Coq: What is it doing, and the object-meta swap

CS3234

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Proofs

- We’ve now shown you two (related) ways of drawing proofs: trees and 3-column format
- Actually, the 3-column format is encoding a tree (really a directed acyclic graph, i.e., you can reuse lines without copying them out multiple times)

A proof (3 column)

- Consider this very simple proof:

1.	$P \wedge Q$	(assumption)
2.	P	($\wedge$e1, 1)
3.	Q	($\wedge$e2, 1)
4.	$Q \wedge P$	($\wedge$i, 3, 2)
5.	$(P \wedge Q) \rightarrow (Q \wedge P)$	($\rightarrow$i, 1—5)

A proof (tree)

- The same proof in tree form

$$\dfrac{\boxed{\dfrac{\dfrac{\dfrac{}{P \wedge Q}\,\text{assumption}}{Q}\,\wedge\text{e2} \qquad \dfrac{\dfrac{}{P \wedge Q}\,\text{assumption}}{P}\,\wedge\text{e1}}{Q \wedge P}\,\wedge\text{i}}\; P \wedge Q}{(P \wedge Q) \rightarrow (Q \wedge P)}\,\rightarrow\text{i}$$

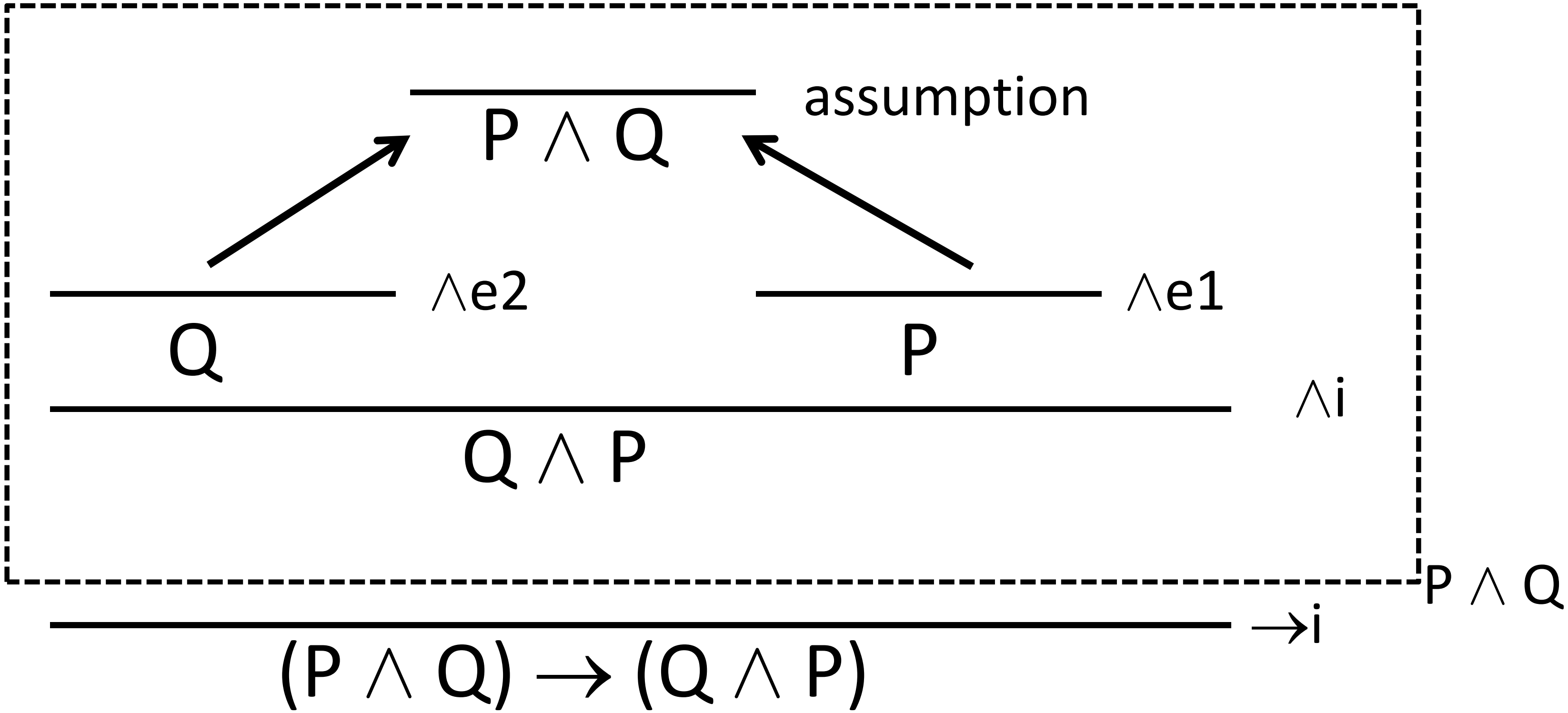
DAG

- The 3-column did not have this duplication:

$$\dfrac{\boxed{\dfrac{\dfrac{\dfrac{}{P \wedge Q}\ \text{assumption}}{Q}\ \wedge e2 \qquad \dfrac{\dfrac{}{P \wedge Q}\ \text{assumption}}{P}\ \wedge e1}{Q \wedge P}\ \wedge i}\ P \wedge Q}{(P \wedge Q) \to (Q \wedge P)}\ \to i$$

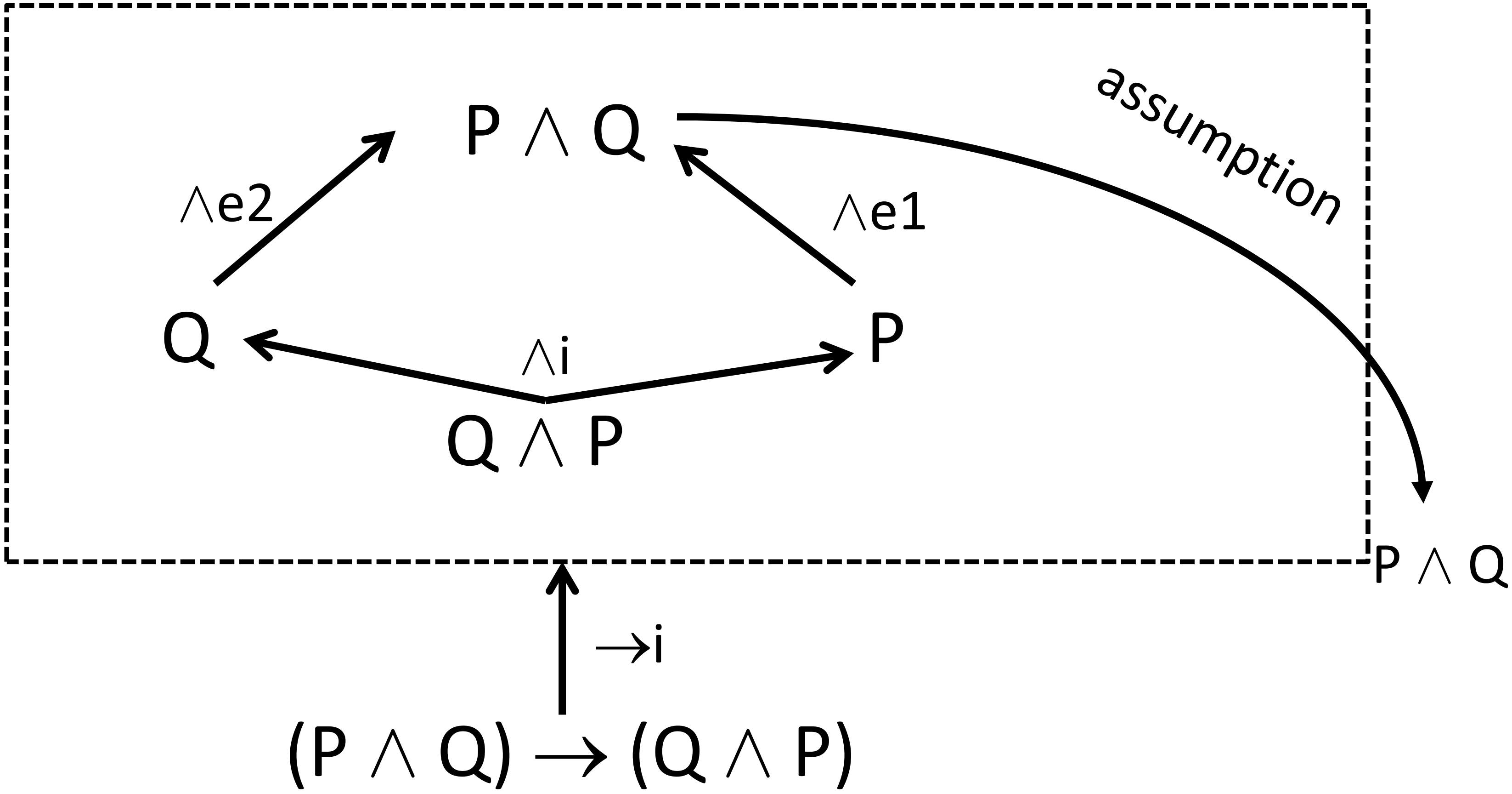
DAG

- In some sense the structure looks like this:



DAG

- Or more accurately like this:



Coq

- Coq is helping you build these trees/DAGs starting from the “bottom up”:

$$\begin{tikzcd} ? \\ (P \land Q) \to (Q \land P) \arrow[u, "?"'] \end{tikzcd}$$

What it actually looks like

```
1 subgoal
v : Valuation
P : PropFormula
Q : PropFormula
______________________________________(1/1)
holds v (propImpl (propConj P Q) (propConj Q P))
```

Now, we want to "`apply Impl_I`"

What it actually looks like

```
1 subgoal
v : Valuation
P : PropFormula
Q : PropFormula
______________________________________(1/1)
holds v (propImpl (propConj P Q) (propConj Q P))
```

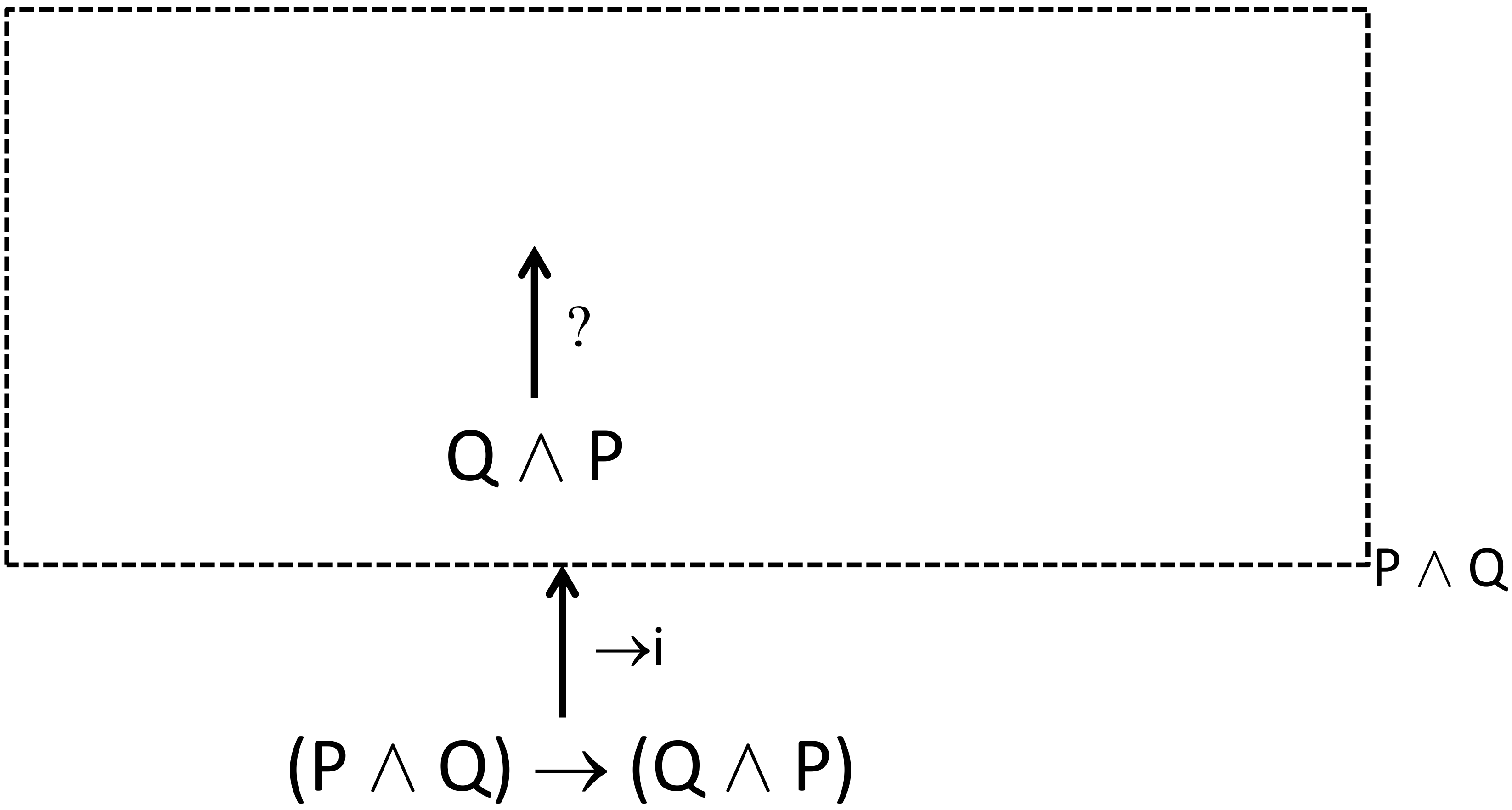
Now, we "`apply Impl_I`" (then `intro`)

What it actually looks like

```
1 subgoal
v : Valuation
P : PropFormula
Q : PropFormula
H : holds v (propConj P Q)
______________________________________(1/1)
holds v (propConj Q P)
```

What happened to the tree?

Coq



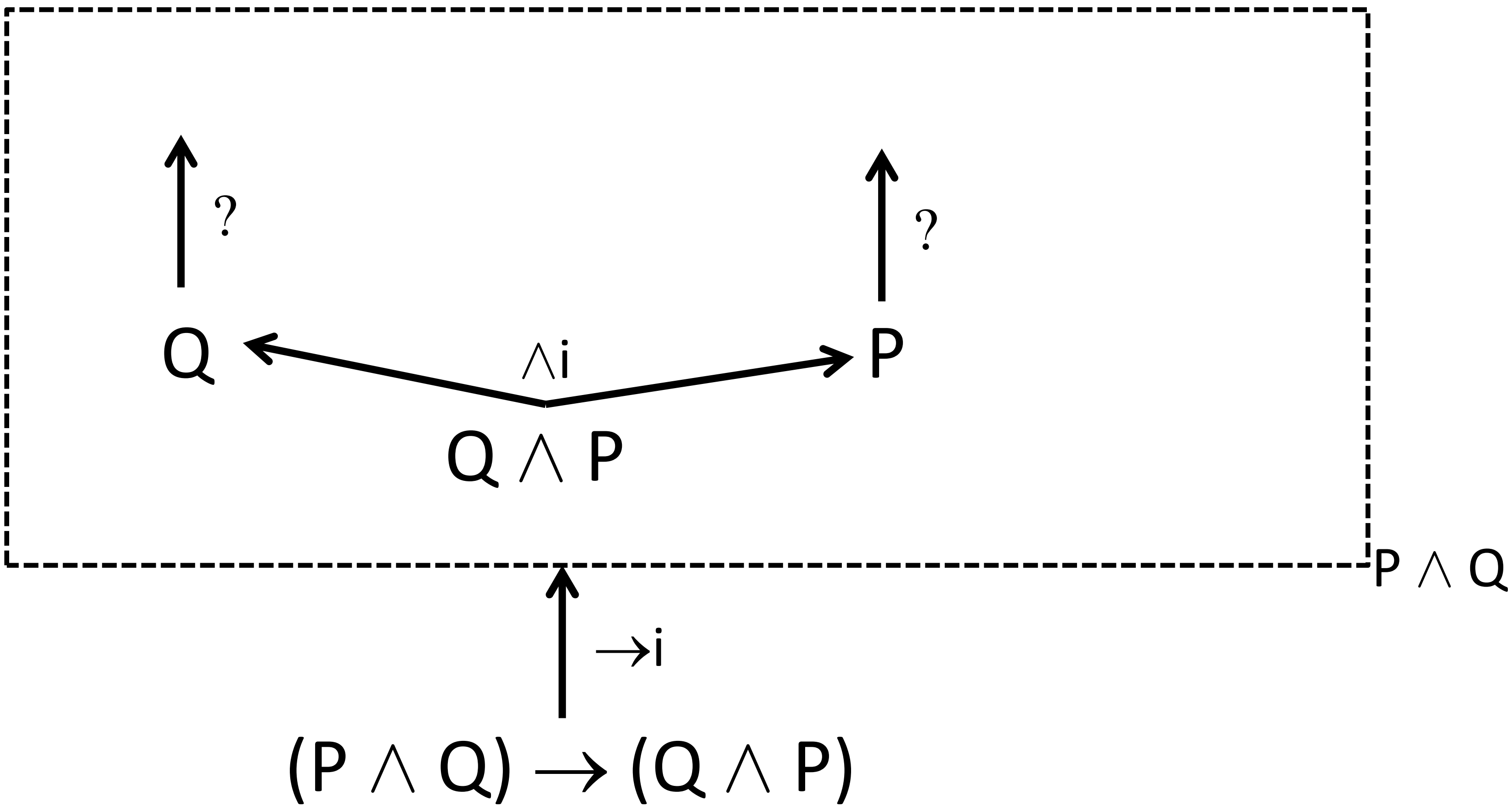
What it actually looks like

Now, we want to “`apply Conj_I`”

```
2 subgoals
v : Valuation
P : PropFormula
Q : PropFormula
H : holds v (propConj P Q)
______________________________________(1/2)
holds v Q


______________________________________(2/2)
holds v P
```

Coq



What it actually looks like

Now, we “`apply Conj_E2 with P`”

```
2 subgoals
v : Valuation
P : PropFormula
Q : PropFormula
H : holds v (propConj P Q)
______________________________________(1/2)
holds v (propConj P Q)


______________________________________(2/2)
holds v P
```

What it actually looks like

Now, we "`apply Conj_E2 with P`"

The old goal was "`holds v Q`"

Can't guess P from the goal

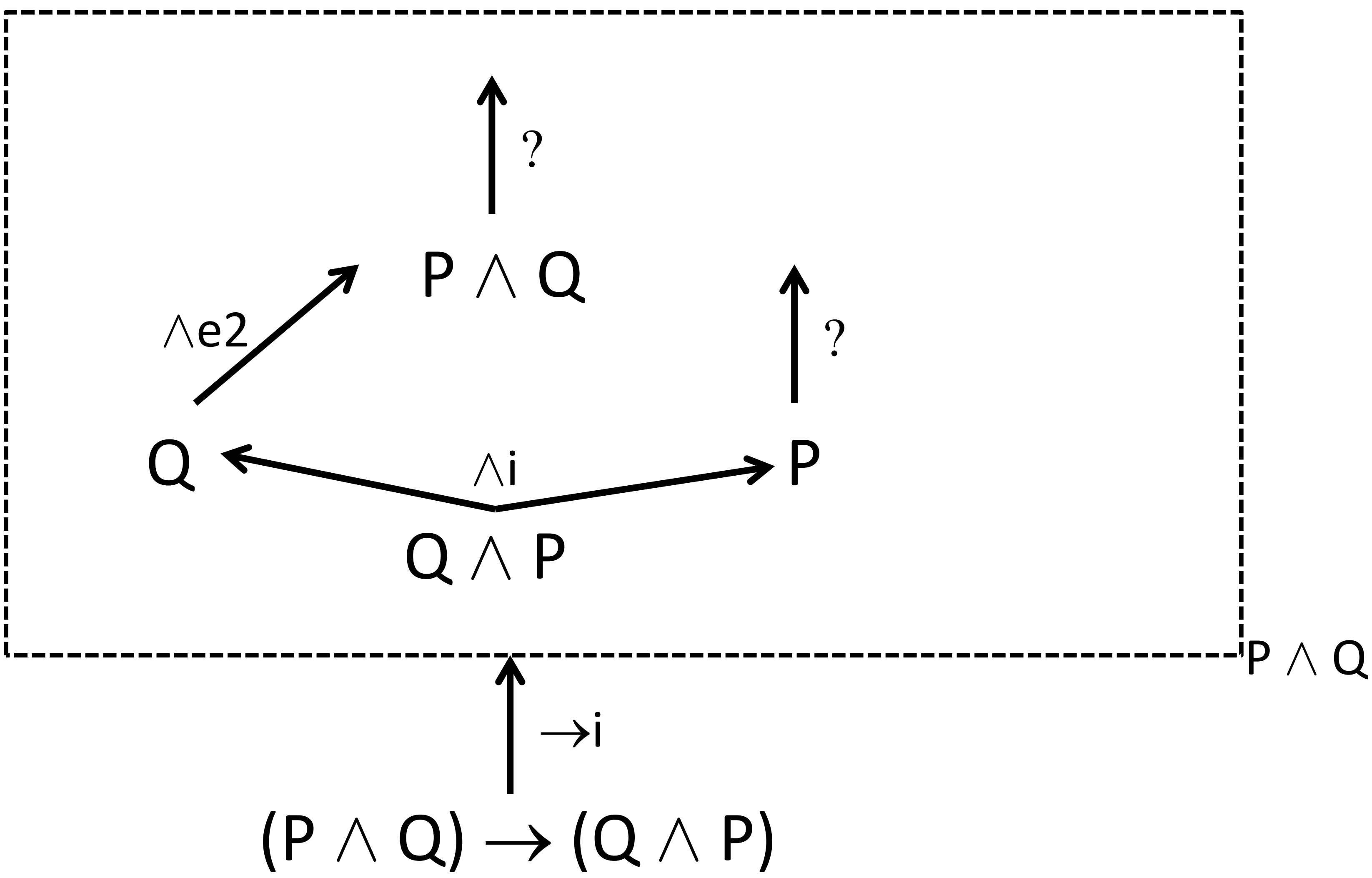
What it actually looks like

Now, we "`apply Conj_E2 with P`"

```
2 subgoals
v : Valuation
P : PropFormula
Q : PropFormula
H : holds v (propConj P Q)
______________________________________(1/2)
holds v (propConj P Q)


______________________________________(2/2)
holds v P
```

Coq

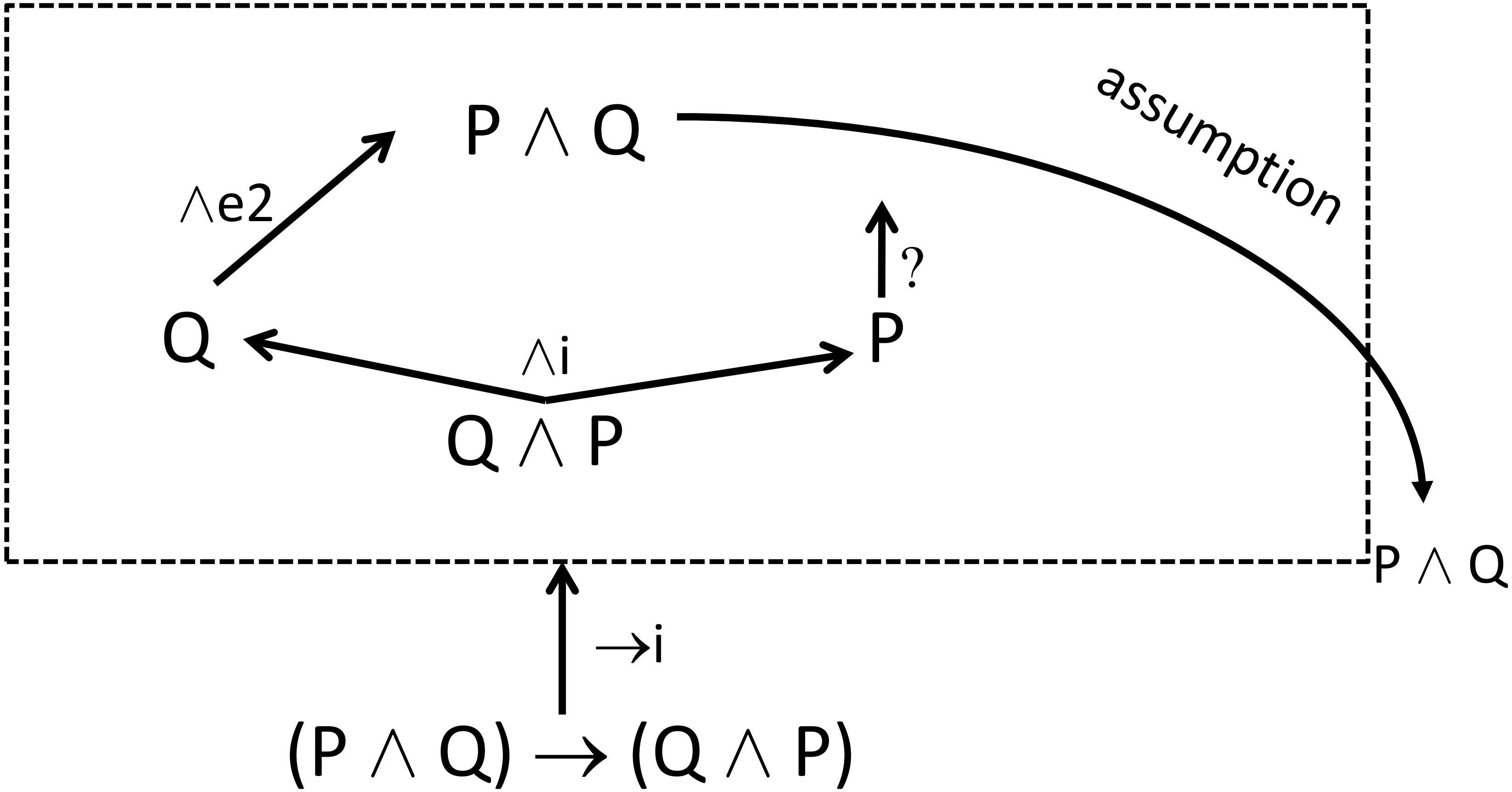


What it actually looks like

Now, we “`assumption`”

```
1 subgoals
v : Valuation
P : PropFormula
Q : PropFormula
H : holds v (propConj P Q)
_____________________________________(1/1)
holds v P
```

DAG



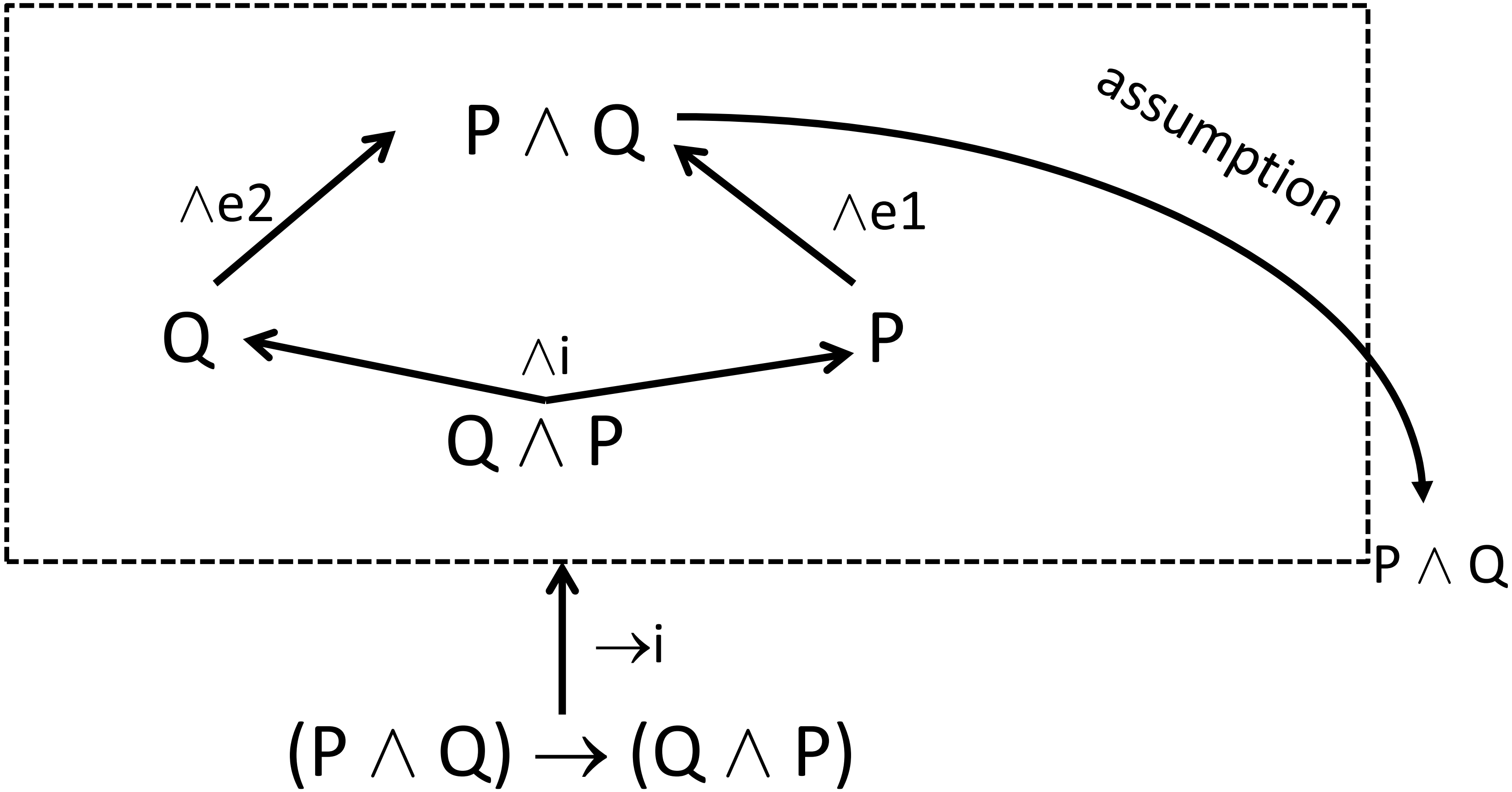
What it actually looks like

Now, we "`apply ConjE1 with Q`" and "`trivial`"

```
Proof completed.
```

DAG

- More-or-less like this:



That’s how proofs work in Coq

- The details can get messy, but more-or-less that’s how Coq is working.

- Now we want to talk about something else you may have wondered about.

What it actually looks like

```
1 subgoal
v : Valuation
P : PropFormula
Q : PropFormula
______________________________________(1/1)
holds v (propImpl (propConj P Q) (propConj Q P))
```

Now, we "`apply Impl_I`" (then `intro`)

Why do we need to do this?

We’ve been cheating

- We wanted to emphasize that formulas were just “data structures” – so we built them explicitly as (inductively defined) data:

```
Inductive PropFormula : Type :=
  | propTop : PropFormula
  | propBot : PropFormula
  | propAtom : Atom -> PropFormula
  | propNeg : PropFormula -> PropFormula
  | propConj : PropFormula -> PropFormula -> PropFormula
  | propDisj : PropFormula -> PropFormula -> PropFormula
  | propImpl : PropFormula -> PropFormula -> PropFormula.
```

Built-in operators

- But Coq has all of these already built-in, and so we had to use some of the built-in operators from time-to-time.

- You've even seen some of them! For example, `->` instead of `propImpl`. Coq also has:
  - `/\` instead of `propConj`
  - `\/` instead of `propDisj`
  - `~` instead of `propNeg`
  - `True`/`False` for `propTop`/`propBot`

Since it's built in, it's much shorter:

```
Lemma Conj_Comm:
  forall v P Q,
  holds v (propImpl
  (propConj P Q)
  (propConj Q P)).
Proof.
intros.
apply Impl_I. intro.
apply Conj_I.
apply Conj_E2 with P.
trivial.
apply Conj_E1 with Q.
trivial.
Qed.
```

```
Lemma Conj_Comm2:
  forall P Q,
  P /\ Q -> Q /\ P.
Proof.
  intros.
  destruct H.
  split.
  trivial.
  trivial.
Qed.
```

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Qed.
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Qed.
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Proof.
  intros.
  destruct H.
  split.
  trivial.
  trivial.
Qed.
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Proof.
intros.
apply Impl_I. intro.
apply Conj_I.
apply Conj_E2 with P.
trivial.
apply Conj_E1 with Q.
trivial.
Qed.
```

```
Lemma Conj_Comm2:
  forall P Q,
  P /\ Q -> Q /\ P.
Proof.
  intros.
  destruct H.
  split.
  trivial.
  trivial.
Qed.
```

Or even (the wonders of automation)

```
Lemma Conj_Comm3:
  forall P Q,
   P /\ Q -> Q /\
  P.
Proof.
   intros.
   destruct H.
   split; trivial.
Qed.
```

Or even (the wonders of automation)

```
Lemma Conj_Comm3:
  forall P Q,
   P /\ Q -> Q /\
  P.
Proof.
   intros.
   destruct H.
   split; trivial.
Qed.
```

```
Lemma Conj_Comm4:
  forall P Q,
   P /\ Q -> Q /\
  P.
Proof.
   tauto.
Qed.
```

A quick tactic list (not complete!)

$\rightarrow$ i	`intro (intros)`
$\rightarrow$ e	`apply (apply in, generalize, spec)`
$\wedge$ i	`split`
$\wedge$ e12	`destruct`
$\vee$ i1	`left`
$\vee$ i2	`right`
$\vee$ e	`destruct`

Going “meta”

- From now on (not counting this week’s Coq HW or next week’s quiz), we will always present the Coq using the built-in operators

- This should save you considerable time and let you write shorter proofs…

- The real advantage comes when we move to predicate logic

Tactics for predicate logic

- Actually, you have already seen some of these… and they are not very hard.

$\forall$i	`intros`
$\forall$e	`spec, generalize (apply…)`
$\exists$i	`exists`
$\exists$e	`destruct`

Coq HW (not until next week)

- We will give some homework next week that will involve these tactics for your practice

- Don't Panic: The quiz next week will not cover them (except `intros` which you have seen lots of), so it's just a "sneak peek" now.

- Often it is much easier to do predicate logic in the theorem prove than on paper.

The horrible side of predicate logic

- You may recall the (horrible) issues of variable hiding and variable capture.
- *Almost guaranteed* to be on midterm & final.
- The great thing is Coq does all of this for you.

Demo

- We have two lemmas that demonstrate some of the issues involved in variable scoping.

Questions?