08—Modal Logic

CS 3234: Logic and Formal Systems

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1 Motivation
2 Basic Modal Logic
3 Logic Engineering
Motivation

Basic Modal Logic

Logic Engineering
Necessity

You are crime investigator and consider different suspects. You know that the victim Ms Smith had called the police.
Necessity

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  - Maybe the cook did it before dinner?
**Necessity**

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- Maybe the cook did it before dinner?
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You are crime investigator and consider different suspects. You know that the victim Ms Smith had called the police.
  - Maybe the cook did it before dinner?
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But: “The victim Ms Smith made a phone call before she was killed.” is necessarily true.
Necessity

- You are crime investigator and consider different suspects. You know that the victim Ms Smith had called the police.
  - Maybe the cook did it before dinner?
  - Maybe the maid did it after dinner?
- But: “The victim Ms Smith made a phone call before she was killed.” is necessarily true.
- “Necessarily” means in all possible scenarios (worlds) under consideration.
Often, it is not enough to distinguish between “true” and “false”.
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We need to consider modalities if truth, such as:

- necessity (“in all possible scenarios”)
- morality/law (“in acceptable/legal scenarios”)
- time (“forever in the future”)
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- necessity (“in all possible scenarios”)
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Modal logic constructs a framework using which modalities can be formalized and reasoning methods can be established.
1 Motivation

2 Basic Modal Logic
   - Syntax
   - Semantics
   - Equivalences

3 Logic Engineering
Syntax of Basic Modal Logic

\[ \phi ::= \top | \bot | p | (\neg \phi) | (\phi \land \phi) \]
\[ | (\phi \lor \phi) | (\phi \rightarrow \phi) \]
\[ | (\Box \phi) | (\Diamond \phi) \]
Pronunciation

If we want to keep the meaning open, we simply say “box” and “diamond”.
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If we want to appeal to our intuition, we may say “necessarily” and “possibly” (or “forever in the future” and “sometime in the future”)
Pronunciation and Examples

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Examples

\((p \land \diamond(p \rightarrow \square \neg r))\)
Pronunciation

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If we want to appeal to our intuition, we may say “necessarily” and “possibly” (or “forever in the future” and “sometime in the future”)

Examples

\((p \land \Diamond(p \rightarrow \Box \neg r))\)

\(\Box((\Diamond q \land \neg r) \rightarrow \Box p)\)
A model $\mathcal{M}$ of propositional modal logic over a set of propositional atoms $A$ is specified by three things:
Kripke Models

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Definition

A model $\mathcal{M}$ of propositional modal logic over a set of propositional atoms $A$ is specified by three things:

1. A set $W$ of worlds;
2. A relation $R$ on $W$, meaning $R \subseteq W \times W$, called the accessibility relation;
3. A function $L : W \to A \to \{T, F\}$, called labeling function.
Who is Kripke?

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At Princeton Kripke taught philosophy from 1977 onwards.

Contributions include modal logic, naming, belief, truth, the meaning of “I”
Example

\[ W = \{ x_1, x_2, x_3, x_4, x_5, x_6 \} \]

\[ R = \{ (x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_4, x_5), (x_5, x_4), (x_5, x_6) \} \]

\[ L = \{ (x_1, \{ q \}), (x_2, \{ p, q \}), (x_3, \{ p \}), (x_4, \{ q \}), (x_5, \{ \}), (x_6, \{ p \}) \} \]
When is a formula true in a possible world?

Definition

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and $\phi$ a formula in basic modal logic. We define $x \models \phi$ via structural induction:
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Definition

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and $\phi$ a formula in basic modal logic. We define $x \models \phi$ via structural induction:

- $x \models \top$
- $x \not\models \bot$
Motivation
Basic Modal Logic
Logic Engineering
Syntax
Semantics
Equivalences

When is a formula true in a possible world?

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Let $\mathcal{M} = (W, R, L)$, $x \in W$, and $\phi$ a formula in basic modal logic. We define $x \models \phi$ via structural induction:

- $x \models \top$
- $x \not\models \bot$
- $x \models p$ iff $L(x)(p) = T$
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- $x \models \top$
- $x \not\models \bot$
- $x \models p$ iff $L(x)(p) = T$
- $x \models \neg \phi$ iff $x \not\models \phi$
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- $x \models \phi \land \psi$ iff $x \models \phi$ and $x \models \psi$
- $x \models \phi \lor \psi$ iff $x \models \phi$ or $x \models \psi$
- ...

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When is a formula true in a possible world?

**Definition (continued)**

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and $\phi$ a formula in basic modal logic. We define $x \models \phi$ via structural induction:

- ... 
- $x \models \phi \rightarrow \psi$ iff $x \models \psi$, whenever $x \models \phi$
When is a formula true in a possible world?

Definition (continued)

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and $\phi$ a formula in basic modal logic. We define $x \models \phi$ via structural induction:

- $\ldots$
- $x \models \phi \rightarrow \psi$ iff $x \models \psi$, whenever $x \models \phi$
- $x \models \Box \phi$ iff for each $y \in W$ with $R(x, y)$, we have $y \models \phi$
When is a formula true in a possible world?

Definition (continued)

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and $\phi$ a formula in basic modal logic. We define $x \vDash \phi$ via structural induction:

- $x \vDash \phi \rightarrow \psi$ iff $x \vDash \psi$, whenever $x \vDash \phi$
- $x \vDash \Box \phi$ iff for each $y \in W$ with $R(x, y)$, we have $y \vDash \phi$
- $x \vDash \Diamond \phi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \vDash \phi$. 
Motivation
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Logic Engineering

Syntax
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Equivalences

Example
Example

\[ x_1 \models q \]

\[ p, q \]

\[ p \]

\[ q \]

\[ x_2 \]

\[ x_3 \]

\[ x_4 \]

\[ x_5 \]

\[ x_6 \]
Example

- $x_1 \models q$
- $x_1 \models \diamond q$, $x_1 \not\models \Box q$
Example

- $\models q$
- $\models \Diamond q, \models \neg \Box q$
- $\models \Box p, \models \Box q, \models \Box p \lor \Box q, \models \Box(p \lor q)$
Example

- $x_1 \models q$
- $x_1 \models \Diamond q$, $x_1 \not\models \Box q$
- $x_5 \not\models \Box p$, $x_5 \not\models \Box q$, $x_5 \not\models \Box p \lor \Box q$, $x_5 \models \Box (p \lor q)$
- $x_6 \models \Box \phi$ holds for all $\phi$, but $x_6 \not\models \Diamond \phi$ regardless of $\phi$

Example

```
\begin{align*}
\text{Motivation} & \quad \text{Syntax} \\
\text{Basic Modal Logic} & \quad \text{Semantics} \\
\text{Logic Engineering} & \quad \text{Equivalences}
\end{align*}
```

Motivation

Basic Modal Logic

Logic Engineering

Syntax

Semantics

Equivalences

Example

```
\begin{align*}
& \text{x}_1 \models q \\
& \text{x}_1 \models \Diamond q, \text{x}_1 \not\models \Box q \\
& \text{x}_5 \not\models \Box p, \text{x}_5 \not\models \Box q, \text{x}_5 \not\models \Box p \lor \Box q, \text{x}_5 \models \Box (p \lor q) \\
& \text{x}_6 \models \Box \phi \text{ holds for all } \phi, \text{ but } \text{x}_6 \not\models \Diamond \phi \text{ regardless of } \phi
\end{align*}
```

08—Modal Logic
We said $x_6 \models \Box \phi$ holds for all $\phi$, but $x_6 \not\models \Diamond \phi$ regardless of $\phi$. 
Formula Schemes

Example
We said $x_6 \models \Box \phi$ holds for all $\phi$, but $x_6 \not\models \Diamond \phi$ regardless of $\phi$.

Notation
Greek letters denote formulas, and are not propositional atoms.
We said $x_6 \models \Box \phi$ holds for all $\phi$, but $x_6 \not\models \Diamond \phi$ regardless of $\phi$.

Greek letters denote formulas, and are not propositional atoms.

Terms where Greek letters appear instead of propositional atoms are called *formula schemes*. 
Entailment and Equivalence

**Definition**

A set of formulas $\Gamma$ entails a formula $\psi$ of basic modal logic if, in any world $x$ of any model $\mathcal{M} = (W, R, L)$, we have $x \models \psi$ whenever $x \models \phi$ for all $\phi \in \Gamma$. We say $\Gamma$ entails $\psi$ and write $\Gamma \models \psi$. 
Entailment and Equivalence

Definition
A set of formulas $\Gamma$ entails a formula $\psi$ of basic modal logic if, in any world $x$ of any model $\mathcal{M} = (W, R, L)$, we have $x \vDash \psi$ whenever $x \vDash \phi$ for all $\phi \in \Gamma$. We say $\Gamma$ entails $\psi$ and write $\Gamma \models \psi$.

Equivalence
We write $\phi \equiv \psi$ if $\phi \models \psi$ and $\psi \models \phi$. 
Some Equivalences

- De Morgan rules: $\neg \Box \phi \equiv \Diamond \neg \phi$, $\neg \Diamond \phi \equiv \Box \neg \phi$. 
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- De Morgan rules: $\neg \Box \phi \equiv \Diamond \neg \phi$, $\neg \Diamond \phi \equiv \Box \neg \phi$.
- Distributivity of $\Box$ over $\land$:
  
  \[
  \Box (\phi \land \psi) \equiv \Box \phi \land \Box \psi
  \]
Some Equivalences

- **De Morgan rules:**
  \[ \neg \Box \phi \equiv \Diamond \neg \phi, \quad \neg \Diamond \phi \equiv \Box \neg \phi. \]

- **Distributivity of \( \Box \) over \( \land \):**
  \[ \Box (\phi \land \psi) \equiv \Box \phi \land \Box \psi \]

- **Distributivity of \( \Diamond \) over \( \lor \):**
  \[ \Diamond (\phi \lor \psi) \equiv \Diamond \phi \lor \Diamond \psi \]
Some Equivalences

- **De Morgan rules:** \(\neg \Box \phi \equiv \Diamond \neg \phi\), \(\neg \Diamond \phi \equiv \Box \neg \phi\).
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  \]
- **Distributivity of \(\Diamond\) over \(\lor\):**
  \[
  \Diamond(\phi \lor \psi) \equiv \Diamond \phi \lor \Diamond \psi
  \]
- \(\Box T \equiv T\), \(\Diamond \bot \equiv \bot\)
Definition

A formula $\phi$ is valid if it is true in every world of every model, i.e.
iff $\models \phi$ holds.
Examples of Valid Formulas

- All valid formulas of propositional logic
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- All valid formulas of propositional logic
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- \( \Box (\phi \land \psi) \rightarrow \Box \phi \land \Box \psi \)
- \( \Diamond (\phi \lor \psi) \rightarrow \Diamond \phi \lor \Diamond \psi \)
- Formula \( K: \Box (\phi \rightarrow \psi) \rightarrow \Box \phi \rightarrow \Box \psi. \)
Motivation

Basic Modal Logic

Logic Engineering

Valid Formulas wrt Modalities
Properties of $R$
Correspondence Theory
Preview: Some Modal Logics
A Range of Modalities

In a particular context $\Box \phi$ could mean:
A Range of Modalities

In a particular context $\square \phi$ could mean:

- It is necessarily true that $\phi$
A Range of Modalities

In a particular context $\Box \phi$ could mean:

- It is necessarily true that $\phi$
- It will always be true that $\phi$
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- Agent $Q$ believes that $\phi$
- Agent $Q$ knows that $\phi$
- After any execution of program $P$, $\phi$ holds.
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In a particular context $\Box \phi$ could mean:

- It is necessarily true that $\phi$
- It will always be true that $\phi$
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- Agent $Q$ knows that $\phi$
- After any execution of program $P$, $\phi$ holds.

Since $\lozenge \phi \equiv \neg \Box \neg \phi$, we can infer the meaning of $\lozenge$ in each context.
From the meaning of $\Box \phi$, we can conclude the meaning of $\Diamond \phi$, since $\Diamond \phi \equiv \neg \Box \neg \phi$:

<table>
<thead>
<tr>
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It is necessarily true that $\phi$
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\[
\begin{array}{c}
\square \phi \\
\hline
\Diamond \phi
\end{array}
\]

It is necessarily true that $\phi$  
It is possibly true that $\phi$
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\begin{array}{c|c|c}
\square \phi & \Diamond \phi \\
\hline 
\text{It is necessarily true that } \phi & \text{It is possibly true that } \phi \\
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</tr>
</tbody>
</table>
A Range of Modalities

From the meaning of $\square \phi$, we can conclude the meaning of $\Diamond \phi$, since $\Diamond \phi \equiv \neg \square \neg \phi$:

<table>
<thead>
<tr>
<th>$\square \phi$</th>
<th>$\Diamond \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>It is necessarily true that $\phi$</td>
<td>It is possibly true that $\phi$</td>
</tr>
<tr>
<td>It will always be true that $\phi$</td>
<td>Sometime in the future $\phi$</td>
</tr>
<tr>
<td>It ought to be that $\phi$</td>
<td>It is permitted to be that $\phi$</td>
</tr>
<tr>
<td>Agent $Q$ believes that $\phi$</td>
<td>$\phi$ is consistent with $Q$’s beliefs</td>
</tr>
<tr>
<td>Agent $Q$ knows that $\phi$</td>
<td>For all $Q$ knows, $\phi$</td>
</tr>
<tr>
<td>After any run of $P$, $\phi$ holds.</td>
<td>After some run of $P$, $\phi$ holds</td>
</tr>
</tbody>
</table>
### Formula Schemes that hold wrt some Modalities

<p>| □φ | □φ → □φ | □φ → □□φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ | □φ → □φ |
|-----|---------|-----------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| □φ  | ✓       | ✓         | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓       | ✓�|</p>
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<thead>
<tr>
<th>$\square \phi$</th>
<th>$R(x, y)$</th>
</tr>
</thead>
<tbody>
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<td>It is necessarily true that $\phi$</td>
<td>$y$ is possible world according to info at $x$</td>
</tr>
<tr>
<td>It will always be true that $\phi$</td>
<td>$y$ is a future world of $x$</td>
</tr>
<tr>
<td>It ought to be that $\phi$</td>
<td>$y$ is an acceptable world according to the information at $x$</td>
</tr>
<tr>
<td>Agent Q believes that $\phi$</td>
<td>$y$ could be the actual world according to Q’s beliefs at $x$</td>
</tr>
<tr>
<td>Agent Q knows that $\phi$</td>
<td>$y$ could be the actual world according to Q’s knowledge at $x$</td>
</tr>
<tr>
<td>After any execution of P, $\phi$ holds</td>
<td>$y$ is a possible resulting state after execution of P at $x$</td>
</tr>
</tbody>
</table>
Possible Properties of $R$

- reflexive: for every $w \in W$, we have $R(x, x)$. 
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- Euclidean: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$.
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- Euclidean: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$.
- functional: for each $x$ there is a unique $y$ such that $R(x, y)$.
- linear: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$ or $y = z$ or $R(z, y)$. 
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- symmetric: for every $x, y \in W$, we have $R(x, y)$ implies $R(y, x)$.
- serial: for every $x$ there is a $y$ such that $R(x, y)$.
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- Euclidean: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$.
- functional: for each $x$ there is a unique $y$ such that $R(x, y)$.
- linear: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$ or $y = z$ or $R(z, y)$.
- total: for every $x, y \in W$, we have $R(x, y)$ and $R(y, x)$. 
Possible Properties of $R$

- reflexive: for every $w \in W$, we have $R(x, x)$.
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- total: for every $x, y \in W$, we have $R(x, y)$ and $R(y, x)$.
- equivalence: reflexive, symmetric and transitive.
Example

Consider the modality in which $\Box \phi$ means “it ought to be that $\phi$.”
Example

Consider the modality in which $\Box \phi$ means “it ought to be that $\phi$”.

- Should $R$ be reflexive?
Consider the modality in which \( \Box \phi \) means “it ought to be that \( \phi \”).

- Should \( R \) be reflexive?
- Should \( R \) be serial?
Guess

$R$ is reflexive if and only if $\Box \phi \rightarrow \phi$ is valid.
We would like to establish that some formulas hold whenever $R$ has a particular property.
We would like to establish that some formulas hold whenever $R$ has a particular property.

Ignore $L$, and only consider the $(W, R)$ part of a model, called *frame*. 
Motivation

- We would like to establish that some formulas hold whenever $R$ has a particular property.
- Ignore $L$, and only consider the $(W, R)$ part of a model, called frame.
- Establish formula schemes based on properties of frames.
Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

- $R$ is reflexive;
- $\mathcal{F}$ satisfies $\Box \phi \rightarrow \phi$;
- $\mathcal{F}$ satisfies $\Box p \rightarrow p$ for any atom $p$.
## Reflexivity and Transitivity

### Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

- $R$ is reflexive;
- $\mathcal{F}$ satisfies $\Box \phi \rightarrow \phi$;
- $\mathcal{F}$ satisfies $\Box p \rightarrow p$ for any atom $p$.

### Theorem 2

The following statements are equivalent:

- $R$ is transitive;
- $\mathcal{F}$ satisfies $\Box \phi \rightarrow \Box \Box \phi$;
- $\mathcal{F}$ satisfies $\Box p \rightarrow \Box \Box p$ for any atom $p$. 
Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

1. $R$ is reflexive;
2. $\mathcal{F}$ satisfies $\square \phi \rightarrow \phi$;
3. $\mathcal{F}$ satisfies $\square p \rightarrow p$ for any atom $p$.
Proof of Theorem 1

Let \( \mathcal{F} = (W, R) \) be a frame. The following statements are equivalent:

1. \( R \) is reflexive;
2. \( \mathcal{F} \) satisfies \( \Box \phi \rightarrow \phi \);
3. \( \mathcal{F} \) satisfies \( \Box p \rightarrow p \) for any atom \( p \)

1 \( \Rightarrow \) 2: Let \( R \) be reflexive.
Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

1. $R$ is reflexive;
2. $\mathcal{F}$ satisfies $\Box \phi \rightarrow \phi$;
3. $\mathcal{F}$ satisfies $\Box p \rightarrow p$ for any atom $p$

1 $\Rightarrow$ 2: Let $R$ be reflexive. Let $L$ be any labeling function; $\mathcal{M} = (W, R, L)$. 
Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

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1 $\Rightarrow$ 2: Let $R$ be reflexive. Let $L$ be any labeling function; $\mathcal{M} = (W, R, L)$. Need to show for any $x$: $x \models \square \phi \rightarrow \phi$
Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

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1 $\Rightarrow$ 2: Let $R$ be reflexive. Let $L$ be any labeling function; $\mathcal{M} = (W, R, L)$. Need to show for any $x$:

$x \models \square \phi \rightarrow \phi$ Suppose $x \not\models \square \phi$. 

Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

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$x \models \Box \phi \rightarrow \phi$ Suppose $x \not\models \Box \phi$.

Since $R$ is reflexive, we have $x \not\models \phi$. 

Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

1. $R$ is reflexive;
2. $\mathcal{F}$ satisfies $\Box \phi \rightarrow \phi$;
3. $\mathcal{F}$ satisfies $\Box p \rightarrow p$ for any atom $p$

1 $\Rightarrow$ 2: Let $R$ be reflexive. Let $L$ be any labeling function; $\mathcal{M} = (W, R, L)$. Need to show for any $x$: $x \models \Box \phi \rightarrow \phi$ Suppose $x \not\models \Box \phi$.
Since $R$ is reflexive, we have $x \not\models \phi$.
Using the semantics of $\rightarrow$: $x \not\models \Box \phi \rightarrow \phi$
Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

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Proof of Theorem 1

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1. $R$ is reflexive;
2. $\mathcal{F}$ satisfies $\square \phi \rightarrow \phi$;
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$2 \Rightarrow 3$: Just set $\phi$ to be $p$
Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

1. $R$ is reflexive;
2. $\mathcal{F}$ satisfies $\square \phi \rightarrow \phi$;
3. $\mathcal{F}$ satisfies $\square p \rightarrow p$ for any atom $p$
Proof of Theorem 1

Let $F = (W, R)$ be a frame. The following statements are equivalent:

1. $R$ is reflexive;
2. $F$ satisfies $\square \phi \rightarrow \phi$;
3. $F$ satisfies $\square p \rightarrow p$ for any atom $p$

$3 \Rightarrow 1$: Suppose the frame satisfies $\square p \rightarrow p$. 
Proof of Theorem 1

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3 $\Rightarrow$ 1: Suppose the frame satisfies $\Box p \rightarrow p$. Take any world $x$ from $W$. 

Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

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3. $\mathcal{F}$ satisfies $\Box p \rightarrow p$ for any atom $p$

3 $\Rightarrow$ 1: Suppose the frame satisfies $\Box p \rightarrow p$. Take any world $x$ from $W$. Choose a labeling function $L$ such that $L(x)(p) = F$, but $L(y)(p) = T$ for all $y$ with $y \neq x$
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Proof by contradiction: Assume $(x, x) \notin R$. 
Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

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2. $\mathcal{F}$ satisfies $\square \phi \rightarrow \phi$;
3. $\mathcal{F}$ satisfies $\square p \rightarrow p$ for any atom $p$

$3 \Rightarrow 1$: Suppose that the frame satisfies $\square p \rightarrow p$. Take any world $x$ from $W$. Choose a labeling function $L$ such that $L(x)(p) = F$, but $L(y)(p) = T$ for all $y$ with $y \neq x$. Proof by contradiction: Assume $(x, x) \notin R$. Then we would have $x \vDash \square p$, but not $x \vDash p$. Contradiction!
### Formula Schemes and Properties of $R$

<table>
<thead>
<tr>
<th>name</th>
<th>formula scheme</th>
<th>property of $R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$\Box \phi \rightarrow \phi$</td>
<td>reflexive</td>
</tr>
<tr>
<td>B</td>
<td>$\phi \rightarrow \Box \Diamond \phi$</td>
<td>symmetric</td>
</tr>
<tr>
<td>D</td>
<td>$\Box \phi \rightarrow \Diamond \phi$</td>
<td>serial</td>
</tr>
<tr>
<td>4</td>
<td>$\Box \phi \rightarrow \Box \Box \phi$</td>
<td>transitive</td>
</tr>
<tr>
<td>5</td>
<td>$\Diamond \phi \rightarrow \Box \Diamond \phi$</td>
<td>Euclidean</td>
</tr>
<tr>
<td></td>
<td>$\Box \phi \rightarrow \Diamond \phi \land \Diamond \phi \rightarrow \Box \phi$</td>
<td>functional</td>
</tr>
<tr>
<td></td>
<td>$\Box (\phi \land \Box \phi \rightarrow \psi) \lor \Box (\psi \land \Box \psi \rightarrow \phi)$</td>
<td>linear</td>
</tr>
</tbody>
</table>
Which Formula Schemes to Choose?

Definition

Let $\mathcal{L}$ be a set of formula schemes and $\Gamma \cup \{\psi\}$ a set of formulas of basic modal logic.
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A set of formula schemes is said to be closed iff it contains all substitution instances of its elements.
Which Formula Schemes to Choose?

Definition
Let $\mathcal{L}$ be a set of formula schemes and $\Gamma \cup \{\psi\}$ a set of formulas of basic modal logic.

- A set of formula schemes is said to be *closed* iff it contains all substitution instances of its elements.
- Let $\mathcal{L}_c$ be the smallest closed superset of $\mathcal{L}$. 
Definition

Let \( \mathcal{L} \) be a set of formula schemes and \( \Gamma \cup \{ \psi \} \) a set of formulas of basic modal logic.

- A set of formula schemes is said to be \textit{closed} iff it contains all substitution instances of its elements.
- Let \( \mathcal{L}_c \) be the smallest closed superset of \( \mathcal{L} \).
- \( \Gamma \) entails \( \psi \) in \( \mathcal{L} \) iff \( \Gamma \cup \mathcal{L}_c \) semantically entails \( \psi \). We say \( \Gamma \models_{\mathcal{L}} \psi \).
Examples of Modal Logics: K

K is the weakest modal logic, $\mathcal{L} = \emptyset$. 
Examples of Modal Logics: KT45

\[ \mathcal{L} = \{ T, 4, 5 \} \]
Examples of Modal Logics: KT45

\[ \mathcal{L} = \{ T, 4, 5 \} \]

Used for reasoning about knowledge.

- T: Truth: agent Q only knows true things.
- 4: Positive introspection: If Q knows something, he knows that he knows it.
- 5: Negative introspection: If Q doesn’t know something, he knows that he doesn’t know it.
Next Week

- Examples of Modal Logic
- Natural deduction in modal logic
- Modal logic in Coq