01—Introduction to CS5209; Propositional Calculus I

CS 5209: Foundation in Logic and AI

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January 14, 2010

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2 Brief Introduction to CS5209

3 Administrative Matters

4 Propositional Calculus: Declarative Sentences

5 Propositional Calculus: Natural Deduction

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What is logic?

1. the branch of philosophy dealing with forms and processes of thinking, especially those of inference and scientific method,

2. a particular system or theory of logic [according to 1].

(from "The World Book Dictionary")

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Origins of Mathematical Logic

Greek origins

The ancient Greek formulated rules of logic as *syllogisms*, which can be seen as precursors of formal logic frameworks.

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Example of Syllogism

Premise

All men are mortal.

Premise

Socrates is a man.

Conclusion

Therefore, Socrates is mortal.

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Historical Notes

Logic traditions in Ancient Greece

Stoic logic: Centers on propositional logic; can be traced back to Euclid of Megara (400 BCE)

Peripatetic logic: Precursor of predicate logic; founded by Artistotle (384–322 BCE), focus on syllogisms

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Logic Throughout the World

Indian logic: Nyaya school of Hindu philosophy, culminating with Dharmakirti (7th century CE), and Gangea Updhyya of Mithila (13th century CE), formalized inference

Chinese logic: Gongsun Long (325–250 BCE) wrote on logical arguments and concepts; most famous is the "White Horse Dialogue"; logic typically rejected as trivial by later Chinese philosophers

Islamic logic: Further development of Aristotelian logic, culminating with Algazel (1058–1111 CE)

Medieval logic: Aristotelian; culminating with William of Ockham (1288–1348 CE)

Traditional logic: Port-Royal Logic, influential logic textbook first published in 1665

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Remarks on Ockham

Ockham's razor (in his own words)

For nothing ought to be posited without a reason given, unless it is self-evident or known by experience or proved by the authority of Sacred Scripture.

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Ockham's razor (popular version, not found in his writings)

Entia non sunt multiplicanda sine necessitate.

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Built-in Skepticism

As a result of this *ontological parsimony*, Ockham states that human reason cannot prove the immortality of the soul nor the existence, unity, and infinity of God.

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Propositional Calculus

Study of atomic propositions

Propositions are built from sentences whose internal structure
is not of concern.

Building propositions

Boolean operators are used to construct propositions out of
simpler propositions.

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Example for Propositional Calculus

Atomic proposition

One plus one equals two.

Atomic proposition

The earth revolves around the sun.

Combined proposition

One plus one equals two *and* the earth revolves around the sun.

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Goals and Main Result

Meaning of formula

Associate meaning to a set of formulas by assigning a value
true or *false* to every formula in the set.

Proofs

Symbol sequence that formally establishes whether a formula
is always true.

Soundness and completeness

The set of provable formulas is the same as the set of formulas
which are always true.

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Uses of Propositional Calculus

Hardware design

The production of logic circuits uses propositional calculus at all phases; specification, design, testing.

Verification

Verification of hardware and software makes extensive use of propositional calculus.

Problem solving

Decision problems (scheduling, timetabling, etc) can be expressed as satisfiability problems in propositional calculus.

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Predicate Calculus: Central ideas

Richer language

Instead of dealing with atomic propositions, predicate calculus provides the formulation of statements involving sets, functions and relations on these sets.

Quantifiers

Predicate calculus provides statements that all or some elements of a set have specified properties.

Compositionality

Similar to propositional calculus, formulas can be built from composites using logical connectives.

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Progamming Language Semantics

The meaning of programs such as

```
if x >= 0 then y := sqrt(x) else y := abs(x)
```

can be captured with formulas of predicate calculus:

$$\forall x \forall y (x' = x \land (x \geq 0 \rightarrow y' = \sqrt{x}) \land (\neg(x \geq 0) \rightarrow y' = |x|))$$

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Other Uses of Predicate Calculus

Specification: Formally specify the purpose of a program in order to serve as input for software design,

Verification: Prove the correctness of a program with respect to its specification.

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Example for Specification

Let *P* be a program of the form

```
while a <> b do
   if a > b then a := a - b else a:= b - a;
```

The specification of the program is given by the formula

$$\{a \geq 0 \land b \geq 0\} \ P \ \{a = gcd(a, b)\}$$

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Theorem Proving and Logic Programming

Theorem proving

Formal logic has been used to design programs that can automatically prove mathematical theorems.

Logic programming

Research in theorem proving has led to an efficient way of proving formulas in predicate calculus, called *resolution*, which forms the basis for *logic programming*.

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Other Systems of Logic

Three-valued logic

A third truth value (denoting "don't know" or "undetermined") is often useful.

Intuitionistic logic

A mathematical object is accepted only if a finite construction can be given for it.

Temporal logic

Integrates time-dependent constructs such as ("always" and "eventually") explicitly into a logic framework; useful for reasoning about real-time systems.

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Style: Broad, elementary, rigorous
Method: From Theory to Practice
Overview of Module Content

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Style: Broad, elementary, rigorous
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Overview of Module Content

Style: Broad, elementary, rigorous

Broad: Cover a good number of logical frameworks

Elementary: Focus on a minimal subset of each framework

Rigorous: Cover topics formally, preparing students for advanced studies in logic in computer science

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Style: Broad, elementary, rigorous
Method: From Theory to Practice
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Method: From Theory to Practice

Cover theory and back it up with practical excercises that apply the theory and give new insights.

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Style: Broad, elementary, rigorous
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1. Propositional calculus (3 lectures, including today)
2. Predicate calculus (3 lectures)
3. Verification by Model Checking (1 lectures)
4. Program Verification (2 lectures)
5. Modal Logics (2 lectures; to be confirmed)

Administrative Matters

- Use www.comp.nus.edu.sg/~cs5209 and IVLE
- Textbook
- Assignments (one per week, starting next week; marked)
- Self-assessments (occasional; not marked)
- Discussion forums (IVLE)
- Announcements (IVLE)
- Webcast (IVLE)
- Blog (IVLE, just for fun)
- Tutorials (one per week); register!

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Declarative Sentences

The language of propositional logic is based on *propositions* or *declarative sentences*.

Declarative Sentences

Sentences which one can—in principle—argue as being true or false.

Examples

1. The sum of the numbers 3 and 5 equals 8.
2. Jane reacted violently to Jack's accusations.
3. Every natural number > 2 is the sum of two prime numbers.
4. All Martians like pepperoni on their pizza.

Not Examples

- Could you please pass me the salt?
- Ready, steady, go!
- May fortune come your way.

Putting Propositions Together

Example 1.1

If *the train arrives late* and
there are no taxis at the station then
John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, *there were taxis at the station.*

Putting Propositions Together

Example 1.2

If *it is raining* and
Jane does not have her umbrella with her then
she will get wet.

Jane is not wet.

It is raining.

Therefore, *Jane has her umbrella with her.*

Focus on Structure

We are primarily concerned about the structure of arguments in this class, not the validity of statements in a particular domain.

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We therefore simply abbreviate sentences by letters such as p, q, r, p_1, p_2 etc.

From Concrete Propositions to Letters

Example 1.1

If *the train arrives late* and
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becomes

From Concrete Propositions to Letters

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becomes

Letter version

If *p* and not *q*, then *r*. Not *r*. *p*. Therefore, *q*.

From Concrete Propositions to Letters

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From Concrete Propositions to Letters

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has

the same letter version

If *p* and not *q*, then *r*. Not *r*. *p*. Therefore, *q*.

Putting Propositions Together

Sentences like "If *p* and not *q*, then *r*." occur frequently. Instead of English words such as "if...then", "and", "not", it is more convenient to use symbols such as →, ∧, ¬.

Logical Connectives

$\neg$: negation of p is denoted by $\neg p$

$\vee$: disjunction of p and r is denoted by $p \vee r$, meaning at least one of the two statements is true.

$\wedge$: conjunction of p and r is denoted by $p \wedge r$, meaning both are true.

$\rightarrow$: implication between p and r is denoted by $p \rightarrow r$, meaning that r is a logical consequence of p. p is called the *antecedent*, and r the *consequent*.

Example 1.1 Revisited

From Example 1.1

If *the train arrives late* and
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Symbolic Propositions

We replaced "*the train arrives late*" by *p* etc

The statement becomes: If *p* and not *q*, then *r*.

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Symbolic Propositions

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The statement becomes: If *p* and not *q*, then *r*.

Symbolic Connectives

With symbolic connectives, the statement becomes:

$p \wedge \neg q \rightarrow r$

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Introduction

Objective

We would like to develop a *calculus* for reasoning about propositions, so that we can establish the validity of statements such as Example 1.1.

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Objective

We would like to develop a *calculus* for reasoning about propositions, so that we can establish the validity of statements such as Example 1.1.

Idea

We introduce *proof rules* that allow us to derive a formula ψ from a number of other formulas $\phi_1, \phi_2, \ldots \phi_n$.

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We introduce *proof rules* that allow us to derive a formula ψ from a number of other formulas $\phi_1, \phi_2, \ldots \phi_n$.

Notation

We write a *sequent* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$
to denote that we can derive ψ from $\phi_1, \phi_2, \ldots, \phi_n$.

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Example 1.1 Revisited

English

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Sequent

$$p \land \neg q \rightarrow r, \neg r, p \vdash q$$

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What Next?

Sequent

$$p \wedge \neg q \rightarrow r, \neg r, p \vdash q$$

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What Next?

Sequent

$$p \wedge \neg q \to r, \neg r, p \vdash q$$

Remaining task

Develop a set of proof rules that allows us to establish such sequents.

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Rules for Conjunction

Introduction of Conjunction

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}[\wedge i]$$

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Rules for Conjunction

Introduction of Conjunction

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \, [\wedge i]$$

Elimination of Conjunction

$$\frac{\phi \wedge \psi}{\phi} \, [\wedge e_1] \qquad \frac{\phi \wedge \psi}{\psi} \, [\wedge e_2]$$

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Example of Proof

To show

$$p \wedge q, r \vdash q \wedge r$$

How to start?

$$p \wedge q \qquad\qquad r$$

$$q \wedge r$$

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Proof Step-by-Step

1. $p \wedge q$ (premise)

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Proof Step-by-Step

1. $p \land q$ (premise)
2. r (premise)

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Proof Step-by-Step

1. $p \land q$ (premise)
2. r (premise)
3. q (by using Rule $\land e_2$ and Item 1)

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Proof Step-by-Step

1. $p \wedge q$ (premise)
2. r (premise)
3. q (by using Rule $\wedge e_2$ and Item 1)
4. $q \wedge r$ (by using Rule $\wedge i$ and Items 3 and 2)

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Graphical Representation of Proof

$$\frac{p \wedge q}{q} [\wedge e_2] \qquad r$$

$$\frac{}{q \wedge r} [\wedge i]$$

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Graphical Representation of Proof

$$\frac{p \land q}{q} [\land e_2] \qquad r$$

$$\frac{\qquad\qquad\qquad\qquad}{q \land r} [\land i]$$

Find the parts of the corresponding sequent:

$$p \land q, r \vdash q \land r$$

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Graphical Representation of Proof

$$\frac{p \land q}{q}[\land e_2] \qquad r$$

$$\frac{}{q \land r}[\land i]$$

Find the parts of the corresponding proof:

1. $p \land q$ (premise)
2. r (premise)
3. q (by using Rule $\land e_2$ and Item 1)
4. $q \land r$ (by using Rule $\land i$ and Items 3 and 2)

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Where are we heading with this?

- We would like to prove sequents of the form
  $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

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- We would like to prove sequents of the form
  $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$
- We introduce rules that allow us to form "legal" proofs

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  somehow "valid", or "meaningful"?
- What does it mean to be meaningful?
- Can we say that any meaningful sequent has a valid proof?
- ...but first back to the proof rules...

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Rules of Double Negation

$$\frac{\neg\neg\phi}{\phi}[\neg\neg e] \qquad \frac{\phi}{\neg\neg\phi}[\neg\neg i]$$

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Rule for Eliminating Implication

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}[\rightarrow e]$$

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Rule for Eliminating Implication

$$\frac{\phi \qquad \phi \to \psi}{\psi}[\to e]$$

Example

> p: It rained.
>
> $p \to q$: If it rained, then the street is wet.

We can conclude from these two that the street is indeed wet.

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Another Rule for Eliminating Implication

The rule

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}[\rightarrow e]$$

is often called "Modus Ponens" (or MP)

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Another Rule for Eliminating Implication

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$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}[\rightarrow e]$$

is often called "Modus Ponens" (or MP)

Origin of term

"Modus ponens" is an abbreviation of the Latin "modus ponendo ponens" which means in English "mode that affirms by affirming". More precisely, we could say "mode that affirms the antecedent of an implication".

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The Twin Sister of Modus Ponens

The rule

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}[\rightarrow e]$$

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The Twin Sister of Modus Ponens

The rule

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi} [\rightarrow e]$$

is called "Modus Ponens" (or MP)
A similar rule

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi} [MT]$$

is called "Modus Tollens" (or MT).

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The Twin Sister of Modus Ponens

The rule

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}[MT]$$

is called "Modus Tollens" (or MT).

Origin of term

"Modus tollens" is an abbreviation of the Latin "modus tollendo tollens" which means in English "mode that denies by denying". More precisely, we could say "mode that denies the consequent of an implication".

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Example

$$p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$$

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Example

$$p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$$

1    $p \rightarrow (q \rightarrow r)$ premise

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Example

$$p \to (q \to r), p, \neg r \vdash \neg q$$

1	$p \to (q \to r)$	premise
2	p	premise

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Example

$$p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$$

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise

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Example

$$p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$$

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \rightarrow r$	$\rightarrow_e$ 1,2

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Example

$$p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$$

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \rightarrow r$	$\rightarrow_e$ 1,2
5	$\neg q$	MT 4,3

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How to *introduce* implication?

Compare the sequent (MT)

$$p \rightarrow q, \neg q \vdash \neg p$$

with the sequent

$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

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How to *introduce* implication?

Compare the sequent (MT)

$$p \rightarrow q, \neg q \vdash \neg p$$

with the sequent

$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

The second sequent should be provable, but we don't have a rule to introduce implication yet!

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A Proof We Would Like To Have

$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \rightarrow \neg p$	$\rightarrow_i$ 2–3

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We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.

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A Proof We Would Like To Have

$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

1	$p \rightarrow q$	premise

| 2 | $\neg q$ | assumption |
| 3 | $\neg p$ | MT 1,2 |

| 4 | $\neg q \rightarrow \neg p$ | $\rightarrow_i$ 2–3 |

We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.

We cannot use assumptions from inside the box in rules outside the box.

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Rule for Introduction of Implication

$$\begin{array}{c} \boxed{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}} \\ \hline \hspace{2cm} [\to i] \\ \phi \to \psi \end{array}$$

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Rules for Introduction of Disjunction

$$\frac{\phi}{\phi \lor \psi}\,[\lor i_i] \qquad \frac{\psi}{\phi \lor \psi}\,[\lor i_2]$$

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Rule for Elimination of Disjunction

$$\phi \vee \psi \qquad \boxed{\begin{array}{c} \phi \\ \vdots \\ \chi \end{array}} \qquad \boxed{\begin{array}{c} \psi \\ \vdots \\ \chi \end{array}}$$

$$\overline{\hspace{6cm}} [\vee e]$$

$$\chi$$

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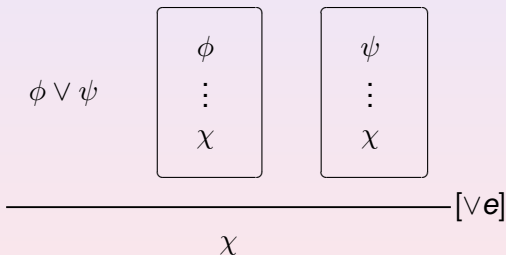
Example

1	$p \wedge (q \vee r)$	premise
2	p	$\wedge e_1$ 1
3	$q \vee r$	$\wedge e_2$ 1

4	q	assumption
5	$p \wedge q$	$\wedge i$ 2,4
6	$(p \wedge q) \vee (p \wedge r)$	$\vee i_1$ 5

7	r	assumption
8	$p \wedge r$	$\wedge i$ 2,7
9	$(p \wedge q) \vee (p \wedge r)$	$\vee i_2$ 8

| 10 | $(p \wedge q) \vee (p \wedge r)$ | $\vee e$ 3, 4–6, 7–9 |

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Summary

- Interested in relationships between propositions, not the content of individual propositions

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- Interested in relationships between propositions, not the content of individual propositions
- Build propositions ($p \wedge q$) out of primitive ones p and q

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Summary

- Interested in relationships between propositions, not the content of individual propositions
- Build propositions ($p \wedge q$) out of primitive ones p and q
- Introduce rules that allow us construct proofs

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Remaining tasks:

- What are formulas? (syntax)

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- What is the meaning of formulas? (validity; semantics)

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Remaining tasks:

- What are formulas? (syntax)
- What is the meaning of formulas? (validity; semantics)
- What is the relationship between provable formulas and valid formulas?

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Next Week

- More rules for negation
- Excursion: Intuitionistic logic
- Propositional logic as a formal language
- Semantics of propositional logic