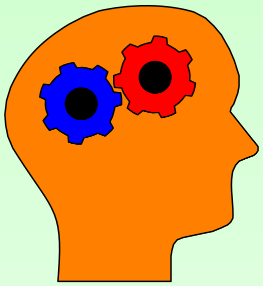
CS6202: Advanced Topics in Programming Languages and Systems

Lecture 1 : **Lambda Calculus**



Lambda Calculus

- Untyped Lambda Calculus
- Evaluation Strategy
- Techniques - encoding, extensions, recursion
- Operational Semantics
- Explicit Typing
- Type Rules and Type Assumption
- Progress, Preservation, Erasure

Introduction to Lambda Calculus:
http://www.inf.fu-berlin.de/lehre/WS03/alpi/lambda.pdf
http://www.cs.chalmers.se/Cs/Research/Logic/TypesSS05/Extra/geuvers.pdf

Untyped Lambda Calculus

- Extremely simple programming language which captures *core* aspects of computation and yet allows programs to be treated as mathematical objects.
- Focused on *functions* and applications.
- Invented by Alonzo (1936,1941), used in programming (Lisp) by John McCarthy (1959).

Functions without Names

Usually functions are given a name (e.g. in language C):

```
int plusone(int x) { return x+1; }
…plusone(5)…
```

However, function names can also be dropped:

```
(int (int x) { return x+1;} ) (5)
```

Notation used in untyped lambda calculus:

$(\lambda$ x. x+1) (5)

Syntax

In purest form (no constraints, no built-in operations), the lambda calculus has the following syntax.

t ::=		terms
	x	variable
	λ x . t	abstraction
	t t	application

This is simplest universal programming language!

Conventions

- Parentheses are used to avoid ambiguities.
  e.g. x y z can be either (x y) z or x (y z)
- Two conventions for avoiding too many parentheses:
  - Applications associates to the left
    e.g. x y z stands for (x y) z
  - Bodies of lambdas extend as far as possible.
    e.g. λ x. λ y. x y x stands for λ x. (λ y. ((x y) x)).
- Nested lambdas may be collapsed together.
  e.g. λ x. λ y. x y x can be written as λ x y. x y x

Scope

- An occurrence of variable x is said to be *bound* when it occurs in the body t of an abstraction λ x . t
- An occurrence of x is *free* if it appears in a position where it is not bound by an enclosing abstraction of x.
- Examples: x y
  λy. x y
  λ x. x (identity function)
  (λ x. x x) (λ x. x x) (non-stop loop)
  (λ x. x) y
  (λ x. x) x

Alpha Renaming

- Lambda expressions are equivalent up to bound variable renaming.
  e.g. λ x. x $=_\alpha$ λ y. y
  λ y. x y $=_\alpha$ λ z. x z

  But NOT:
  λ y. x y $=_\alpha$ λ y. z y
- Alpha renaming rule:

  λ x . E $=_\alpha$ λ z . [x $\mapsto$ z] E (z is not free in E)

Beta Reduction

- An application whose LHS is an abstraction, evaluates to the body of the abstraction with parameter substitution.
  e.g. (λ x. x y) z $\to_\beta$ z y
  (λ x. y) z $\to_\beta$ y
  (λ x. x x) (λ x. x x) $\to_\beta$ (λ x. x x) (λ x. x x)

- Beta reduction rule (operational semantics):

  $$(\lambda\, x\, .\, t_1)\, t_2 \quad \to_\beta \quad [x \mapsto t_2]\, t_1$$

  Expression of form $(\lambda\, x\, .\, t_1)\, t_2$ is called a *redex* (reducible expression).

Evaluation Strategies

- A term may have many redexes. Evaluation strategies can be used to limit the number of ways in which a term can be reduced.
- An evaluation strategy is *deterministic*, if it allows reduction with at most one redex, for any term.
- Examples:
  - full beta reduction
  - normal order
  - call by name
  - call by value, etc

Full Beta Reduction

- Any redex can be chosen, and evaluation proceeds until no more redexes found.
- Example: (λx.x) ((λx.x) (λz. (λx.x) z))
  denoted by id (id (λz. id z))

  Three possible redexes to choose:

  <u>id (id (λz. id z))</u>
  id (<u>id (λz. id z)</u>)
  id (id (λz. <u>id z</u>))
- Reduction:
  id (id (λz. <u>id z</u>))
  → id (<u>id (λz.z)</u>)
  → <u>id (λz.z)</u>
  → λz.z
  $\not\to$

Normal Order Reduction

- Deterministic strategy which chooses the *leftmost, outermost* redex, until no more redexes.
- Example Reduction:

  <u>id (id (λz. id z))</u>
  → <u>id (λz. id z))</u>
  → λz.<u>id z</u>
  → λz.z
  $\not\to$

Call by Name Reduction

- Chooses the *leftmost, outermost* redex, but *never* reduces inside abstractions.
- Example:

  id (id (λz. id z))
  → id (λz. id z))
  → λz.id z
  ↛

Call by Value Reduction

- Chooses the *leftmost, innermost* redex whose RHS is a value; and never reduces inside abstractions.
- Example:

  id (id (λz. id z))
  → id (λz. id z)
  → λz.id z
  ↛

Strict vs Non-Strict Languages

- *Strict* languages always evaluate all arguments to function before entering call. They employ call-by-value evaluation (e.g. C, Java, ML).
- *Non-strict* languages will enter function call and only evaluate the arguments as they are required. *Call-by-name* (e.g. Algol-60) and *call-by-need* (e.g. Haskell) are possible evaluation strategies, with the latter avoiding the re-evaluation of arguments.
- In the case of call-by-name, the evaluation of argument occurs with each parameter access.

Programming Techniques in λ-Calculus

- Multiple arguments.
- Church Booleans.
- Pairs.
- Church Numerals.
- Enrich Calculus.
- Recursion.

Multiple Arguments

- Pass multiple arguments one by one using lambda abstraction as intermediate results. The process is also known as *currying*.

- Example:

$f = \lambda(x,y).s \quad \Longrightarrow \quad f = \lambda\ x.\ (\lambda\ y.\ s)$

Application:

$f(v,w)$ — *requires pairs as primitve types*

$(f\ v)\ w$ — *requires higher order feature*

Church Booleans

- Church's encodings for true/false type with a conditional:

  true = λ t. λ f. t
  false = λ t. λ f. f
  if = λ l. λ m. λ n. l m n

- Example:

  if true v w
  = (λ l. λ m. λ n. l m n) true v w
  $\rightarrow$ true v w
  = (λ t. λ f. t) v w
  $\rightarrow$ v

- Boolean and operation can be defined as:

  and = λ a. λ b. if a b false
  = λ a. λ b. (λ l. λ m. λ n. l m n) a b false
  = λ a. λ b. a b false

Pairs

- Define the functions pair to construct a pair of values, fst to get the first component and snd to get the second component of a given pair as follows:

  pair = λ f. λ s. λ b. b f s
  fst = λ p. p true
  snd = λ p. p false

- Example:

  snd (pair c d)
  = (λ p. p false) ((λ f. λ s. λ b. b f s) c d)
  $\rightarrow$ (λ p. p false) (λ b. b c d)
  $\rightarrow$ (λ b. b c d) false
  $\rightarrow$ false c d
  $\rightarrow$ d

Church Numerals

- Numbers can be encoded by:

  c_0 = λ s. λ z. z
  c_1 = λ s. λ z. s z
  c_2 = λ s. λ z. s (s z)
  c_3 = λ s. λ z. s (s (s z))
  :

Church Numerals

- Successor function can be defined as:

succ = λ n. λ s. λ z. s (n s z)

Example:

succ c_1
= (λ n. λ s. λ z. s (n s z)) (λ s. λ z. s z)
→ λ s. λ z. s ((λ s. λ z. s z) s z)
→ λ s. λ z. s (s z)

succ c_2
= λ n. λ s. λ z. s (n s z) (λ s. λ z. s (s z))
→ λ s. λ z. s ((λ s. λ z. s (s z)) s z)
→ λ s. λ z. s (s (s z))

Church Numerals

- Other Arithmetic Operations:

plus = λ m. λ n. λ s. λ z. m s (n s z)
times = λ m. λ n. m (plus n) c_0
iszero = λ m. m (λ x. false) true

- Exercise : Try out the following.

plus c_1 x
times c_0 x
times x c_1
iszero c_0
iszero c_2

Enriching the Calculus

- We can add constants and built-in primitives to enrich λ-calculus. For example, we can add boolean and arithmetic constants and primitives (e.g. true, false, if, zero, succ, iszero, pred) into an enriched language we call λNB:
- Example:

λ x. succ (succ x) ∈ λNB
λ x. true ∈ λNB

Recursion

- Some terms go into a loop and do not have normal form. Example:

(λ x. x x) (λ x. x x)
→ (λ x. x x) (λ x. x x)
→ …

- However, others have an interesting property

fix = λ f. (λ x. f (λ y. x x y)) (λ x. f (λ y. x x y))

which returns a fix-point for a given functional.

Given x = h x ← x *is fix-point of* h
= fix h

That is: fix h → h (fix h) → h (h (fix h)) → …

Example - Factorial

- We can define factorial as:

fact = λ n. if (n<=1) then 1 else times n (fact (pred n))

= (λ h. λ n. if (n<=1) then 1 else times n (h (pred n))) fact

= fix (λ h. λ n. if (n<=1) then 1 else times n (h (pred n)))

Example - Factorial

- Recall:
  fact = fix (λ h. λ n. if (n<=1) then 1 else times n (h (pred n)))

- Let g = (λ h. λ n. if (n<=1) then 1 else times n (h (pred n)))

Example reduction:

fact 3 = fix g 3
= g (fix g) 3
= times 3 ((fix g) (pred 3))
= times 3 (g (fix g) 2)
= times 3 (times 2 ((fix g) (pred 2)))
= times 3 (times 2 (g (fix g) 1))
= times 3 (times 2 1)
= 6

Formal Treatment of Lambda Calculus

- Let V be a countable set of variable names. The set of terms is the smallest set T such that:
  1. $x \in T$ for every $x \in V$
  2. if $t_1 \in T$ and $x \in V$, then $\lambda\, x.\, t_1 \in T$
  3. if $t_1 \in T$ and $t_2 \in T$, then $t_1\ t_2 \in T$

- Recall syntax of lambda calculus:

t ::=		terms
	x	variable
	λ x.t	abstraction
	t t	application

Free Variables

- The set of free variables of a term t is defined as:

$FV(x) = \{x\}$

$FV(\lambda\, x.t) = FV(t) \setminus \{x\}$

$FV(t_1\ t_2) = FV(t_1) \cup FV(t_2)$

Substitution

- Works when free variables are replaced by term that does not clash:

  $[x \mapsto \lambda\, z.\, z\, w]\ (\lambda\, y.x) = (\lambda\, y.\ \lambda\, x.\, z\, w)$

- However, problem if there is name capture/clash:

  $[x \mapsto \lambda\, z.\, z\, w]\ (\lambda\, x.x) \neq (\lambda\, x.\ \lambda\, z.\, z\, w)$

  $[x \mapsto \lambda\, z.\, z\, w]\ (\lambda\, w.x) \neq (\lambda\, w.\ \lambda\, z.\, z\, w)$

Formal Defn of Substitution

$[x \mapsto s]\ x$	=	s	if $y=x$
$[x \mapsto s]\ y$	=	y	if $y \neq x$
$[x \mapsto s]\ (t_1\ t_2)$	=	$([x \mapsto s]\ t_1)\ ([x \mapsto s]\ t_2)$	
$[x \mapsto s]\ (\lambda\, y.t)$	=	$\lambda\, y.t$	if $y=x$
$[x \mapsto s]\ (\lambda\, y.t)$	=	$\lambda\, y.\ [x \mapsto s]\ t$	if $y \neq x \land y \notin FV(s)$
$[x \mapsto s]\ (\lambda\, y.t)$	=	$[x \mapsto s]\ (\lambda\, z.\ [y \mapsto z]\ t)$ if $y \neq x \land y \in FV(s) \land$ fresh z	

Syntax of Lambda Calculus

- Term:

t ::=		terms
	x	variable
	λ x.t	abstraction
	t t	application

- Value:

t ::=		terms
	λ x.t	abstraction value

Call-by-Value Semantics

premise → $t_1 \to t'_1$

conclusion → $t_1\ t_2 \to t'_1\ t_2$

$$\frac{t_1 \to t'_1}{t_1\ t_2 \to t'_1\ t_2} \quad \text{(E-App1)}$$

$$\frac{t_2 \to t'_2}{v_1\ t_2 \to v_1\ t'_2} \quad \text{(E-App2)}$$

$$(\lambda\, x.t)\ v \to [x \mapsto v]\ t \quad \text{(E-AppAbs)}$$

Getting Stuck

- Evaluation can get stuck. (Note that only values are λ-abstraction)
  e.g. (x y)
- In extended lambda calculus, evaluation can also get stuck due to the absence of certain primitive rules.

  $(\lambda\ x.\ \text{succ}\ x)\ \text{true} \rightarrow \text{succ true} \not\rightarrow$

Boolean-Enriched Lambda Calculus

- Term:

t ::=		terms
	x	variable
	λ x.t	abstraction
	t t	application
	true	constant true
	false	constant false
	if t then t else t	conditional

- Value:

v ::=		value
	λ x.t	abstraction value
	true	true value
	false	false value

Key Ideas

- Exact typing impossible.

  if <long and tricky expr> then true else (λ x.x)

- Need to introduce function type, but need argument and result types.

  if true then (λ x.true) else (λ x.x)

Simple Types

- The set of simple types over the type Bool is generated by the following grammar:
- T ::=

T ::=		types
	Bool	type of booleans
	T → T	type of functions

- → is right-associative:

  $T_1 \rightarrow T_2 \rightarrow T_3$ denotes $T_1 \rightarrow (T_2 \rightarrow T_3)$

Implicit or Explicit Typing

- Languages in which the programmer declares all types are called *explicitly typed*. Languages where a typechecker infers (almost) all types is called *implicitly typed*.
- Explicitly-typed languages places onus on programmer but are usually better documented. Also, compile-time analysis is simplified.

Explicitly Typed Lambda Calculus

- t ::= — terms
  - …
  - λ x : T.t — abstraction
  - …
- v ::= — value
  - λ x : T.t — abstraction value
  - …
- T ::= — types
  - Bool — type of booleans
  - T → T — type of functions

Examples

true

λ x:Bool . x

(λ x:Bool . x) true

if false then (λ x:Bool . True) else (λ x:Bool . x)

Erasure

- The erasure of a simply typed term t is defined as:

erase(x)	=	x
erase(λx :T.t)	=	λ x. erase(t)
erase(t_1 t_2)	=	erase(t_1) erase(t_2)

- A term m in the untyped lambda calculus is said to be *typable* in $\lambda_{\rightarrow}$ (simply typed λ-calculus) if there are some simply typed term t, type T and context Γ such that:

$$\text{erase}(t)=m \;\wedge\; \Gamma \vdash t : T$$

Typing Rule for Functions

- First attempt:

$$\frac{t_2 : T_2}{\lambda\, x{:}T_1 .\, t_2 : T_1 \to T_2}$$

- But $t_2{:}T_2$ can assume that x has type T_1

Need for Type Assumptions

- Typing relation becomes ternary

$$\frac{x{:}T_1 \vdash t_2 : T_2}{\lambda\, x{:}T_1 . t_2 : T_1 \to T_2}$$

- For nested functions, we may need several assumptions.

Typing Context

- A *typing context* is a finite map from *variables to their types*.
- Examples:

  $x : Bool$

  $x : Bool,\ y : Bool \to Bool,\ z : (Bool \to Bool) \to Bool$

Type Rule for Abstraction

Shall use Γ to denote typing context.

$$\frac{\Gamma, x{:}T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda\, x{:}T_1 . t_2 : T_1 \to T_2} \quad \text{(T-Abs)}$$

Other Type Rules

- Variable

$$\frac{x{:}T \in \Gamma}{\Gamma \vdash x : T} \quad \text{(T-Var)}$$

- Application

$$\frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\, t_2 : T_2} \quad \text{(T-App)}$$

- Boolean Terms.

Typing Rules

True : Bool (T-true) False : Bool (T-false) 0 : Nat (T-Zero)

$$\frac{t_1{:}\text{Bool} \quad t_2{:}T \quad t_3{:}T}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T} \quad \text{(T-If)}$$

$$\frac{t : \text{Nat}}{\text{succ } t : \text{Nat}}\text{(T-Succ)} \qquad \frac{t : \text{Nat}}{\text{pred } t : \text{Nat}}\text{(T-Pred)} \qquad \frac{t : \text{Nat}}{\text{iszero } t : \text{Bool}} \text{ (T-Iszero)}$$

Example of Typing Derivation

$$\cfrac{\cfrac{\cfrac{x : \text{Bool} \in x : \text{Bool}}{x : \text{Bool} \vdash x : \text{Bool}}\text{ (T-Var)}}{\vdash (\lambda\, x : \text{Bool}.\ x) : \text{Bool} \rightarrow \text{Bool}}\text{ (T-Abs)} \qquad \cfrac{}{\vdash \text{true} : \text{Bool}}\text{ (T-True)}}{\vdash (\lambda\, x : \text{Bool}.\ x)\ \text{true}\ : \text{Bool}}\text{ (T-App)}$$

Canonical Forms

- If v is a value of type Bool, then v is either true or false.
- If v is a value of type $T_1 \rightarrow T_2$, then $v = \lambda\, x{:}T_1.\ t_2$ where $t{:}T_2$

Progress

Suppose t is a closed well-typed term (that is $\{\} \vdash t : T$ for some T).

Then either t is a value or else there is some t' such that $t \to t'$.

Preservation of Types (under Substitution)

If $\Gamma, x{:}S \vdash t : T$ and $\Gamma \vdash s : S$

then $\Gamma \vdash [x \mapsto s]t : T$

Preservation of Types (under reduction)

If $\Gamma \vdash t : T$ and $t \to t'$

then $\Gamma \vdash t' : T$

Motivation for Typing

- Evaluation of a term either results in a *value* or *gets stuck*!
- Typing can *prove* that an expression cannot get stuck.
- Typing is *static* and can be checked at compile-time.

Normal Form

A term t is a *normal form* if there is no t' such that $t \to t'$.

The multi-step evaluation relation $\to^*$ is the reflexive, transitive closure of one-step relation.

pred (succ(pred 0))
$\to$
pred (succ 0)
$\to$
0

pred (succ(pred 0))
$\to^*$
0

Stuckness

Evaluation may fail to reach a value:

succ (if true then false else true)
$\to$
succ (false)
$\not\to$

A term is *stuck* if it is a normal form but not a value.

Stuckness is a way to characterize *runtime errors*.

Safety = Progress + Preservation

- Progress : A well-typed term is not stuck. Either it is a value, or it can take a step according to the evaluation rules.

  Suppose t is a well-typed term (that is t:T for some T). Then either t is a value or else there is some t' with $t \to t'$

Safety = Progress + Preservation

- Preservation : If a well-typed term takes a step of evaluation, then the resulting term is also well-typed.

  If $t{:}T \;\wedge\; t \to t'$ then $t'{:}T$.