of atoms is a mapping from A to $\{T, F\}$.

Inductive Definition

Definition

For a given set A of propositional atoms, the set of *well-formed formulas in propositional logic* is the least set F that fulfills the following rules:

- The constant symbols $\bot$ and $\top$ are in F.
- Every element of A is in F.
- If ϕ is in F, then $(\neg\phi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \wedge \psi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \vee \psi)$ is also in F.
- If ϕ and ψ are in F, then $(\phi \rightarrow \psi)$ is also in F.

Parse trees

A formula

$$(((\neg p) \land q) \to (p \land (q \lor (\neg r))))$$

Parse trees

A formula

$$(((\neg p) \land q) \rightarrow (p \land (q \lor (\neg r))))$$

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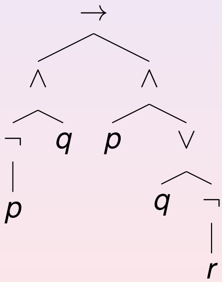

Evaluation of Formulas

Definition

The result of *evaluating* a well-formed propositional formula ϕ with respect to a valuation v, denoted $v(\phi)$ is defined as follows:

- If ϕ is the constant $\bot$, then $v(\phi) = F$.
- If ϕ is the constant $\top$, then $v(\phi) = T$.
- If ϕ is an propositional atom p, then $v(\phi) = p^v$.
- If ϕ has the form $(\neg\psi)$, then $v(\phi) = \backslash v(\psi)$.
- If ϕ has the form $(\psi \land \tau)$, then $v(\phi) = v(\psi) \& v(\tau)$.
- If ϕ has the form $(\psi \lor \tau)$, then $v(\phi) = v(\psi) \mid v(\tau)$.
- If ϕ has the form $(\psi \to \tau)$, then $v(\phi) = v(\psi) \Rightarrow v(\tau)$.

Valid and Satisfiable Formulas

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A formula is called *satisfiable* if it evaluates to T with respect to at least one valuation.

Questions about Propositional Formula

- Is a given formula valid?
- Is a given formula satisfiable?
- Is a given formula invalid?
- Is a given formula unsatisfiable?
- Are two formulas equivalent?

Decision Problems

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Examples

The question whether a given propositional formula is satisifiable (unsatisfiable, valid, invalid) is a decision problem.

The question whether two given propositional formulas are equivalent is also a decision problem.

How to Solve the Decision Problem?

Question

How do you decide whether a given propositional formula is satisfiable/valid?

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The good news

We can construct a truth table for the formula and check if some/all rows have T in the last column.

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An algorithm for satisifiability

Using a truth table, we can implement an *algorithm* that returns *"yes"* if the formula is satisifiable, and that returns *"no"* if the formula is unsatisfiable.

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Decidability of satisfiability

The question, whether a given propositional formula is satisifiable, is decidable.

The Bad News

Concern

In practice, propositional formulas can be large. Example:
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Techniques so far inadequate

Proving satisfiability/validity using truth tables or natural deduction is impractical for large formulas.

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Is there a *polynomial-time* algorithm that decides whether a given formula is satisfiable?

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Question

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Is there a *polynomial-time* algorithm that decides whether a given formula is satisfiable?

Answer

We do not know!

What *do* we know about satisfiability?

Truth assignment as witness

If the answer is “yes”, then a satisfying truth assignment can serve as a proof that the answer is indeed “yes”.

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Witness for satisfiability

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Checking the witness

We can quickly check whether indeed the witness assignment makes the formula true. This can be done in time proportional to the size of the formula.

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NP stands for “**N**on-deterministic **P**olynomial time”.

Original definition

NP is the set of decision problems solvable in polynomial time by a non-deterministic Turing machine.

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- In 1972, Richard Karp presented 21 mutually equivalent problems in NP, for which no polynomial time algorithms was known.
- Cook and Leonid Levin proved independently that propositional satisifiability is in this class (called NP-complete).

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- Many computer scientists assume $P \neq \mathrm{NP}$, and therefore consider NP-complete problems as "intractable".
- Many "proofs" for one or the other answer have been proposed, and subsequently rejected, most recently by Vinay Deolalikar (a researcher at HP), in August 2010.

Conjunctive Normal Form

Definition

A literal L is either an atom p or the negation of an atom $\neg p$. A formula C is in *conjunctive normal form* (CNF) if it is a conjunction of clauses, where each clause is a disjunction of literals:

$$\begin{aligned} L & ::= & p \mid \neg p \\ D & ::= & L \mid L \vee D \\ C & ::= & D \mid D \wedge C \end{aligned}$$

Examples

$(\neg p \vee q \vee r) \wedge (\neg q \vee r) \wedge (\neg r)$ is in CNF.
$(\neg p \vee q \vee r) \wedge ((p \wedge \neg q) \vee r) \wedge (\neg r)$ is not in CNF.
$(\neg p \vee q \vee r) \wedge \neg(\neg q \vee r) \wedge (\neg r)$ is not in CNF.

Usefulness of CNF

Lemma

A disjunction of literals $L_1 \lor L_2 \lor \cdots \lor L_m$ is valid iff there are $1 \leq i, j \leq m$ such that L_i is $\neg L_j$.

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Proposition

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Satisfiability test

We can test satisfiability of ϕ by transforming $\neg\phi$ into CNF, and show that some clause is not valid.

Transformation to CNF

Theorem

Every formula in the propositional calculus can be transformed into an equivalent formula in CNF.

Algorithm for CNF Transformation

1. Eliminate implication using:
   $A \to B \equiv \neg A \lor B$
2. Push all negations inward using De Morgan's laws:
   $\neg(A \land B) \equiv (\neg A \lor \neg B)$
   $\neg(A \lor B) \equiv (\neg A \land \neg B)$
3. Eliminate double negations using the equivalence $\neg\neg A \equiv A$
4. The formula now consists of disjunctions and conjunctions of literals. Use the distributive laws
   $A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$
   $(A \land B) \lor C \equiv (A \lor C) \land (B \lor C)$
   to eliminate conjunctions within disjunctions.

Example

$$
\begin{aligned}
(\neg p \rightarrow \neg q) \rightarrow (p \rightarrow q) & \equiv \neg(\neg\neg p \vee \neg q) \vee (\neg p \vee q) \\
& \equiv (\neg\neg\neg p \wedge q) \vee (\neg p \vee q) \\
& \equiv (\neg p \wedge q) \vee (\neg p \vee q) \\
& \equiv (\neg p \vee \neg p \vee q) \wedge (q \vee \neg p \vee q)
\end{aligned}
$$

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