Advanced Automata Theory 5
Infinite Games

Frank Stephan

Department of Computer Science

Department of Mathematics

National University of Singapore

fstephan@comp.nus.edu.sg

Repetition 1

Here games are two-player games.
Anke versus Boris.
Anke starts to play and then Boris and Anke move alternately.

Game in Graph.
"Board of the game" is a finite graph $(\mathbf{G}, \mathbf{E})$.
Players move a marker around in the graph.
The player who moves the marker into the target wins.

Although many games are not defined that way, they can be represented as a game moving a marker on a graph.

Example.
Game of reducing digits in numbers.

Repetition 2

A winning strategy is an algorithm or table which tells Anke in each position how to move (in dependence of the prior moves which occurred in the game) such that Anke will eventually win (if possible).

A node v is a winning position for Anke iff there is a winning strategy which tells Anke how to win, provided that the game starts from the node v.

Similarly one defines winning strategies and positions for Boris.

For each game, either there is a winning strategy for Anke or a winning strategy for Boris or a draw strategy for both players. Similarly, for each node in the graph v and each player p to move, the game is in the situation (v, p) either winnable for Anke or winnable for Boris or draw (both players can enforce a draw).

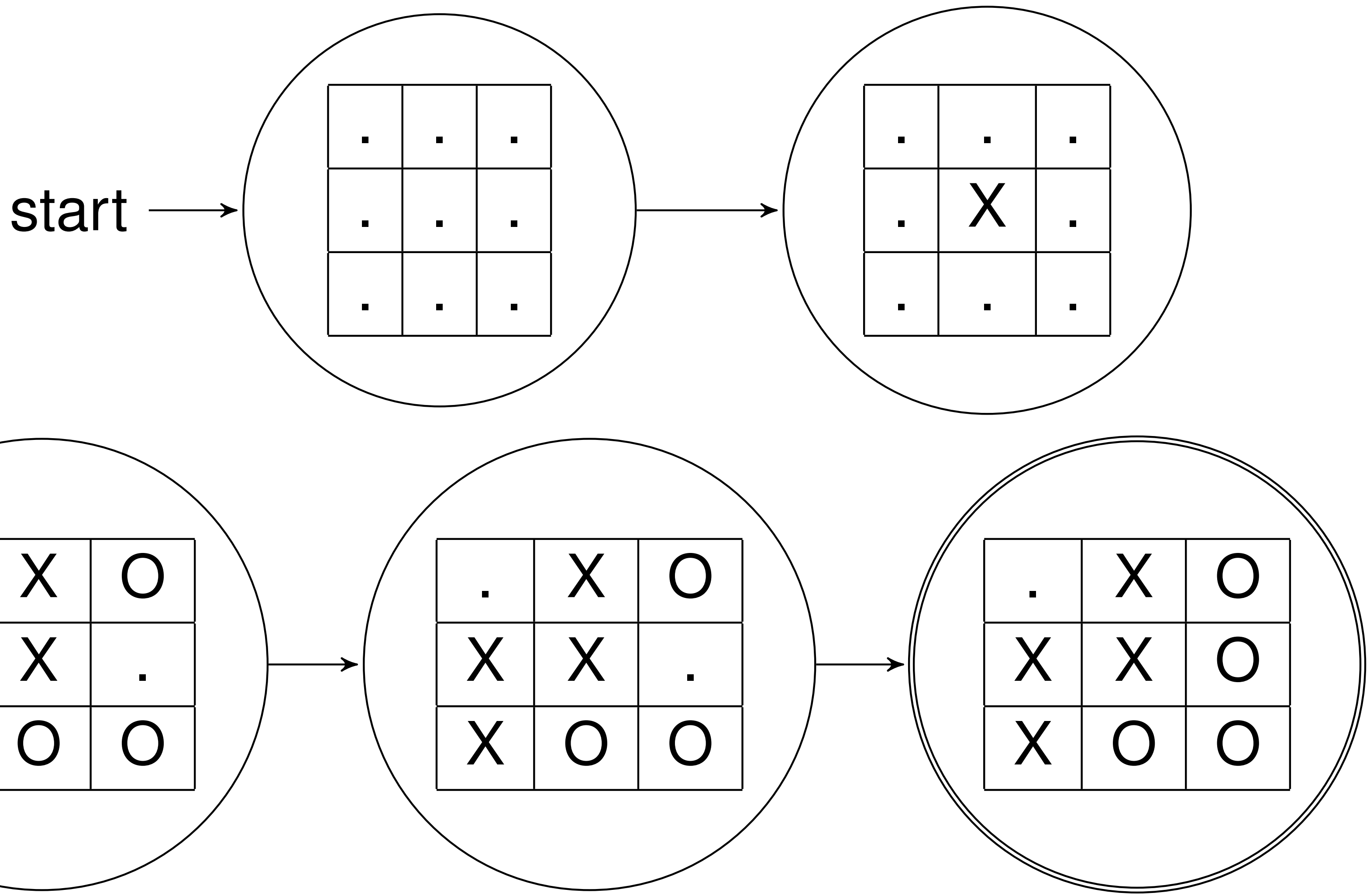
Repetition 3

Theorem. There is an algorithm which determines which player has a winning strategy. The algorithm runs in time polynomial in the size of the graph.

Proof. Let $\mathbf{Q}$ be the set of all nodes and $\mathbf{T}$ be the set of target nodes. The games starts in some node in $\mathbf{Q} - \mathbf{T}$.

1. Let $\mathbf{T_0} = \mathbf{T}$ and $\mathbf{S_0} = \emptyset$ and $\mathbf{n} = \mathbf{0}$.
2. Let $\mathbf{S_{n+1}} = \mathbf{S_n} \cup \{\mathbf{q} \in \mathbf{Q} - (\mathbf{T_n} \cup \mathbf{S_n})$: one can go in one step from $\mathbf{q}$ to a node in $\mathbf{T_n}\}$.
3. Let $\mathbf{T_{n+1}} = \mathbf{T_n} \cup \{\mathbf{q} \in \mathbf{Q} - (\mathbf{T_n} \cup \mathbf{S_{n+1}})$: if one goes from $\mathbf{q}$ one step then one ends up in $\mathbf{S_{n+1}}\}$.
4. If $\mathbf{S_{n+1}} \neq \mathbf{S_n}$ or $\mathbf{T_{n+1}} \neq \mathbf{T_n}$ then let $\mathbf{n} = \mathbf{n} + \mathbf{1}$ and goto 2.
5. $\mathbf{S_n}$ are Winning, $\mathbf{T_n} - \mathbf{T}$ are losing and $\mathbf{Q} - (\mathbf{T_n} \cup \mathbf{S_n})$ are draw positions.

Repetition 4



Repetition 5

For strategic games with two alternately moving players without random aspects, there are three possibilities (plus unknown). Here what is known for famous games.

- The first player has a winning strategy: Connect Four, Hex (on $n * n$ board).
- The second player has a winning strategy: 4$*$4 Othello, 6$*$6 Othello, 15$*$15 Gomoku.
- Both players have a draw strategy: Draughts (Checkers), Nine Men's Morris, Tic Tac Toe.
- Unknown: Chess, Go, 19$*$19 Gomoku (conjecture: second player wins), 8$*$8 Othello (conjecture: draw).

`http://en.wikipedia.org/wiki/Solved_game`

Repetition 6

Alternating Finite Automaton. Anke and Boris decide on moves in nfa while processing a word $\mathbf{w}$. Three possibilities for pairs $(\mathbf{q}, \mathbf{a})$ of states $\mathbf{q}$ and symbols $\mathbf{a}$:

- $(\mathbf{q}, \mathbf{a}) \to \mathbf{r}$: Next state is $\mathbf{r}$;
- $(\mathbf{q}, \mathbf{a}) \to \mathbf{r} \vee \mathbf{p}$: Anke picks $\mathbf{r}$ or $\mathbf{p}$;
- $(\mathbf{q}, \mathbf{a}) \to \mathbf{r} \wedge \mathbf{p}$: Boris picks $\mathbf{r}$ or $\mathbf{p}$.

The afa accepts a word $\mathbf{w}$ iff Anke has a winning strategy.

Example. States $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$; alphabet $\{\mathbf{0}, \mathbf{1}\}$; language $\{\mathbf{0}, \mathbf{1}\}^* \cdot \mathbf{1}$.

state	type	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{p}$	start, rejecting	$\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$	$\mathbf{q} \vee \mathbf{r}$
$\mathbf{q}$	accepting	$\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$	$\mathbf{p} \vee \mathbf{r}$
$\mathbf{r}$	accepting	$\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$	$\mathbf{p} \vee \mathbf{q}$

Survival Games

Motivation

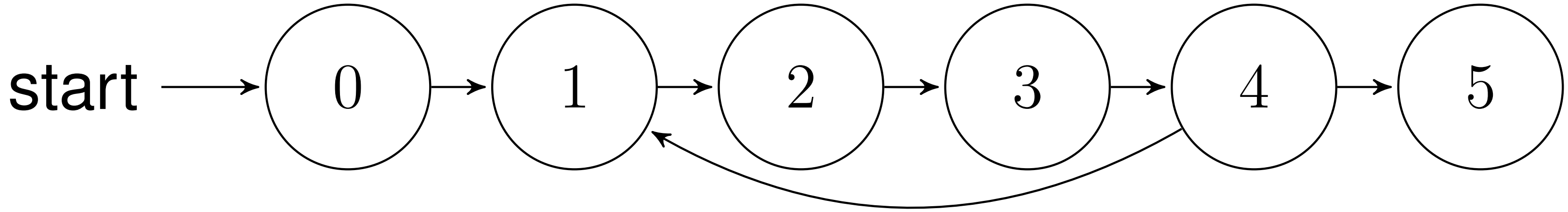
A human makes an expedition, say into Outer Space. There the human has to keep the immediate environment (temperature, pressure, oxygen) inside the spaceship such that survival is possible. When a mistake is made, the human dies.

The Game

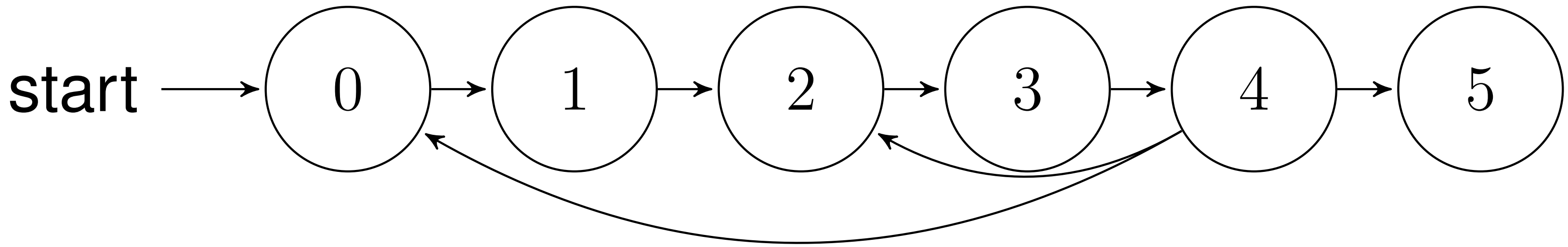
In a survival game, Anke represents the human who influences with her decisions the environment such that she can go on. The opposing player Boris represents the environment's reactions to her decisions.
Anke wins if the game runs forever and Boris wins if the game ends up in a deadend, that is, a node from which onwards no move is possible.

Example 5.2

The first game, Anke can win. In node 4, she always moves back to 1.

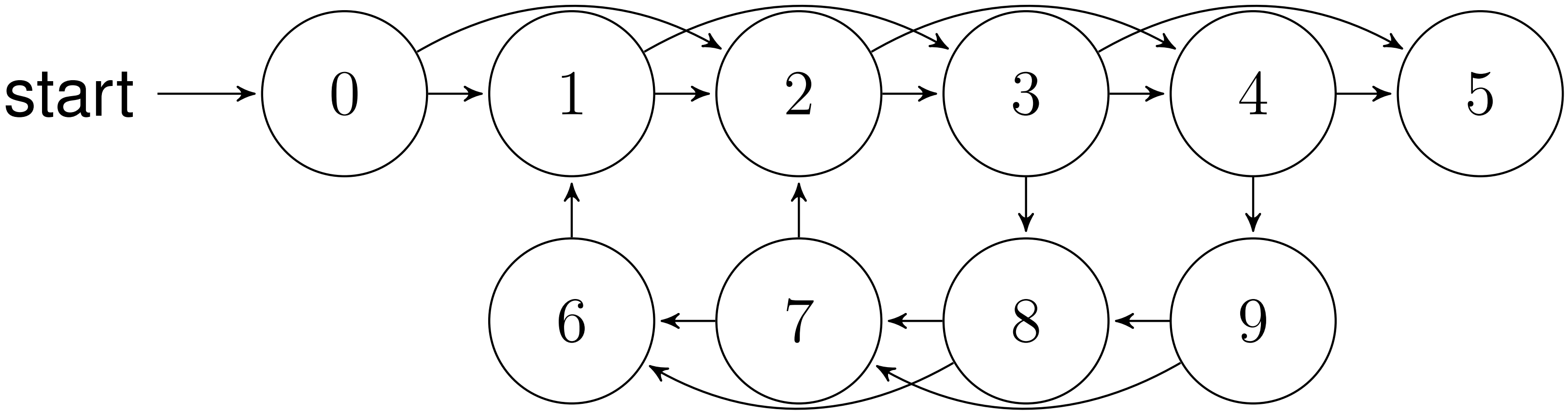


The second game, Boris can win. When the game comes to node 4 for the second time, it is Boris' turn to move and he moves to 5.



Both winning strategies are memoryless, that is, for each node and player, there is a fixed best move which is independent on the prior game history.

Quiz 5.3



Which player has a winning strategy?
Give a memoryless winning strategy for the player.

Theorem 5.4

There is an algorithm which can check which player wins a survival game (when playing optimally).

Let $\mathbf{f}(\mathbf{v}, \mathbf{Anke}) = \mathbf{f}(\mathbf{v}, \mathbf{Boris}) = \mathbf{0}$ for nodes without successor.
Let $\mathbf{f}(\mathbf{v}, \mathbf{Anke}) = \mathbf{n}$ for least $\mathbf{n}$ with $\mathbf{f}(\mathbf{w}, \mathbf{Boris}) < \mathbf{n}$ for all successors $\mathbf{w}$ of $\mathbf{v}$.
Let $\mathbf{f}(\mathbf{v}, \mathbf{Boris}) = \mathbf{n}$ for least $\mathbf{n}$ with $\mathbf{f}(\mathbf{w}, \mathbf{Anke}) < \mathbf{n}$ for some successor $\mathbf{w}$ of $\mathbf{v}$.
Let $\mathbf{f}(\mathbf{v}, \mathbf{p}) = \infty$ whenever $\neg \mathbf{f}(\mathbf{v}, \mathbf{p}) \leq \mathbf{2} * |\mathbf{V}|$.

If $\mathbf{f}(\mathbf{s}, \mathbf{Anke}) = \infty$ then Anke has winning strategy by moving from each node $\mathbf{v}$ with $\mathbf{f}(\mathbf{v}, \mathbf{Anke}) = \infty$ to a successor $\mathbf{w}$ with $\mathbf{f}(\mathbf{w}, \mathbf{Boris}) = \infty$.
If $\mathbf{f}(\mathbf{s}, \mathbf{Anke}) < \infty$ then Boris has winning strategy by moving from each node $\mathbf{v}$ with $\mathbf{f}(\mathbf{v}, \mathbf{Boris}) < \infty$ to a successor $\mathbf{w}$ with $\mathbf{f}(\mathbf{w}, \mathbf{Anke}) < \mathbf{f}(\mathbf{v}, \mathbf{Boris})$.

Exercise 5.5

Consider game $\mathbf{G(p,q,u,v,w)}$ played on a graph $\mathbf{G}$ with nodes $\mathbf{u,v,w}$ and $\mathbf{p \in \{Anke, Boris\}}$ and $\mathbf{q \subseteq \{Anke, Boris\}}$; players move alternately and player $\mathbf{p}$ starts in node $\mathbf{u}$.

Anke wins a play in this game if player $\mathbf{p}$ starts to move from node $\mathbf{u}$ and the game goes through node $\mathbf{v}$ at some time and the game ends after finitely many steps in $\mathbf{w}$ with a player in the set $\mathbf{q}$ being the next to move.

Comment: Condition visiting $\mathbf{v}$ is void when $\mathbf{v \in \{u,w\}}$; there might be more vertices in the graph than $\mathbf{u,v,w}$.

Give an algorithm to determine which player in this game has a winning strategy.

Update Games

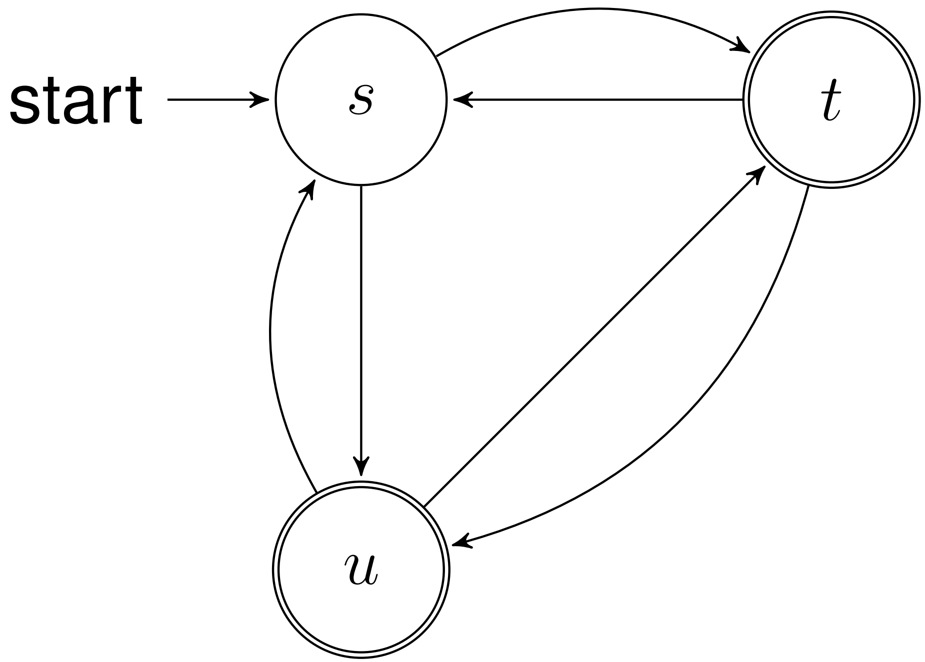
Survival Game: Anke wins iff game has infinitely many moves.

Update Game: Anke wins iff all nodes in a set $\mathbf{W}$ are visited infinitely often.

So game $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \mathbf{W})$ is given by graph $(\mathbf{V}, \mathbf{E})$, starting state $\mathbf{s}$ and non-empty set $\mathbf{W} \subseteq \mathbf{V}$. Anke wins a play of the game iff she starts moving from $\mathbf{s}$ and each node $\mathbf{w} \in \mathbf{W}$ is visited infinitely often in the infinite run of the game.

Questions Investigated
Does either Anke or Boris have a winning strategy?
If so, can the winning strategy be memoryless?
Is there an algorithm to find the winning strategy?

Example 5.7



Anke has a winning strategy but no memoryless one.

Theorem 5.8

There is an algorithm which determines which player has a winning strategy in a given update game $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}\})$.

Use Exercise 5.5 to decide which games $\mathbf{G}(\mathbf{p}, \mathbf{q}, \mathbf{u}, \mathbf{v}, \mathbf{w_k})$ Anke wins with $\mathbf{u}, \mathbf{v}$ being any nodes in the graph.

Anke has a winning strategy iff Case 1 or Case 2 as below:

- Case 1: there is a node $\mathbf{v}$ and a player $\mathbf{p} \in \{\mathbf{Anke}, \mathbf{Boris}\}$ such that Anke can win the game $\mathbf{G}(\mathbf{Anke}, \{\mathbf{p}\}, \mathbf{s}, \mathbf{v}, \mathbf{v})$ and for each $\mathbf{w_k}$, Anke can win the game $\mathbf{G}(\mathbf{p}, \{\mathbf{p}\}, \mathbf{v}, \mathbf{w_k}, \mathbf{v})$.
- Case 2: there is a node $\mathbf{v}$ such that Anke can win the game $\mathbf{G}(\mathbf{Anke}, \{\mathbf{Anke}, \mathbf{Boris}\}, \mathbf{s}, \mathbf{v}, \mathbf{v})$ and for every $\mathbf{w_k}$ and every player $\mathbf{p} \in \{\mathbf{Anke}, \mathbf{Boris}\}$, Anke can win the games $\mathbf{G}(\mathbf{p}, \{\mathbf{Anke}, \mathbf{Boris}\}, \mathbf{v}, \mathbf{w_k}, \mathbf{v})$.

If Anke Wins

Case 1: Anke can force the game to come to a situation where player $\mathbf{p}$ moves from node $\mathbf{v}$ onwards and can also force that, in repeating rounds for $\mathbf{k = 1, 2, \ldots, n, 1, 2, \ldots, n, \ldots}$, the game visits node $\mathbf{w_k}$ and comes back to node $\mathbf{v}$ with player $\mathbf{p}$ to move.

Case 2: Anke can force the game to come to node $\mathbf{v}$ and then to go from $\mathbf{v}$ to $\mathbf{w_k}$ and back to $\mathbf{v}$ for each $\mathbf{w_k}$ infinitely often independently on whose player has to move when the game is in node $\mathbf{v}$.

Strategy needs memory to know which of $\mathbf{v}$ and $\mathbf{w_1, w_2, \ldots, w_n}$ has to be visited next.

If Boris Wins

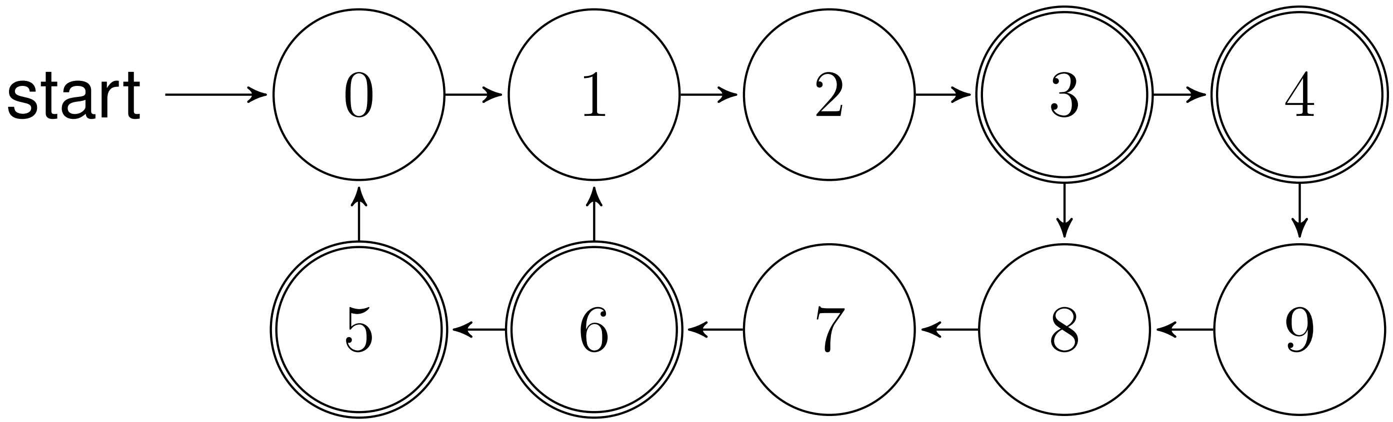
Both Cases fail as the game cannot be forced into some $\mathbf{u} \in \mathbf{W}$. This is a trivial win for Boris.

Both Cases fail in a way that there are $\mathbf{u}, \mathbf{w} \in \mathbf{W}$ and different $\mathbf{p}, \mathbf{p}'$ such that Boris wins both games $\mathbf{G}(\mathbf{Anke}, \{\mathbf{p}\}, \mathbf{s}, \mathbf{u}, \mathbf{u})$ and $\mathbf{G}(\mathbf{p}', \{\mathbf{Anke}, \mathbf{Boris}\}, \mathbf{u}, \mathbf{w}, \mathbf{u})$. Then Boris can enforce that Anke reaches $\mathbf{u}$ only in a way that it is $\mathbf{p}'$'s turn to move and then Boris can enforce that either $\mathbf{w}$ or $\mathbf{u}$ is not visited afterwards.

Both Cases fail in a way that there are $\mathbf{u}, \mathbf{v}, \mathbf{v}', \mathbf{w} \in \mathbf{W}$ and a player $\mathbf{p}$ such that Boris wins $\mathbf{G}(\mathbf{Anke}, \{\mathbf{Anke}\}, \mathbf{u}, \mathbf{v}, \mathbf{u})$ and $\mathbf{G}(\mathbf{Boris}, \{\mathbf{Boris}\}, \mathbf{u}, \mathbf{v}', \mathbf{u})$ and $\mathbf{G}(\mathbf{p}, \{\mathbf{Anke}, \mathbf{Boris}\}, \mathbf{u}, \mathbf{w}, \mathbf{u})$. Then Boris can, when the game reaches $\mathbf{u}$, enforce that after visiting either $\mathbf{v}$ or $\mathbf{v}'$ and returning to $\mathbf{u}$ it is $\mathbf{p}$'s turn to move and then Boris can enforce that either $\mathbf{w}$ or $\mathbf{u}$ is not visited afterwards.

Quiz 5.9

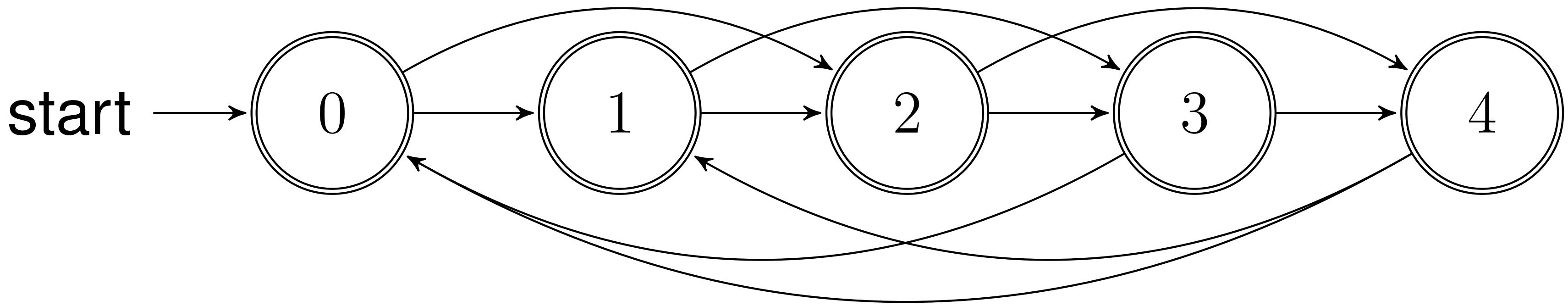
Which player has a winning strategy for the following update game?



How much memory needs the strategy?

Exercise 5.10

Let $\mathbf{n > 4}$, $\mathbf{n}$ be odd, $\mathbf{V = \{m : 0 \leq m < n\}}$ and $\mathbf{E = \{(m, m+1) : m < n-1\} \cup \{(m, m+2) : m < n-2\} \cup \{(n-2, 0), (n-1, 0), (n-1, 1)\}}$ and $\mathbf{s = 0}$. Show that Anke has a winning strategy for the update game $\mathbf{(V, E, s, V)}$ but she does not have a memoryless one.



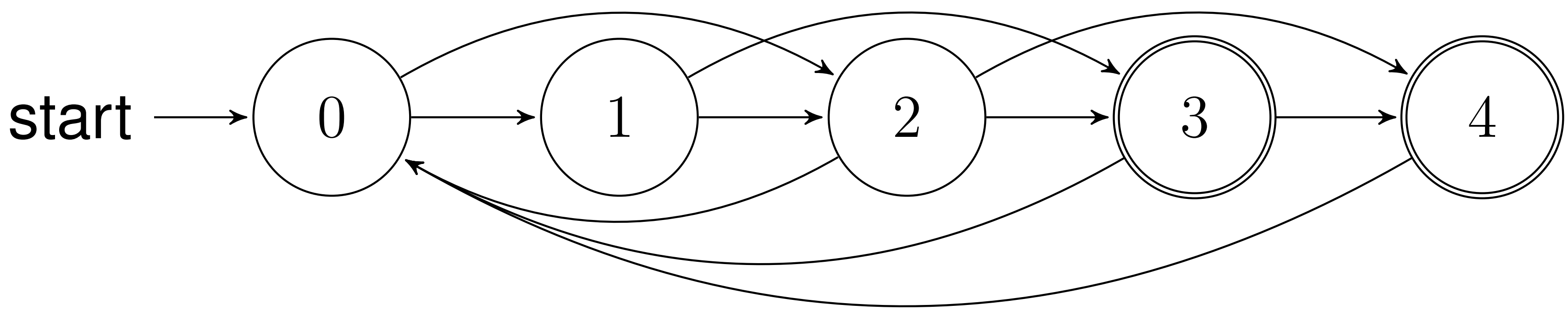
Here the game for $\mathbf{n = 5}$.

Büchi Games (Description 5.11)

Definition

A Büchi game $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \mathbf{W})$ is played on a finite graph with nodes $\mathbf{V}$ and edges $\mathbf{E}$, starting node $\mathbf{s}$ and a special set $\mathbf{W} \subseteq \mathbf{V}$. Anke wins a play in the game iff, when starting in $\mathbf{s}$, the game makes infinitely many moves and during these moves visits one or more nodes in $\mathbf{W}$ infinitely often.

Example



Anke has a winning strategy for this game.

Deciding Büchi Games (Theorem 5.12)

Initialise $\mathbf{f(v,p) = 0}$ for all $\mathbf{v,p}$ and repeat until all done.

- If there are $\mathbf{v \in V - W}$ and $\mathbf{w \in V}$ with $\mathbf{(v,w) \in E}$ and $\mathbf{f(v,Anke) < f(w,Boris) - 1}$ then update $\mathbf{f(v,Anke) = f(v,Anke) + 1}$;
- If there is $\mathbf{v \in V - W}$ with outgoing edge and all $\mathbf{w \in V}$ with $\mathbf{(v,w) \in E}$ satisfy $\mathbf{f(v,Boris) < f(w,Anke) - 1}$ then update $\mathbf{f(v,Boris) = f(v,Boris) + 1}$;
- If there are $\mathbf{v \in W}$ and $\mathbf{w \in V}$ with $\mathbf{f(v,Anke) \leq 30 \cdot |V|^2 - 3 \cdot |V|}$, $\mathbf{(v,w) \in E}$ and $\mathbf{f(v,Anke) \leq f(w,Boris) + 6 \cdot |V|}$ then update $\mathbf{f(v,Anke) = f(v,Anke) + 3 \cdot |V|}$;
- If there is $\mathbf{v \in W}$ with outgoing edge and $\mathbf{f(v,Boris) \leq 30 \cdot |V|^2 - 3 \cdot |V|}$ and all $\mathbf{w \in V}$ with $\mathbf{(v,w) \in E}$ satisfy $\mathbf{f(v,Boris) \leq f(w,Anke) + 6 \cdot |V|}$ then update $\mathbf{f(v,Boris) = f(v,Boris) + 3 \cdot |V|}$.

Winning Strategies

If $\mathbf{f(s, Anke) \geq 30 \cdot |V|^2 - 3 \cdot |V|}$

Then Anke has a winning strategy by always chosing for a node $\mathbf{v}$ that successor $\mathbf{w}$ where $\mathbf{f(w, Boris)}$ is as large as possible

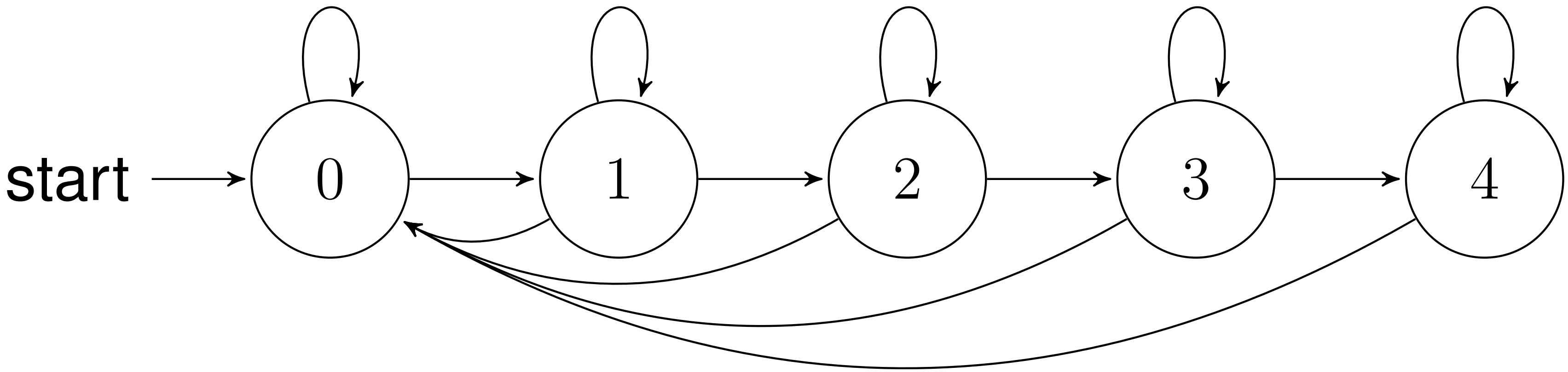
Else Boris has a winning strategy by always chosing for a node $\mathbf{v}$ that successor $\mathbf{w}$ where $\mathbf{f(w, Anke)}$ is as small as possible.

The strategy is memoryless, as it only follows $\mathbf{f}$ and does not inspect the history of the game.

Verification uses properties of $\mathbf{f}$ which come out of the update rules.

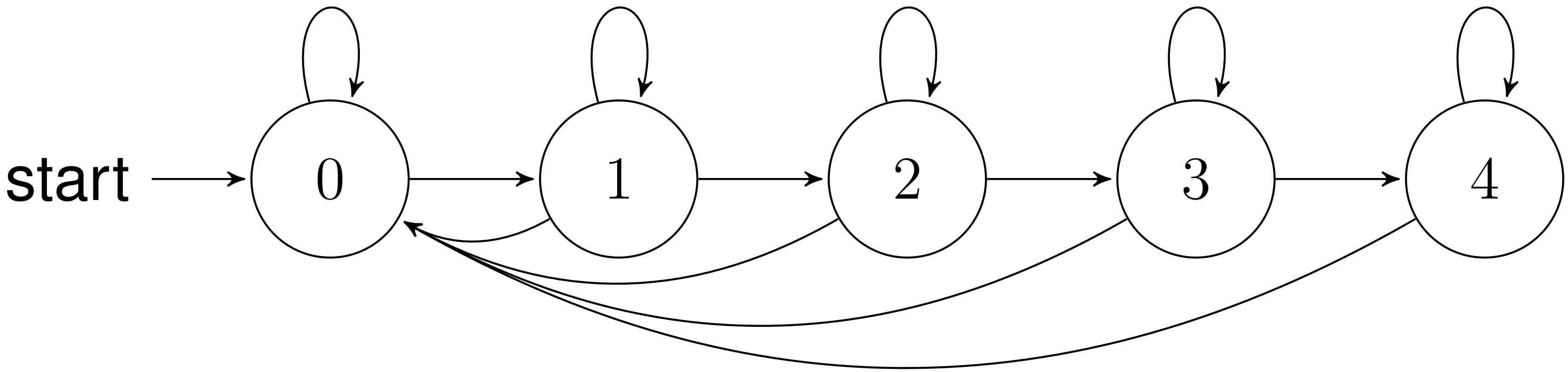
Parity Games (Description 5.13)

A parity game (V, E, s, val) has a function $val : V \to \mathbb{N}$. Anke wins a play $v_0, v_1, v_2, \ldots$ iff $\limsup val(v_k)$ is even.



Example of parity game, node q is labeled with $val(q)$. Play $0 - 1 - 2 - 3 - 0 - 1 - 2 - 3 - 0 - 1 - 2 - 3 - 0 - 1 - \ldots$ is won by Boris.

Anke's Winning Strategy

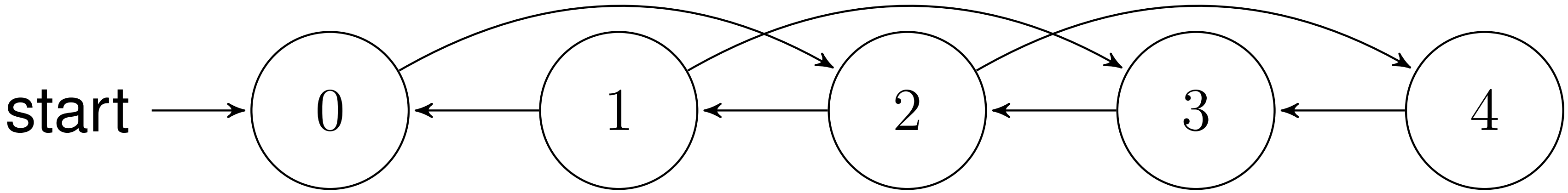


Node	0	1	2	3	4
Anke's Move	0	2	2	4	4

If $\mathbf{maxval}$ is even and one can always go from $\mathbf{n}$ to $\mathbf{n}$ and to $\mathbf{min\{maxval, n + 1\}}$ and to $\mathbf{0}$ then Anke has a winning strategy.

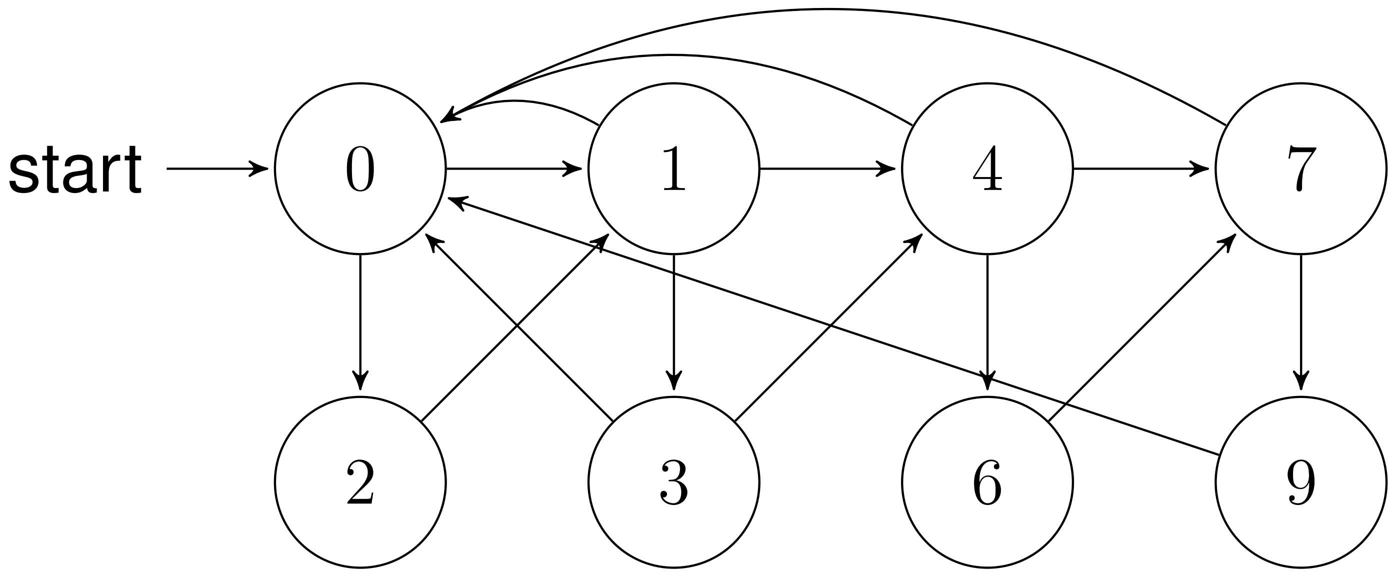
Quiz 5.14

Which player has a winning strategy for the following Parity Game?



Recall that a win in a parity game is an infinite play of the game where both players move alternately and the largest, infinite often occurring number, is even; Anke makes the first move.

Exercise 5.15



Which player has a winning strategy for this parity game?
Give the winning strategy as a table (it is memoryless).

Infinite Games in General

Game (V, E, s, F) given by set V of nodes, possible moves E, starting node s and evaluation function $F : Pow(V) \rightarrow \{Anke, Boris\}$.

Let U be the set of nodes visited infinitely often in a play. Then the player $F(U)$ wins this play.

Survival Game: $F(\emptyset) = Boris$ and $F(U) = Anke$ for non-empty U.

Update Game: If $W \subseteq U$ then $F(U) = Anke$ else $F(U) = Boris$.

Büchi Game: If $W \cap U \neq \emptyset$ then $F(U) = Anke$ else $F(U) = Boris$.

Parity Game: If $U \neq \emptyset$ and $max\{val(w) : w \in U\}$ is even then $F(U) = Anke$ else $F(U) = Boris$.

Exercise 5.17

Determine $\mathbf{F}$ for the following game: $\mathbf{V} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, $\mathbf{E} = \{(\mathbf{0}, \mathbf{1}), (\mathbf{0}, \mathbf{2}), (\mathbf{1}, \mathbf{2}), (\mathbf{1}, \mathbf{3}), (\mathbf{2}, \mathbf{3}), (\mathbf{2}, \mathbf{4}), (\mathbf{3}, \mathbf{4}), (\mathbf{3}, \mathbf{5}), (\mathbf{4}, \mathbf{5}), (\mathbf{4}, \mathbf{0}), (\mathbf{5}, \mathbf{0}), (\mathbf{5}, \mathbf{1})\}$, that is, the successors of $\mathbf{v}$ are $\mathbf{v} + \mathbf{1}, \mathbf{v} + \mathbf{2}$ modulo $\mathbf{6}$. The game starts in $\mathbf{0}$ and the players move alternately. Anke wins the game iff for each infinitely often visited node $\mathbf{v}$, also the node $(\mathbf{v} + \mathbf{3})$ mod $\mathbf{6}$ is infinitely often visited.

Say which $\mathbf{U} \subseteq \mathbf{V}$ are the infinitely often visited nodes of some play and which player would win them.

Which player has a winning strategy for this game? Can this winning strategy be made memoryless?

Exercise 5.18

Let a game $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \mathbf{W})$ given by set $\mathbf{V}$ of nodes, possible moves $\mathbf{E}$, starting node $\mathbf{s}$ and a set of nodes $\mathbf{W}$ to be avoided eventually. Let $\mathbf{U}$ be the infinitely often visited nodes of some play.

Say that if $\mathbf{U} \cap \mathbf{W} = \emptyset$ and $\mathbf{U} \neq \emptyset$ then $\mathbf{Anke}$ wins the game else $\mathbf{Boris}$ wins the game.

Determine an easy way mapping from $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \mathbf{W})$ to $(\mathbf{V}', \mathbf{E}', \mathbf{s}', \mathbf{F}')$ and players $\mathbf{p}$ to $\mathbf{p}'$ such that player $\mathbf{p}$ wins the avoidance game $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \mathbf{W})$ iff $\mathbf{p}'$ wins the game $(\mathbf{V}', \mathbf{E}', \mathbf{s}', \mathbf{F}')$ where the type of $(\mathbf{V}', \mathbf{E}', \mathbf{s}', \mathbf{F}')$ is one of survival game, update game, Büchi game or parity game. Say which type of game it is and how the mapping is done and give reasons why the connection holds.

Exercises 5.19 and 5.20

5.19: Describe an algorithm which transforms a parity game $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \mathbf{val})$ into a new parity game $(\mathbf{V}', \mathbf{E}', \mathbf{s}', \mathbf{val}')$ such that this game never gets stuck and $\mathbf{Anke}$ wins $(\mathbf{V}, \mathbf{E}, \mathbf{s}, \mathbf{val})$ iff $\mathbf{Boris}$ wins $(\mathbf{V}', \mathbf{E}', \mathbf{s}', \mathbf{val}')$; without loss of generality it can be assumed that $\mathbf{V}$ is a finite subset of $\mathbb{N}$ and $\mathbf{val}(\mathbf{v}) = \mathbf{v}$. Do the mapping such that $\mathbf{V}'$ has at most two nodes more than $\mathbf{V}$.

5.20: Consider the following game on the board $\mathbb{N}$: The game terminates at $\mathbf{1}$ and one can move from number $\mathbf{2n+2}$ to $\mathbf{n+1}$ and from number $\mathbf{2n+1}$ to $(\mathbf{2n+1}) \cdot \mathbf{k} + \mathbf{h}$ for some $\mathbf{k}, \mathbf{h} \in \{\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{7}, \mathbf{9}\}$. $\mathbf{Anke}$ wins if the game runs forever and $\mathbf{Boris}$ wins if the game eventually reaches $\mathbf{1}$. Clearly Boris wins if the game starts in $\mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{16}$ and other powers of $\mathbf{2}$. Determine for all further numbers up to $\mathbf{20}$ who has a winning strategy when $\mathbf{Anke}$ starts at this number.

Exercises 5.21 and 5.22

5.21: Consider the game on the board $\mathbb{N}$ with the following moves: One can move from number $\mathbf{2n+2}$ to $\mathbf{n+1}$ and from number $\mathbf{2n+1}$ to one of $\mathbf{2n+2, 2n+4, 6n+6, 6n+8}$. If the game reaches $\mathbf{1}$, $\mathbf{Boris}$ wins; if the game visits each node only finitely often without going to $\mathbf{1}$, $\mathbf{Anke}$ wins; if the game goes through some number infinitely often, it is draw. Give an algorithm which says which starting numbers are wins for $\mathbf{Anke}$, wins for $\mathbf{Boris}$ and draw.

5.22: Consider an Update Game on the board $\mathbf{\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}}$ where players move by adding $\mathbf{1}$ or $\mathbf{3}$ modulo $\mathbf{12}$ to the current position and where $\mathbf{Anke}$ wins when $\mathbf{0, 4, 8}$ are visited infinitely often. Which starting positions are wins for $\mathbf{Anke}$ and which are wins for $\mathbf{Boris}$?

Exercises 5.23 - 5.26

Consider the following game. One can add $\mathbf{1}$ or $\mathbf{2}$ modulo $\mathbf{4}$ to one out of four digits $\mathbf{abcd}$. So from $\mathbf{0123}$, one can move to $\mathbf{1123}, \mathbf{2123}, \mathbf{0223}, \mathbf{0323}, \mathbf{0133}, \mathbf{0103}, \mathbf{0120}, \mathbf{0121}$.
Determine the winners of the following update and Büchi games in dependence of the set $\mathbf{F}$ of nodes which all (update) or where one of them (Büchi) has to be visited infinitely often.

Exercise 5.23. Solve the Büchi game where $\mathbf{F} = \{\mathbf{0000}, \mathbf{1111}, \mathbf{2222}, \mathbf{3333}\}$.

Exercise 5.24. Solve the update game where $\mathbf{F}$ is any nonempty set of nodes.

Exercise 5.25. Solve the Büchi game where $\mathbf{F}$ contains all nodes with three equal digits.

Exercise 5.26. Solve the Büchi game where $\mathbf{F}$ contains all nodes which contain exactly one $\mathbf{0}$.

Exercise 5.27 - 5.28

Consider the following game. One can add $\mathbf{1}$ or $\mathbf{3}$ modulo $\mathbf{4}$ to one out of four digits $\mathbf{abcd}$. So from $\mathbf{0123}$, one can move to $\mathbf{1123}, \mathbf{3123}, \mathbf{0223}, \mathbf{0023}, \mathbf{0133}, \mathbf{0113}, \mathbf{0120}, \mathbf{0122}$. Determine the winners of the following games in dependence of the start state where one considers Büchi games (one node in $\mathbf{F}$ has to be visited infinitely often) and survival games (all nodes in $\mathbf{F}$ have to be avoided all the time).

Exercise 5.27. Let $\mathbf{F}$ contain all nodes which have at least two digits $\mathbf{3}$. Solve both the survival game and the Büchi game.

Exercise 5.28. Let $\mathbf{F}$ contain all nodes which have at least one digit $\mathbf{3}$. Solve both the survival game and the Büchi game.