# Chapter 4

# Lecture 4 - Preliminaries



## Chocolate fish people

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## The extended Euclidean algorithm

$$x * a + y * p = \text{gcd}(x, y)$$

Now - if $p$ is a prime, then gcd$(x, y) = 1$, and so

$$x * a + y * p = 1$$

WRONG!



## The extended Euclidean algorithm

$$x * a + y * p = \text{gcd}(a, p)$$

Now - if $p$ is a prime, then gcd$(a, p) = 1$, and so

$$x * a + y * p = 1$$

RIGHT!

## Last session

- Math preliminaries
  - Fermat's little theorem
  - Euler



## This session

- Physical preliminaries
- Entropy



## This session

- Physical preliminaries
- Entropy

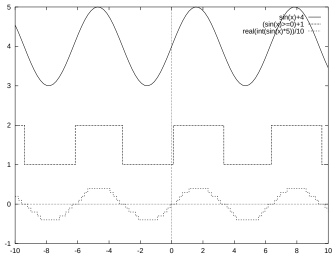


## Preliminaries - physical

Consider:

- Is the data analog or digital?
- What limits are placed on it?
- How is it to be transmitted?
- How can you be sure that it is correct/accurate?

## Analog and digital




The plot is **amplitude versus time**.

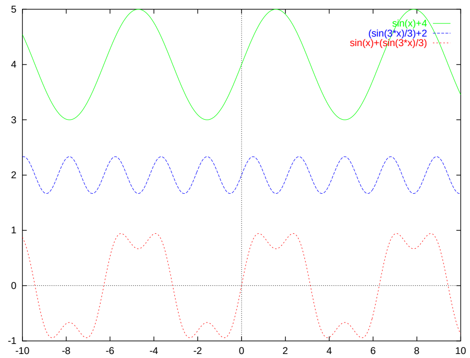


## Analog and digital

- ✔ Repetition rate (if it repeats) is called the *frequency*, and is measured in Hertz
- ✔ The peak to peak signal level is called the *amplitude*.
- ✔ The simplest analog signal is called the sine wave.
- ✔ By mixing we may create any desired periodic waveform.



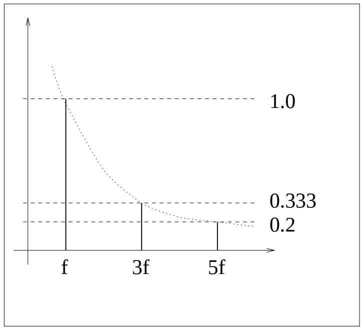
## Analog and digital




The plot is **amplitude versus time**. (Time domain)



## Analog and digital




The plot is **frequency versus time**. (Frequency domain).

## Analog and digital

If we were to continue in the same progression, the resultant waveform would be a square wave:

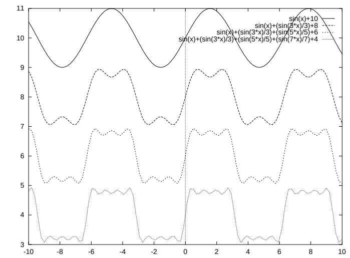
$$\sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n f) \text{ (for odd n)} \Rightarrow \text{ square wave, frequency } f$$

This representation method is known as *Fourier Analysis* after Jean-Baptiste Fourier.



## Fourier analysis

$$\frac{4}{\pi}(sin(2\pi f t) + \frac{1}{3}sin(6\pi f t) + \frac{1}{5}sin(10\pi f t) + \frac{1}{7}sin(14\pi f t) + ...)$$






## Fourier analysis

Transformation between equivalent time domain and frequency domain representations.

> *A piecewise continuously differentiable periodic function in the time domain may be transformed to a discrete aperiodic function in the frequency domain.*

$$\begin{array}{rcl} \text{smooth, repeating} & \leftrightarrow & \text{pointy, notrepeating} \\ f(t) & \leftrightarrow & F(\omega) \end{array}$$



## Fourier analysis

| Time domain | | Frequency domain | Description |
|---|---|---|---|
| Continuous, periodic | $\rightleftarrows$ | Discrete, aperiodic | **Fourier series** |
| Continuous, aperiodic | $\rightleftarrows$ | Continuous, aperiodic | **Fourier transform** |
| Discrete, periodic | $\rightleftarrows$ | Discrete, periodic | **Discrete Fourier series** |
| Discrete, aperiodic | $\rightleftarrows$ | Continuous, periodic | **Discrete Fourier transform** |

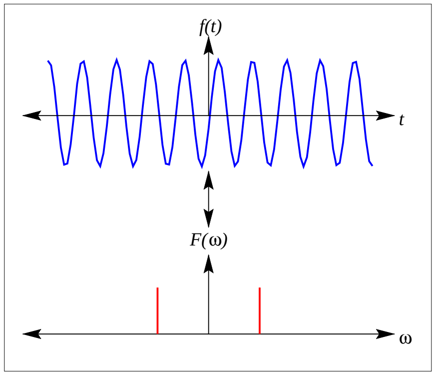
# Accuracy

Relationship between the bandwidth of a channel, and *how accurate* a signal is.

Another way of stating this is to point out that the higher frequency components are important - they are needed to re-create the original signal faithfully. If we had two 1,000Hz signals, one a triangle, one a square wave - if they were both passed through the 1,000Hz bandwidth limited channel above, they would look identical (a sine wave).

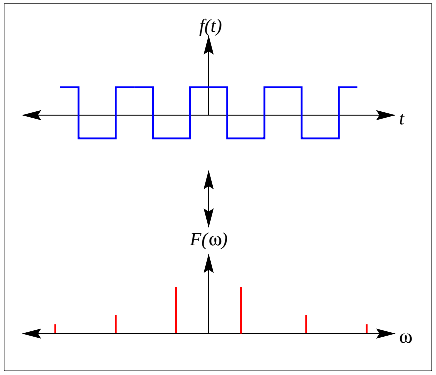


# Example transforms



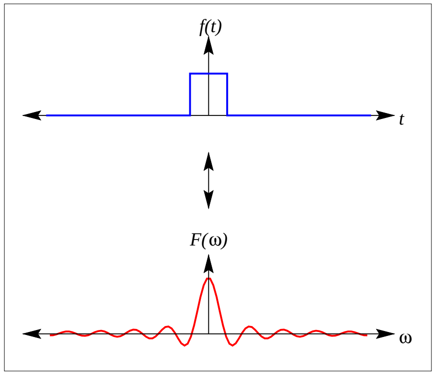



# Example transforms

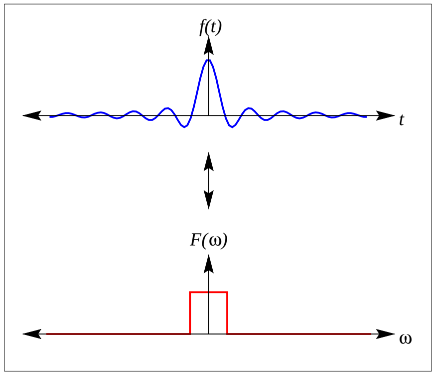





# Example transforms



## Example transforms






## Convolution

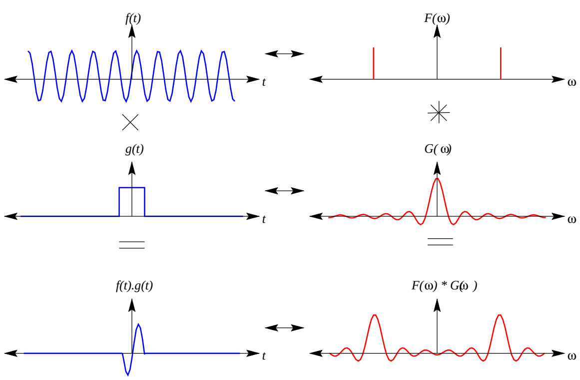
The Fourier transform of the convolution $f(t) \star g(t)$ is the product of the Fourier transforms of the functions $F(\omega)$ and $G(\omega)$, *and vice versa*.

$$
\begin{array}{rcl}
f(t) \star g(t) & \leftrightarrow & F(\omega) \times G(\omega) \\
f(t) \times g(t) & \leftrightarrow & F(\omega) \star G(\omega
\end{array}
$$

We can use convolution to easily predict the functions that result from complex signal filtering or sampling.



## Convolution






## Modulation

A baseband signal is one in which the data is directly converted to a signal and transmitted. When the signal is imposed on another signal, the process is called modulation.

We may *modulate* for several reasons:

- The media may not support the baseband signal
- We may wish to use a single transmission medium to transport many signals

## Modulation methods

- Frequency modulation - frequency shift keying (FSK)
- Amplitude modulation
- Phase modulation - phase shift keying (PSK)
- Combinations of the above (QAM)



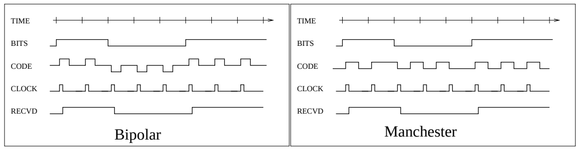
## Baseband digital encoding

The simplest encoding scheme is just to use a *low* level for a *zero* bit, and a *high* level for a *one* bit. As long as both ends of a channel are synchronized in some manner, we can transfer data.

On the other hand, if the ends of the channel are not synchronized we might use a simple encoding scheme, such as *Bipolar* or *Manchester* encoding, to transfer synchronizing (clock) information on the same channel.



## Baseband digital encoding




✔ In Bipolar encoding, a 1 is transmitted with a positive pulse, a 0 with a negative pulse. Sometimes called *return to zero* encoding.

✔ In Manchester encoding, there is a transition in the center of each bit cell.



## Summary

✔ Data commonly transferred digitally

✔ Trade-off between bandwidth, accuracy of any signal

## Information theory

The term ***information*** is commonly understood. Consider the following two sentences:

1. The sun will rise tomorrow.

2. The Fiji rugby team will win against the All Blacks (New Zealand rugby team) the next time they play.

**Question:** Which sentence contains the most information?



## Information theory

✘ Temperature today is OK, Temperature today is OK, Temperature today is OK, Temperature today is OK, Temperature today is OK, Temperature today is OK, Temperature today is OK, Temperature today is OK, Temperature today is OK, Temperature today is OK, ...

... total information here is close to zero!

?

More information means less predictable

Less information means more predictable



## Information theory

Nyquist (1924) and Hartley (1928) laid the foundations:

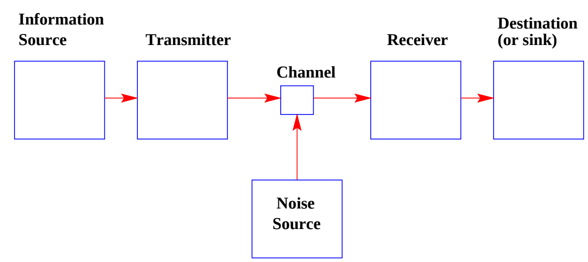
✔ Hartley showed that the information content is proportional to the *logarithm* of the number of possible messages. Integers between $1$ and $n$ need $\log_2 n$ bits.

✔ Shannon developed a mathematical treatment of communication and information in an important paper at

http://cm.bell-labs.com/cm/ms/what/shannonday/paper.html



## Information theory model




The relevance of Shannon to secrecy is in another important paper at

http://www.cs.ucla.edu/˜jkong/research/security/shannon.html

## Entropy

In our communication model, the units of transmission are called *messages*, constructed from an alphabet of (say) $n$ symbols $x \in \{x_1, \ldots, x_n\}$ each with a probability of transmission $P_x$.

We associate with each symbol $x$ a quantity $H_x$ which is a measure of the *information* associated with that symbol.

$$H_x = P_x \log_2 \frac{1}{P_x}$$



## Entropy

$$H_x = P_x \log_2 \frac{1}{P_x}$$

If the probability of occurence of each symbol is the same, we can derive Hartley's result, that the average amount of information transmitted in a single symbol (the source *entropy*) is

$$H(X) = \log_2 n$$

where $X$ is a label referring to each of the source symbols $x_1, \ldots, x_n$.



## Entropy units

Our units for entropy can be *bits/second* or *bits/symbol*, and we also sometimes use unit-less *relative* entropy measures (relative to the entropy of the system if all symbols were equally likely).



## Entropy - same probability

| Symbols | Entropy of each symbol | Bits needed |
|---|---|---|
| 2 | $H_x = \frac{1}{2}\log_2 2 = \frac{1}{2}$ | $2 * \frac{1}{2} = 1$ |
| 4 | $H_x = \frac{1}{4}\log_2 4 = \frac{1}{2}$ | $4 * \frac{1}{2} = 2$ |
| 8 | $H_x = \frac{1}{8}\log_2 8 = \frac{3}{8}$ | $8 * \frac{3}{8} = 3$ |
| 16 | $H_x = \frac{1}{16}\log_2 16 = \frac{4}{16}$ | $16 * \frac{4}{16} = 4$ |
| 21 | $H_x = \frac{1}{21}\log_2 21 = \frac{4.39}{21}$ | $21 * \frac{4.39}{21} = 4.39$ |

## Entropy - different probability

However, if the probability of occurence of each symbol is not the same, we derive the following result, that the source *entropy* is

$$H(X) = \sum_{i=1}^{n} P_{x_i} \log_2 \frac{1}{P_{x_i}}$$

Shannon's paper shows that $H$ determines the channel capacity required to transmit the desired information with the *most* efficient coding scheme.



## Entropy - different probability

If we had a source emitting two symbols, $0$ and $1$, with probabilities of $1$ and $0$, then the entropy of the source is

$$\begin{aligned} H(X) &= \sum_{i=1}^{n} P_{x_i} \log_2 \frac{1}{P_{x_i}} \\ &= \log_2 1 + 0 * \log_2 0 \\ &= 0 \text{ bits/symbol} \end{aligned}$$



## Entropy - different probability

If we were transmitting a sequence of letters *A,B,C,D,E* and *F* with probabilities $\frac{1}{2}$,$\frac{1}{4}$,$\frac{1}{16}$,$\frac{1}{16}$,$\frac{1}{16}$ and $\frac{1}{16}$, the entropy for the system is

$$\begin{aligned} H(X) &= \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{4}{16}\log_2 16 \\ &= 0.5 + 0.5 + 1.0 \\ &= 2 \text{ bits/symbol} \end{aligned}$$



## Encoding the letters

A fixed size 3-bit code, and then a more complex code:

| Symbol | 3-bit code | Complex code |
|---|---|---|
| A | 000 | 0 |
| B | 001 | 10 |
| C | 010 | 1100 |
| D | 011 | 1101 |
| E | 100 | 1110 |
| F | 101 | 1111 |

## Analysis of encoding

The average length of the binary digits needed to encode a typical sequence of symbols using the 3-bit code is

$$
\begin{aligned}
L(X) &= \sum_{i=1}^{n} P_{x_i} \bullet \text{sizeof}(x_i) \\
&= \frac{1}{2} * 3 + \frac{1}{4} * 3 + \frac{4}{16} * 3 \\
&= 1.5 + 0.75 + 0.75 \\
&= 3 \ \text{bits/symbol}
\end{aligned}
$$



## Analysis of encoding

The average length of the binary digits needed to encode a typical sequence of symbols using the complex encoding is

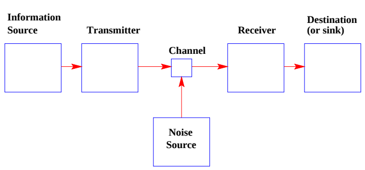
$$
\begin{aligned}
L(X) &= \sum_{i=1}^{n} P_{x_i} \bullet \text{sizeof}(x_i) \\
&= \frac{1}{2} * 1 + \frac{1}{4} * 2 + \frac{4}{16} * 4 \\
&= 0.5 + 0.5 + 1.0 \\
&= 2 \ \text{bits/symbol}
\end{aligned}
$$

i.e. it is more efficient, averaging only 2 bits for each symbol transmitted.



## Entropy and transmission rate

If our source was transmitting $0$ and $1$ bits with equal probability, but the received data was corrupted 50% of the time, we might reason that our rate $r(X)$ of information transmission was $0.5$, because half of our data is getting through correctly.






## Entropy and transmission rate

However, a better argument is to consider the difference between the entropy of the source and the conditional entropy of the received data:

$$r(X) = H(X) - H(X \mid y)$$

where $H(X \mid y)$ is the *conditional* entropy of the received data.

## Entropy and transmission rate

$$\begin{aligned} H(X \mid y) &= 0.5 * \log_2 2 + 0.5 * \log_2 2 \\ &= 1 \\ \text{and} \quad H(X) &= 1 \qquad \text{(shown before)} \\ \text{so} \quad r(X) &= H(X) - H(X \mid y) \\ &= 0 \ \text{bits/symbol} \end{aligned}$$

This is a much better measure of the amount of information transmitted.



## Redundancy

The ratio of the entropy of a source $H(X)$ to what it would be if the symbols had equal probabilities $H'(X)$, is called the *relative* entropy. We use the notation $H_r(X)$, and

$$H_r(X) = \frac{H(X)}{H'(X)}$$

The *redundancy* of the source is $1 - H_r(X)$

$$R(X) = 1 - H_r(X)$$



## Redundancy

- ✔ If we look at English text a symbol at a time[1], the redundancy is about $0.7$.
- ✔ This indicates that it should be simple to compress English text by about 70%.
- ✔ This sort of redundancy is a *unitless* relative redundancy

[1] That is, without considering letter *sequences*.



## Unicity distance

Defined by Shannon - an approximation to the amount of ciphertext such that the the sum of the source entropy and the encryption key entropy is the same as the number of ciphertext bits used.

- ✔ Ciphertexts longer have only one meaningful decryption
- ✔ Ciphertexts shorter may have more than one meaningful decryption (and hence be stronger, as a hacker will not know which one is correct)

## Unicity distance

✔ The longer the unicity distance, the better the cryptosystem

✔ Unicity distance $U$ is the entropy of the key divided by the redundancy of the source, and is approximately

$$U \approx \frac{\log_2 K}{R \log_2 P}$$

($K$ is the key size, $R$ is the redundancy, $P$ is the number of symbols).



## Unicity distance

26 letter alphabet, and 26! keys

$$\begin{aligned} U &\approx \frac{\log_2 26!}{0.5 \log_2 26} \\ &\approx \frac{88}{0.7 * 4.7} \\ &\approx 27 \end{aligned}$$

So given a ciphertext of 27 symbols, a unique decoding is possible.



## Unicity distance

In general

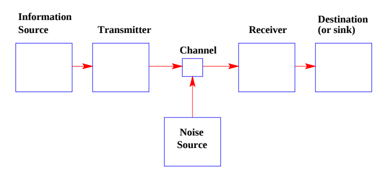
✔ Longer key length then longer unicity distance

✔ Redundancy inversely proportional to unicity distance

✔ Estimates the minimum amount of ciphertext for which there is only a single plaintext solution on doing a brute force attack...



## Shannon and Nyquist




**Maximum BPS** = $W \log_2(1 + \frac{S}{N})$ bits/sec

## Shannon and Nyquist example

If we had a telephone system with a bandwidth of 3,000 Hz, and a S/N of 30db (about 1024:1)

$$
\begin{aligned}
D &= 3000 * \log_2 1025 \\
&\approx 3000 * 10 \\
&\approx 30000 \quad \text{bps}
\end{aligned}
$$

This is a typical maximum bit rate achievable over the telephone network.



## Nyquist

The maximum data rate over a limited bandwidth (W) channel with V discrete levels is:

**Maximum data rate** = $2W \log_2 V$ bits/sec

For example, two-Level data cannot be transmitted over the telephone network faster than 6,000 BPS, because the *bandwidth* of the telephone channel is only about 3,000Hz.



## Nyquist example

If we had a telephone system with a bandwidth of 3,000 Hz, and using 256 levels:

$$
\begin{aligned}
D &= 2 * 3000 * \log_2 256 \\
&= 6000 * 8 \\
&= 48000 \quad \text{bps}
\end{aligned}
$$

In these equations, the assumption is that the relative entropies of the signal and noise are a maximum (that they are random).



## Maximum entropy

In practical systems, signals rarely have maximum entropy, and we can do better - there may be methods to compress the data[2].

[2]**Note:** we must also differentiate between lossy and lossless compression schemes. A signal with an entropy of $0.5$ may not be compressed more than 2:1 unless you use a lossy compression scheme. JPEG and Wavelet compression schemes can achieve huge data size reductions without visible impairment of images, but the restored images are not the same as the original ones - they just look the same. The lossless compression schemes used in PkZip, gzip or GIF files (LZW) cannot achieve compression ratios as high as that found in JPEG.

## Huffman encoding

An immediate question of interest is “*What is the minimum length bit string that may be used to compress a string of symbols?*”.

The Huffman encoding minimizes the bit length given the frequency of occurence of each symbol[3]. The resultant bit string in the best case will be the length predicted from the calculation of the source entropy.

[3] Note that it presupposes knowledge about these frequencies.

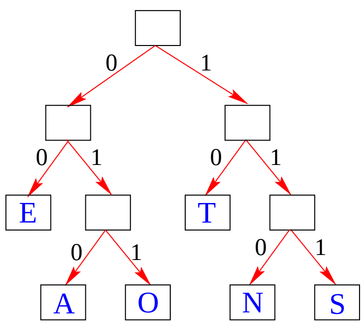


## Huffman encoding

✔ How can we get knowledge about the frequency of (say) the letters in the English language?



## Huffman encoding




Less common characters use longer bit strings.



## Huffman encoding

Our algorithm for encoding is simple - we calculate the tree encoding knowing the frequency of each letter:

| Symbol | Coding |
|---|---|
| E | 00 |
| T | 10 |
| A | 010 |
| O | 011 |
| N | 110 |
| S | 111 |

To decode, traverse the tree taking a left or right path according to the bit. The leaf has our symbol.

## Case study - MNP5 and V.42bis

MNP5 and V42.bis are compression schemes commonly used on modems.

MNP5 suffers from the unfortunate property that it will *expand* data with maximum or near-maximum entropy (instead of compression).

V42.bis does not have this property - it uses a large dictionary, and will not try to compress an already compressed stream.



## MNP5

MNP5 uses two different compression methods, switching between them as appropriate. The methods are:

- Adaptive frequency encoding
- Run-length encoding

Run length encoding sends the bytes with a byte count value, and doubles the size of a data stream with maximum entropy.



## Adaptive frequency encoding

| 3-bit header | Body size | Total code size | Number of codewords |
|---|---|---|---|
| 000 | 1 bit | 4 bits | 2 |
| 001 | 1 bit | 4 bits | 2 |
| 010 | 2 bits | 5 bits | 4 |
| 011 | 3 bits | 6 bits | 8 |
| 100 | 4 bits | 7 bits | 16 |
| 101 | 5 bits | 8 bits | 32 |
| 110 | 6 bits | 9 bits | 64 |
| 111 | 7 bits | 10 bits | 128 |

$\frac{3}{4}$ of our codewords are *larger* than they would be if we did not use this encoding scheme



## Summary of topics

In this section, we introduced the following topics:

- Physical preliminaries, Fourier analysis and convolution
- Entropy
- Encoding

## Further study

- Textbook Chapter 32
- Shannon's paper on secrecy systems [**?**] at
http://www.cs.ucla.edu/˜jkong/research/security/shannon.html.