Sample Math Used in CS4243

The mathematics used in CS4243 are basically linear algebra, calculus, and probability. This summary contains a sampling of the math used.

1. Linear Algebra

- An element $\mathbf{v}$ in an n-dimensional vector space V can be represented as a linear combination of n basis vectors:

$$\mathbf{v} = \sum_{i=1}^{n} a_i \, \mathbf{e}_i \tag{1}$$

  where $\mathbf{e}_i$ is a basis vector and a_i is the corresponding coefficient.

- Given a set of n vector $\mathbf{x}_i$, the mean of the vectors is

$$\mathbf{m} = E\{\mathbf{x}\} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \tag{2}$$

  and the covariance matrix is

$$\mathbf{C} = E\{(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^T\} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T - \mathbf{m}\mathbf{m}^T \tag{3}$$

- An eigensystem relates a matrix $\mathbf{A}$ and a set of vectors $\mathbf{e}_i$ by the equations

$$\mathbf{A} \, \mathbf{e}_i = \lambda_i \, \mathbf{e}_i \ . \tag{4}$$

  The vectors $\mathbf{e}_i$ are the eigenvectors and the corresponding λ_i are the eigenvalues.

- Consider a set of simultaneous equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{5}$$

  where $\mathbf{A}$ is an $N \times N$ matrix, $\mathbf{b}$ and $\mathbf{x}$ are vectors. This equation defines $\mathbf{A}$ as a mapping from the vector space $\mathbf{x}$ to the vector space $\mathbf{b}$.

  - If $\mathbf{A}$ is singular, then there is some subspace $\mathbf{x}$ that is mapped to zero, i.e., $\mathbf{A}\mathbf{x} = 0$. This subspace is called the nullspace, and its dimensionality is called the nullity of $\mathbf{A}$.
  - The non-null subspace that is mapped from $\mathbf{x}$ is called the range of $\mathbf{A}$ and the dimensionality of the range is called the rank of $\mathbf{A}$.
  - So, the rank plus the nullity of $\mathbf{A}$ equals N.

- A 3D point W with world coordinate $\mathbf{X} = (X, Y, Z)^\top$ is mapped onto the camera coordinate system at $\mathbf{X}^c = (X^c, Y^c, Z^c)^\top$ by a rigid transformation

$$\mathbf{X}^c = \mathbf{R}\mathbf{X} + \mathbf{T} \tag{6}$$

where $\mathbf{R} = (\mathbf{R}_1^\top, \mathbf{R}_2^\top, \mathbf{R}_3^\top)^\top$ is the 3×3 rotation matrix and $\mathbf{T} = (T_X, T_Y, T_Z)^\top$ is the translation vector that relate the camera and the world coordinate frames.

- The rotation matrix can be specified in terms of three Euler angles, pitch (vertical angle) ω, yaw (horizontal angle) ϕ, and roll κ:

$$\mathbf{R} = \begin{bmatrix} \cos\phi\cos\kappa & \sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa & -\cos\omega\sin\phi\cos\kappa + \sin\omega\sin\kappa \\ -\cos\phi\sin\kappa & -\sin\omega\sin\phi\sin\kappa + \cos\omega\cos\kappa & \cos\omega\sin\phi\sin\kappa + \sin\omega\cos\kappa \\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi \end{bmatrix} \tag{7}$$

which is an othornormal matrix, i.e.,

$$\mathbf{R}^\top\mathbf{R} = \mathbf{I} \tag{8}$$

where $\mathbf{I}$ is the identity matrix.

2. Calculus

- The gradient ∇f of a function f of two variables x and y is the first partial derivative with respect to x and y

$$\nabla f(x, y) = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} . \tag{9}$$

The magnitude of the gradient is

$$|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} . \tag{10}$$

- The Laplacian $\nabla^2 f$ of a function f of two variables x and y is

$$\nabla^2 f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} . \tag{11}$$

- The image of a moving object changes over time. Assuming small motion, the intensity $E(x, y, t)$ at location (x, y) will remain constant, so that

$$\frac{\partial E}{\partial t} = 0 . \tag{12}$$

Using the chain rule for differentiation, we see that

$$\frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial t} = \frac{\partial E}{\partial t} = 0 . \tag{13}$$

3. Probability

- The notation $P(H|E)$ denote the probability that H occurs given that evidence E has occurred. This conditional probability is defined by the joint probability $P(H,E)$, that is, the probability that both H and E occur:

$$P(H|E) = \frac{P(H,E)}{P(E)} \ . \tag{14}$$

- From the definition of conditional probability, we have

$$P(E|H) = \frac{P(E,H)}{P(H)} = \frac{P(H,E)}{P(H)} \ . \tag{15}$$

Therefore,

$$P(H|E)\, P(E) = P(E|H)\, P(H) \tag{16}$$

or

$$P(H|E) = \frac{P(E|H)\, P(H)}{P(E)} \tag{17}$$

which is known as the *Bayes rule*.

- The entropy E of a probability distribution $P(i)$ is

$$E = -\sum_i P(i) \log P(i) \ . \tag{18}$$