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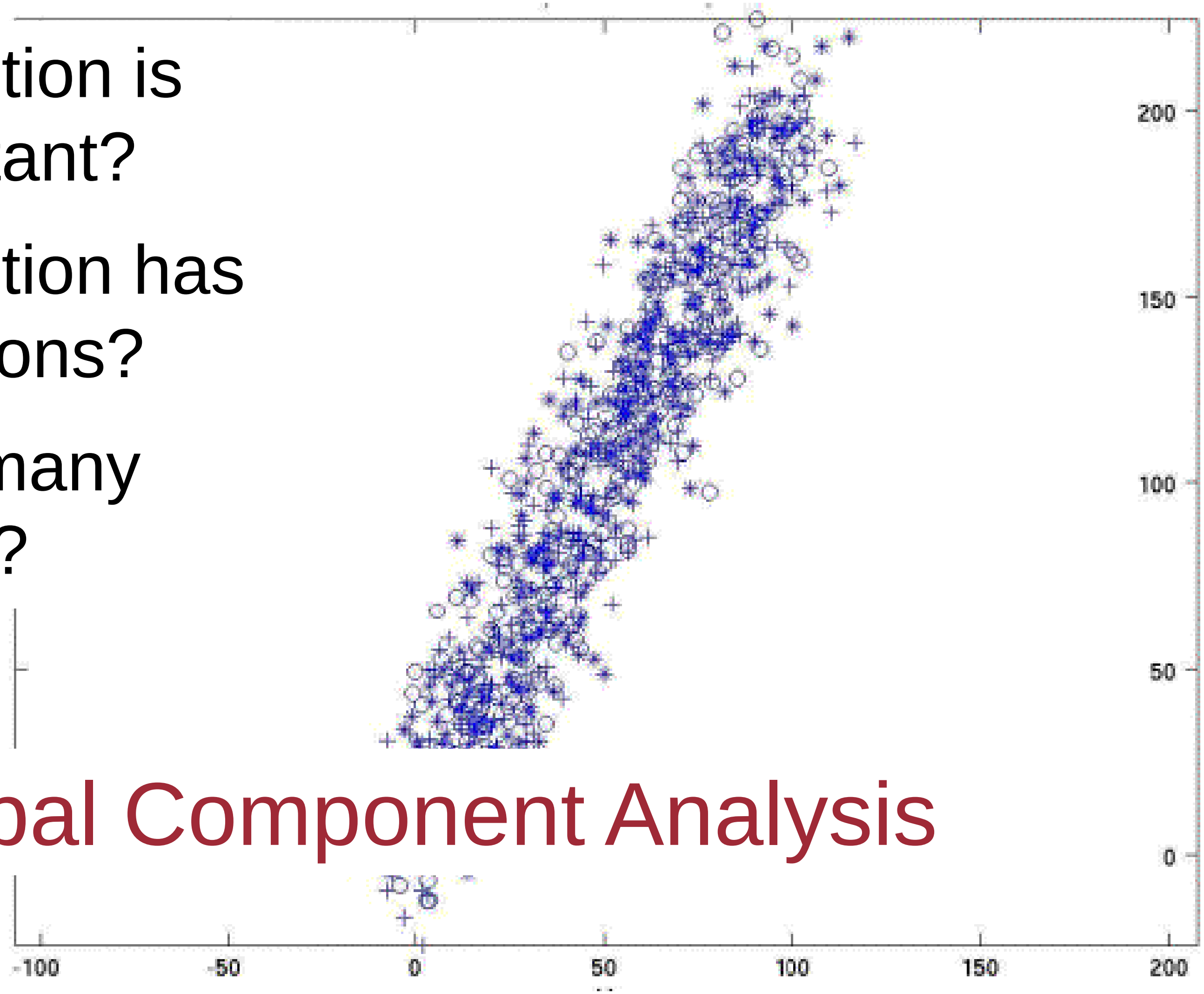
CS4243 Computer Vision and Pattern Recognition

Principal Component Analysis

Given a set of data…

- Which direction is most important?
- Which direction has most variations?
- Need how many dimensions?



Principal Component Analysis

Basics of PCA

- $\mathbf{x}_i$ is m-dimensional vector (data point), $i = 1, \ldots, N$.
- Mean vector $\mathbf{m}$ is

$$\mathbf{m} = E\{\mathbf{x}\} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i$$

- Covariance matrix $\mathbf{R}$ is

Shift vectors so that centroid is at origin

$$\begin{aligned}\mathbf{R} &= E\{(\mathbf{x}-\mathbf{m})(\mathbf{x}-\mathbf{m})^T\} \\ &= \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i-\mathbf{m})(\mathbf{x}_i-\mathbf{m})^T\end{aligned}$$

- $\mathbf{R}$ is real and symmetric.
  - Can apply eigen-decomposition to find q_j, λ_j such that

$$\mathrm{R}\,\mathrm{q}_j = \lambda_j \mathrm{q}_j \quad j = 1, \ldots, m$$

  - eigenvectors q_j are orthonormal

$$\mathrm{q}_j^T \mathrm{q}_j = 1$$
$$\mathrm{q}_j^T \mathrm{q}_k = 0 \text{ for } k \neq j$$

  - eignvalues λ_j are sorted such that $\lambda_j \geq \lambda_{j+1}$

- Assemble eigenvectors into a matrix

$$\mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_m]$$

- Then, can transform $\mathbf{x}_i$ into new vector $\mathbf{y}_i$

$$\mathbf{y}_i = \mathbf{Q}^T(\mathbf{x}_i - \mathbf{m}) = \sum_{j=1}^{m}(\mathbf{x}_i - \mathbf{m})^T\mathbf{q}_j\ \mathbf{q}_j$$

  - So,

$$\mathbf{y}_i = [y_{i1}, \ldots, y_{ij}, \ldots, y_{im}]^T$$

    where y_{ij} is the projection of $\mathbf{x}_i - \mathbf{m}$ on $\mathbf{q}_j$

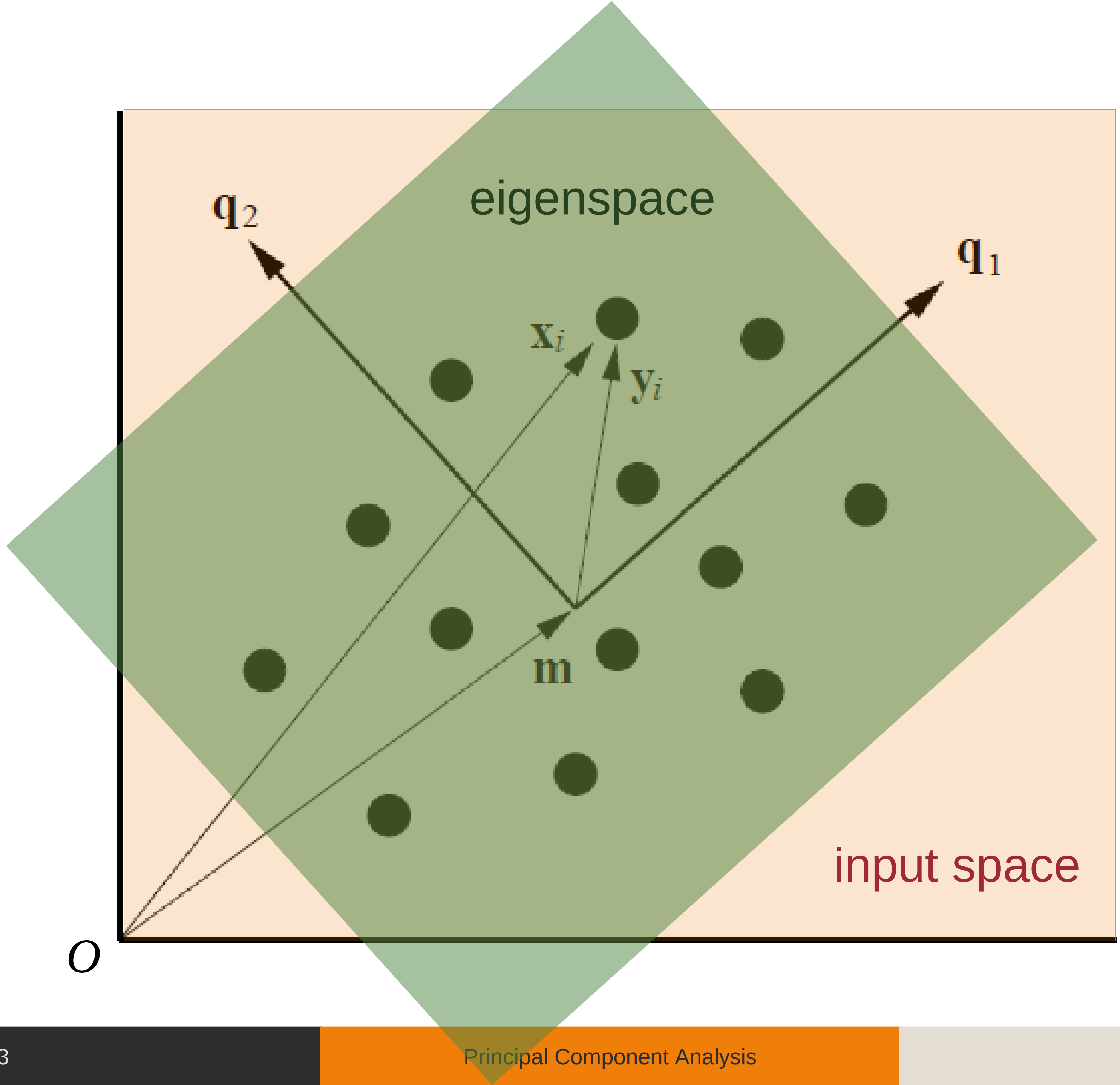
$$y_{ij} = (\mathbf{x}_i - \mathbf{m})^T\mathbf{q}_j$$

  - y_{ij} is principal component of $\mathbf{y}_i$ along $\mathbf{q}_j$
  - y_{ij} are independent or uncorrelated

- Original $\mathbf{x}_i$ can be recovered from $\mathbf{y}_i$

$$\mathbf{x}_i = \mathrm{Q}\,\mathbf{y}_i + \mathbf{m} = \sum_{j=1}^{m} y_{ij}\mathbf{q}_j + \mathbf{m}$$

- Notes:
  - $\mathbf{x}_i \neq \mathbf{y}_i + \mathbf{m}$
  - $\mathbf{x}_i$ is in the input space
  - $\mathbf{y}_i$ is in the eigenspace spanned by $\mathbf{q}_j$


eigenspace
q2
q1
xi
yi
m
input space
O

Properties of PCA

- Mean $\mathbf{m}_y$ over all $\mathbf{y}_i$ is $\mathbf{0}$

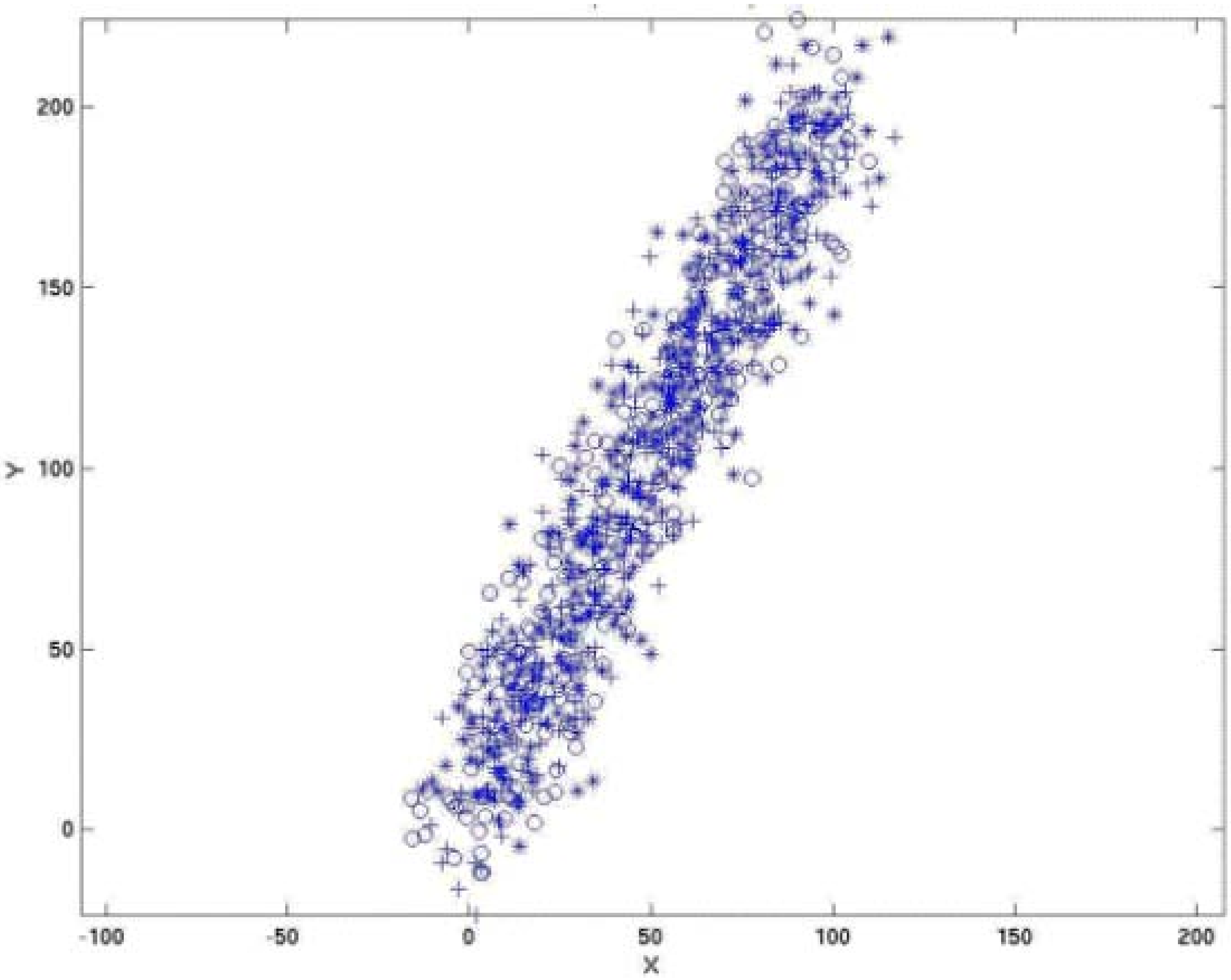
$$\mathbf{m}_y = \frac{1}{N}\sum_{i=1}^{N}\mathbf{y}_i = \mathbf{Q}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i - \mathbf{m}\right) = \mathbf{0}$$

- Variance σ_j^2 along $\mathbf{q}_j$ is λ_j (exercise)

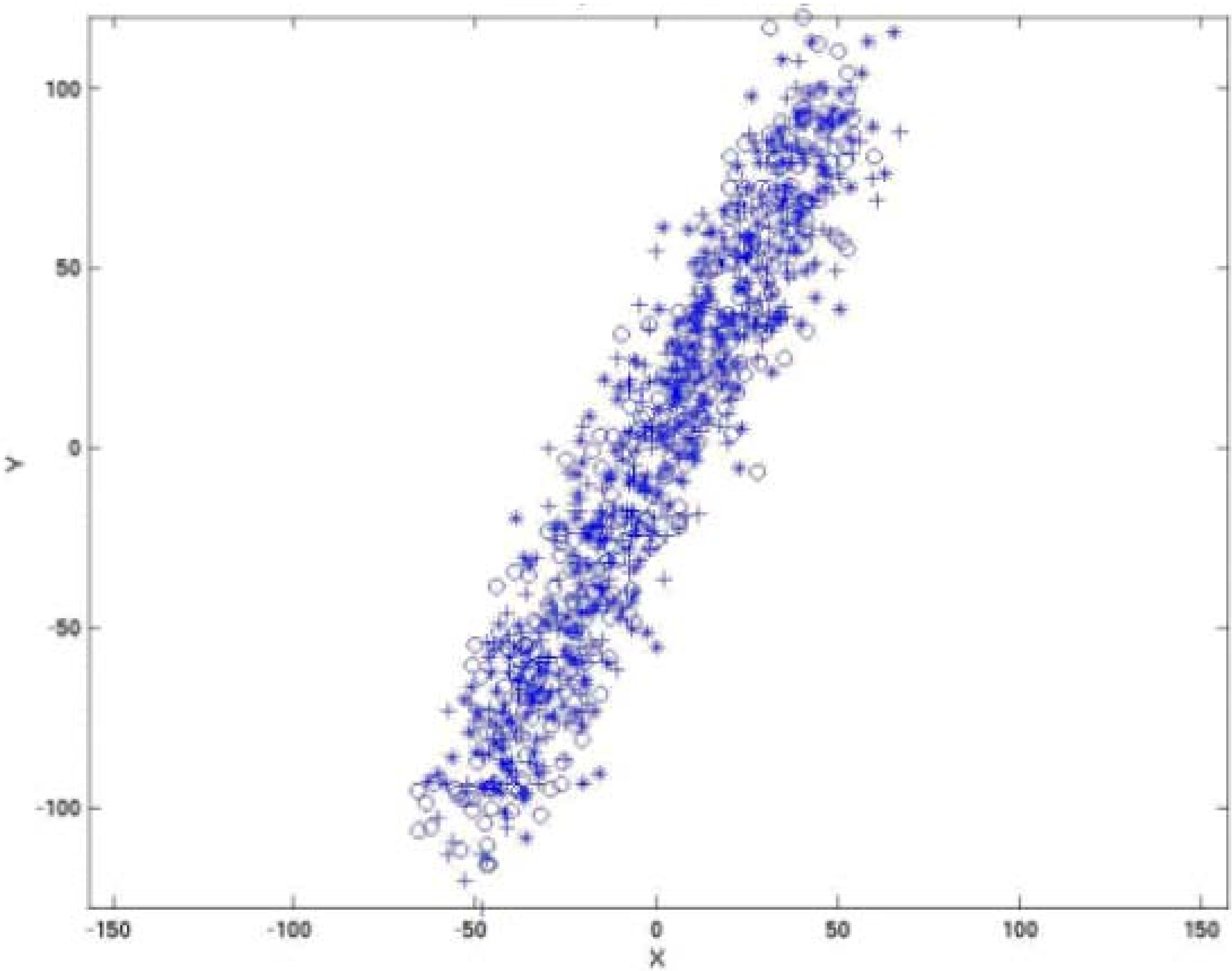
$$\begin{aligned}\sigma_j^2 &= \frac{1}{N}\sum_{i=1}^{N} y_{ij}^2 \\ &= \mathbf{q}_j^T\,\mathbf{R}\,\mathbf{q}_j = \lambda_j\end{aligned}$$

- Since $\lambda_1 \geq \cdots \geq \lambda_m$, so $\sigma_1 \geq \cdots \sigma_m$
  - $\mathbf{q}_1$ gives orientation of largest variation
  - $\mathbf{q}_2$ gives orientation of largest variation orthogonal to $\mathbf{q}_1$ (2nd largest variation)
  - $\mathbf{q}_j$ gives orientation of largest variation orthogonal to $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{j-1}$ (j-th largest variation)
  - $\mathbf{q}_m$ is orthogonal to all other eigenvectors (least variation)

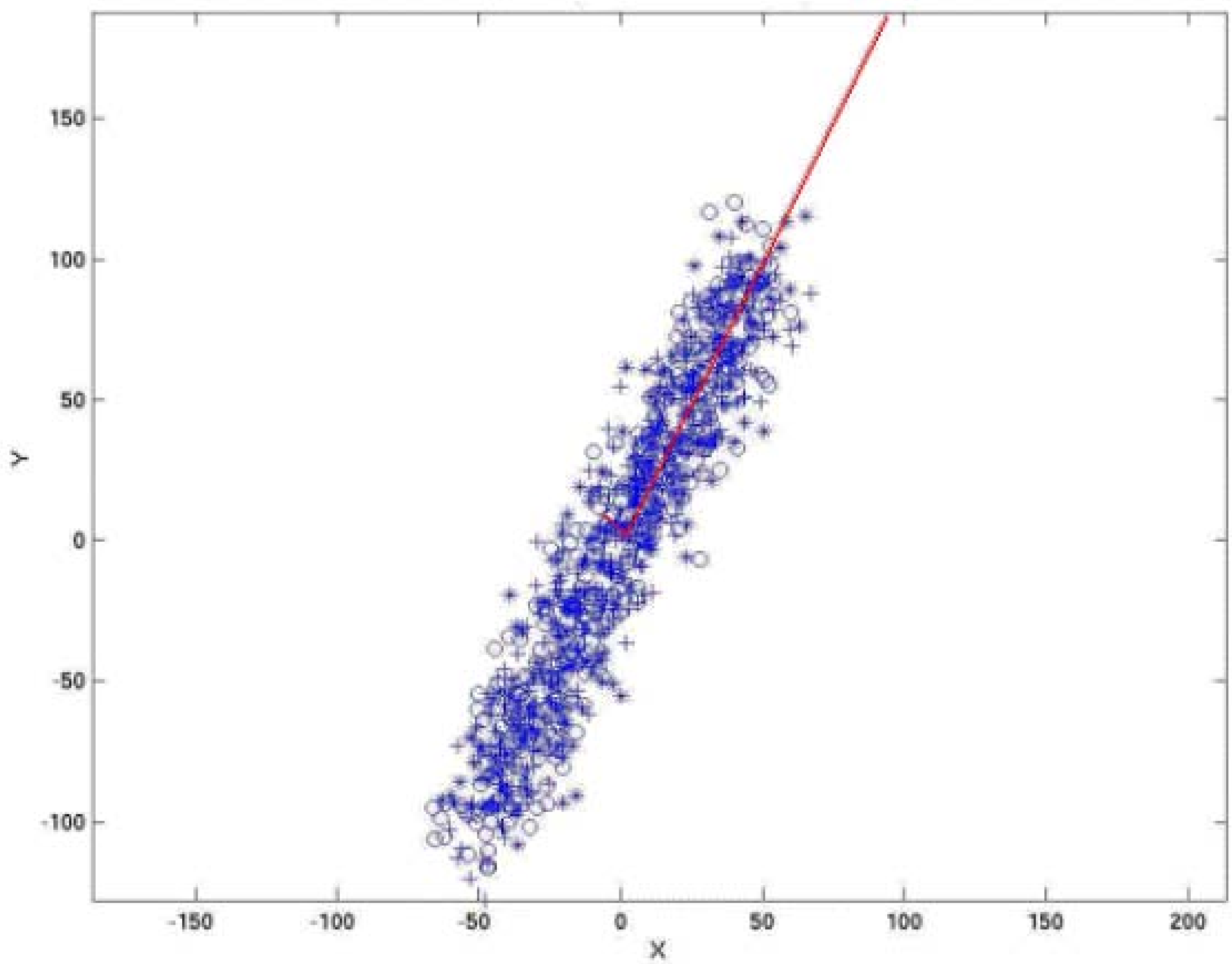
data points



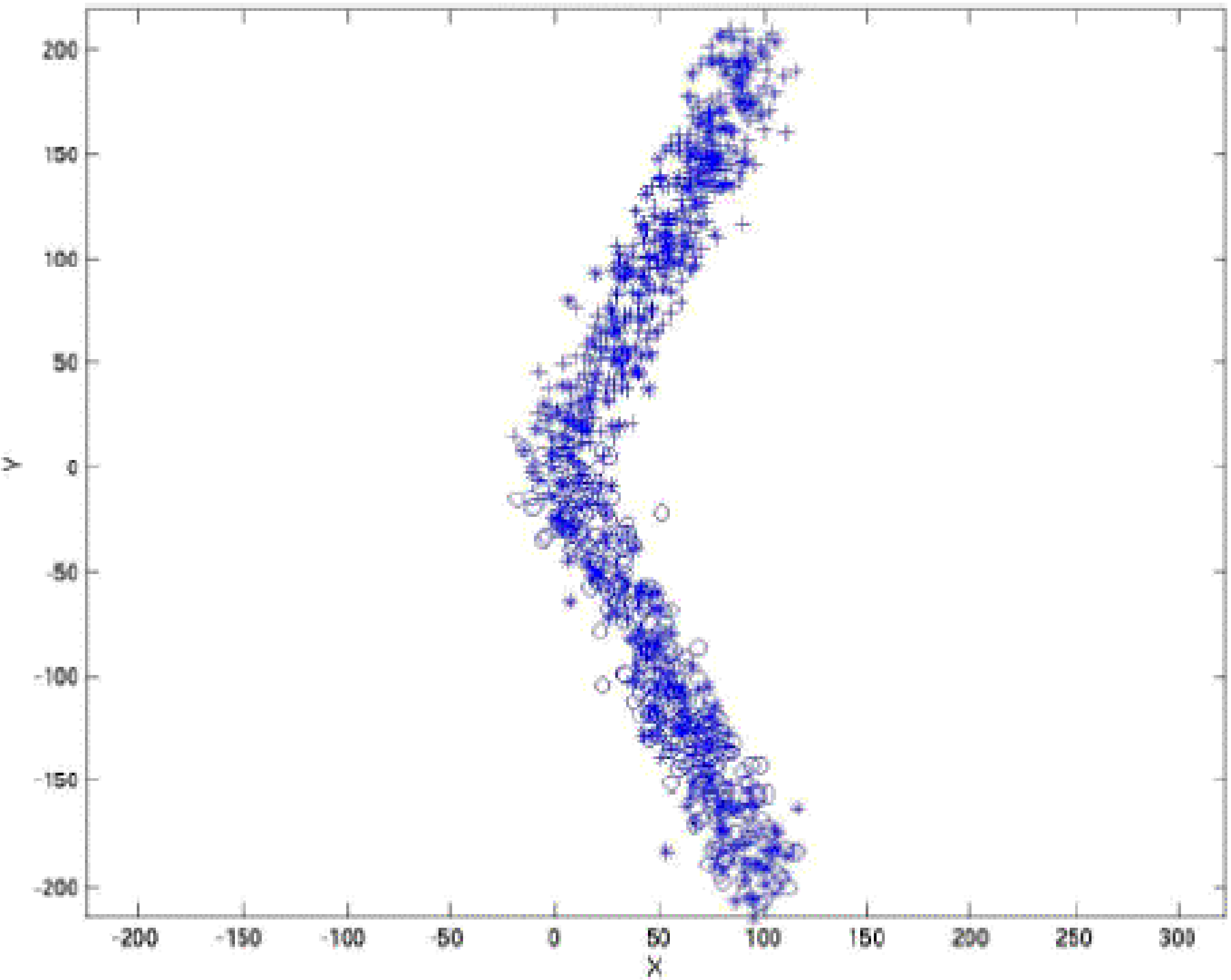
centriod at origin



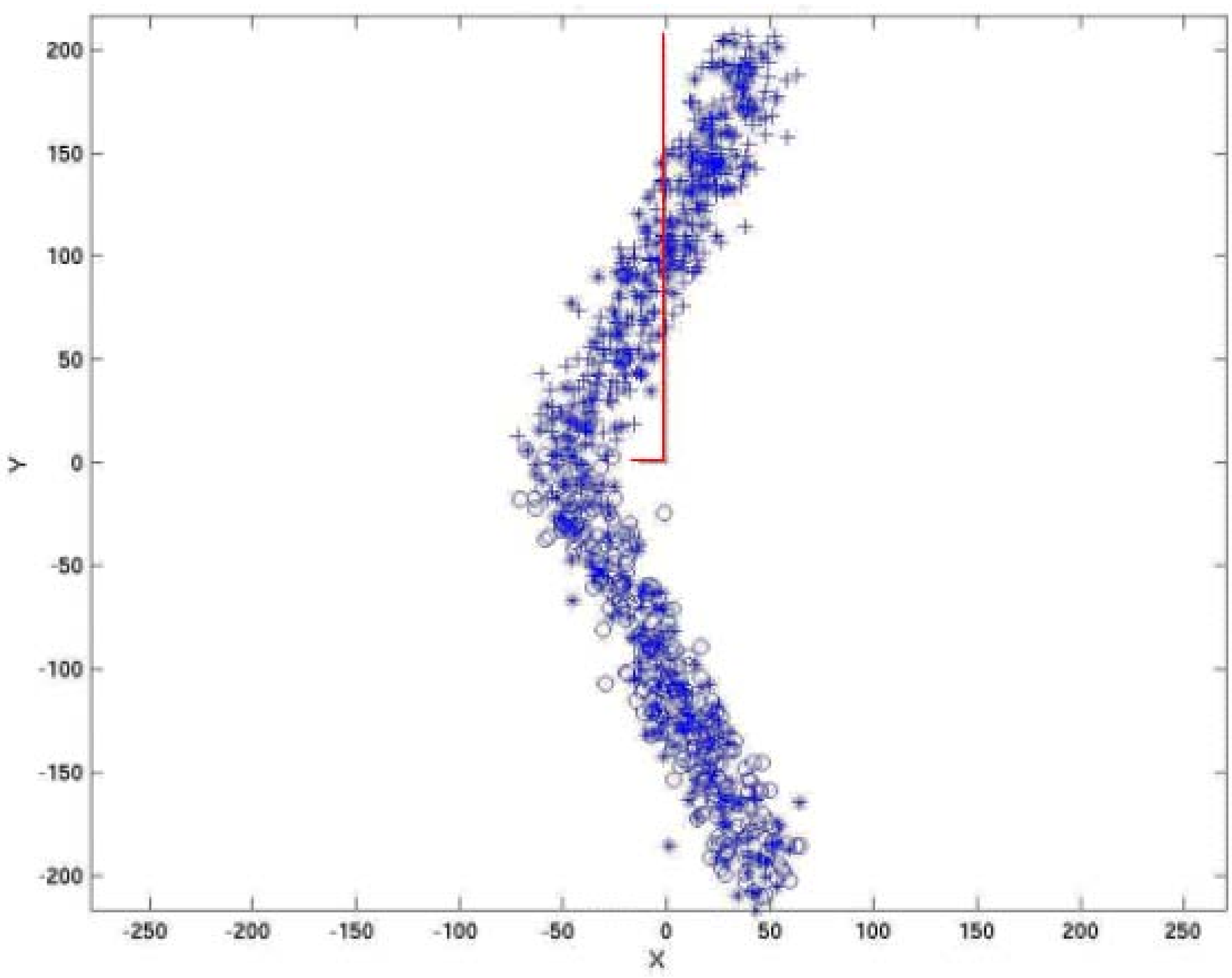
principal components



another example

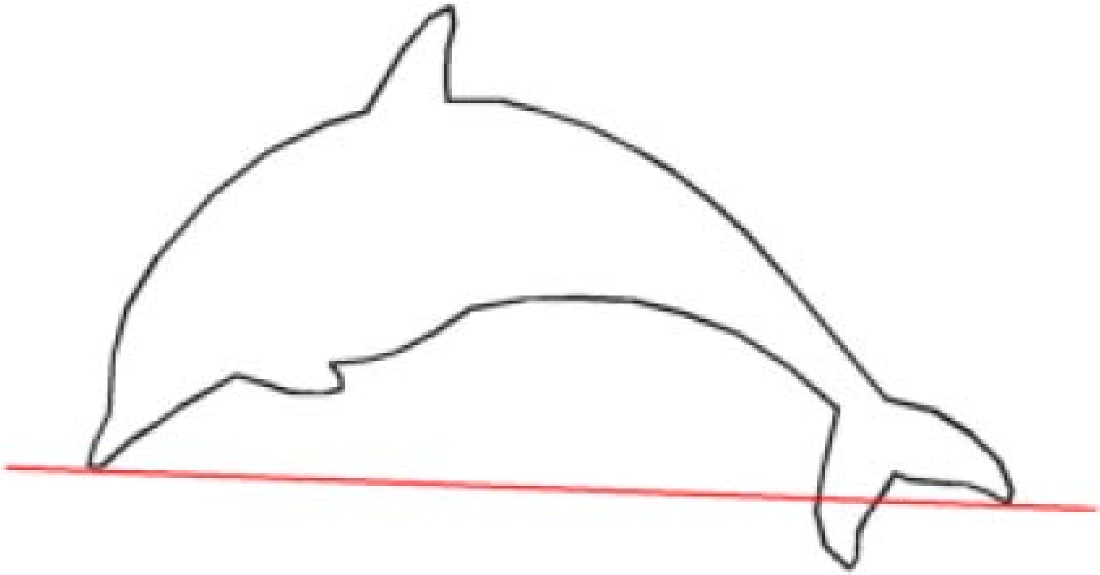


principal components



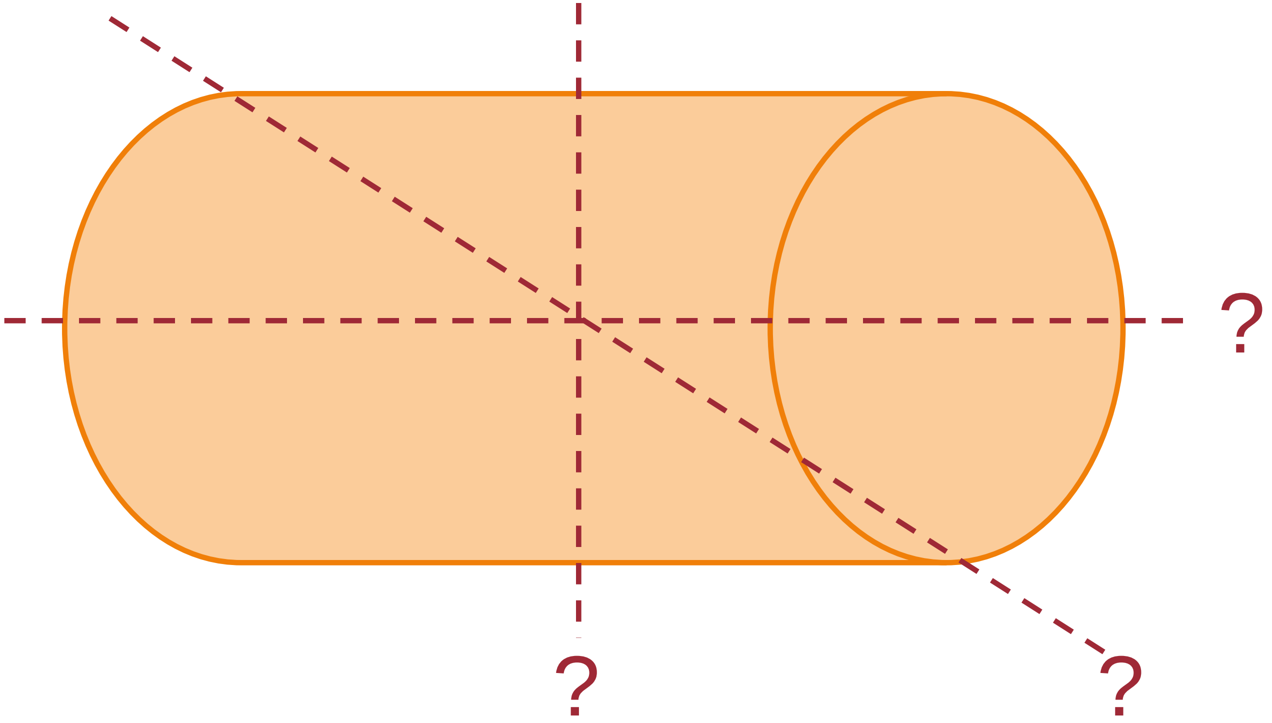
Major Axes

- Can apply PCA to find major axes of objects.



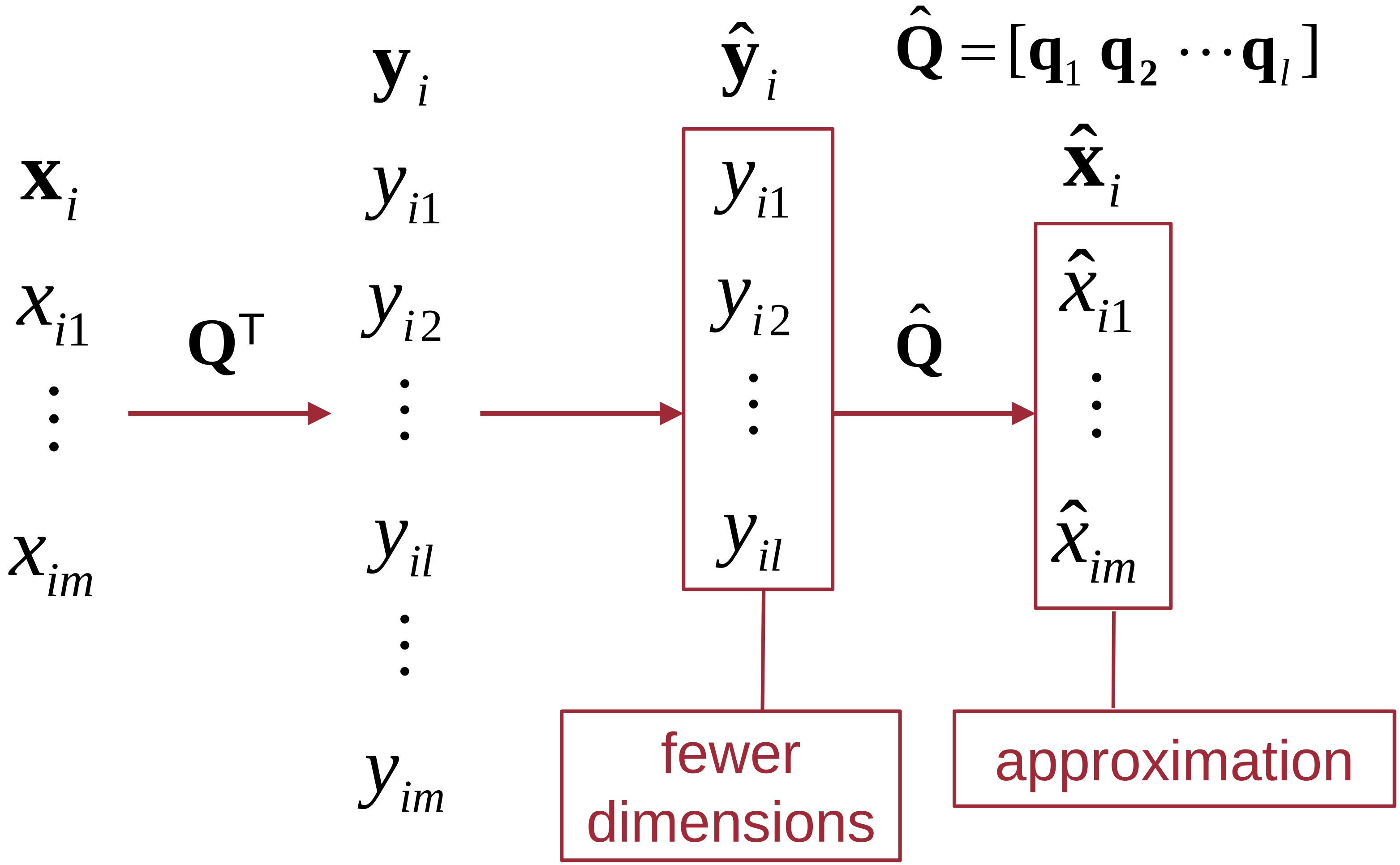
Major Axes

- Be careful!
- What's the first principal orientation of cylinder?



Dimensionality Reduction

- Can just keep the first *l* largest dimensions.



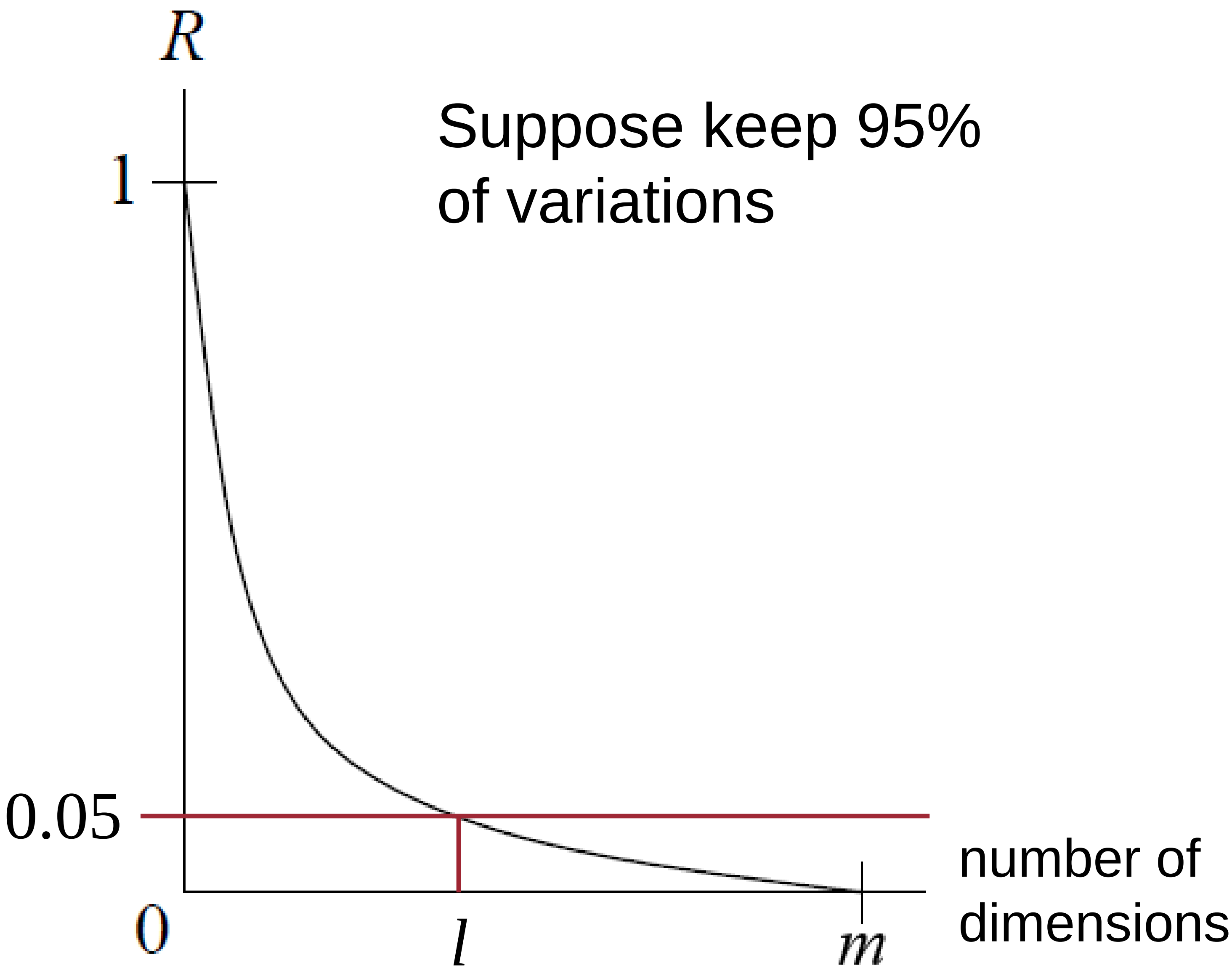
How many dimensions to keep?

- Total variance of $\hat{\mathbf{x}}_i$ is

$$\sum_{j=1}^{l} \sigma_j^2 = \sum_{j=1}^{l} \lambda_j$$

- Keep enough so that ratio R of unaccounted variance is small

$$R = \frac{\sum_{j=l+1}^{m} \sigma_j^2}{\sum_{j=1}^{m} \sigma_j^2} = \frac{\sum_{j=l+1}^{m} \lambda_j}{\sum_{j=1}^{m} \lambda_j}$$



Suppose keep 95% of variations

Summary

- PCA extracts principal components
  - Orientation of largest variations.
  - Most significant dimensions.
- Can be used for
  - Computing principal variations.
  - Identifying major axes.
  - Dimensionality reduction.