Advanced Automata Theory 10 Transducers and Rational Relations

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Repetition: Automatic Structures

A structure $(\mathbf{A}, \mathbf{R_1}, \ldots, \mathbf{R_m}, \mathbf{f_1}, \ldots, \mathbf{f_n}, \mathbf{c_1}, \ldots, \mathbf{c_h})$ is automatic iff $\mathbf{A}$ is a regular set and each relation $\mathbf{R_k}$ is an automatic relation with domain $\mathbf{A}^{\ell_\mathbf{k}}$ and each function $\mathbf{f_k}$ is an automatic function mapping $\mathbf{A}^{\ell'_\mathbf{k}}$ to $\mathbf{A}$; the constants $\mathbf{c_1}, \ldots, \mathbf{c_h}$ are specific members of $\mathbf{A}$.

Examples

$(\mathbb{Z}, +, \mathbf{0})$ is a group with an automatic representation. Indeed, every fully automatic group is by definition an automatic structure.

$(\mathbb{Q}, +, \mathbf{0})$ has no automatic representation.

Repetition: Definability & Automaticity

Khoussainov and Nerode showed that whenever in an automatic structure a relation or function is first-order definable from other automatic relations or functions then it is automatic.

$(\mathbf{0}^*, \mathbf{Succ})$ with $\mathbf{Succ}(\mathbf{w}) = \mathbf{w0}$ is isomorphic to the structure $(\mathbb{N}, \mathbf{x} \mapsto \mathbf{x} + \mathbf{1})$. The addition is not automatic in this structure, hence addition cannot be first-order defined from the successor-relation.

If $(\mathbf{A}, +, \mathbf{0})$ is isomorphic to $(\mathbb{N}, +, \mathbf{0})$ then $\mathbf{1}$ is uniquely determined in $\mathbf{A}$ by the axioms $\mathbf{1} \neq \mathbf{0}$ and $\forall \mathbf{x}, \mathbf{y}\, [\mathbf{x} + \mathbf{y} = \mathbf{1} \Rightarrow \mathbf{x} = \mathbf{0} \vee \mathbf{y} = \mathbf{0}]$. Hence one can define the ordering by $\mathbf{x} < \mathbf{y} \Leftrightarrow \exists \mathbf{z}\, [\mathbf{x} + \mathbf{z} + \mathbf{1} = \mathbf{y}]$.

Repetition: Rings and Semirings

A structure $(\mathbf{A}, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ is called a semiring with $\mathbf{1}$ iff it satisfies the following conditions:

1. $(\mathbf{A}, \oplus, \mathbf{0})$ is a commutative monoid;
2. $(\mathbf{A}, \otimes, \mathbf{1})$ is a monoid;
3. $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{A}\, [\mathbf{x} \otimes (\mathbf{y} \oplus \mathbf{z}) = (\mathbf{x} \otimes \mathbf{y}) \oplus (\mathbf{x} \otimes \mathbf{z})]$ and $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{A}\, [(\mathbf{x} \oplus \mathbf{y}) \otimes \mathbf{z} = (\mathbf{x} \otimes \mathbf{z}) \oplus (\mathbf{y} \otimes \mathbf{z})]$.

If, furthermore, $(\mathbf{A}, \oplus, \mathbf{0})$ is a group then $(\mathbf{A}, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ is called a ring with $\mathbf{1}$.

A semiring / ring is called commutative iff $\forall \mathbf{x}, \mathbf{y}\, [\mathbf{x} \otimes \mathbf{y} = \mathbf{y} \otimes \mathbf{x}]$.

Repetition: Infinite Automatic Ring

Assume that $(\mathbf{F}, +, *, \mathbf{0}, \mathbf{1})$ is a finite ring. Let $\mathbf{G}$ contain those elements $\mathbf{x_1 x_2 \ldots x_n}$ in $\mathbf{F}^*$ which either satisfy $\mathbf{n = 1}$ or $\mathbf{n > 1 \wedge x_{n-1} \neq x_n}$. Intuitively, $\mathbf{02112}$ stands for $\mathbf{021122222}\ldots$ where the last symbol repeats forever.

Now let $\mathbf{x_1 x_2 \ldots x_n + y_1 y_2 \ldots y_m = z_1 z_2 \ldots z_h}$ if for all $\mathbf{k > 0}$, $\mathbf{x_{\min\{n,k\}} + y_{\min\{m,k\}} = z_{\min\{h,k\}}}$. Similarly for multiplication.

Now the member $\mathbf{0}$ of $\mathbf{F}$ is also the additive neutral element in $\mathbf{G}$ and $\mathbf{1}$ is also the multiplicative neutral element in $\mathbf{G}$.

The so generated $(\mathbf{G}, +, *, \mathbf{0}, \mathbf{1})$ is an example of an infinite automatic ring and represents the ring of the eventually constant functions $\mathbf{f} : \mathbb{N} \to \mathbf{F}$ with pointwise operations.

Repetition: Partial and Linear Orders

An ordering $\sqsubset$ on a set $\mathbf{A}$ is a relation satisfying the following two axioms:

1. $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{A}\,[\mathbf{x} \sqsubset \mathbf{y} \wedge \mathbf{y} \sqsubset \mathbf{z} \Rightarrow \mathbf{x} \sqsubset \mathbf{z}]$;
2. $\forall \mathbf{x}\,[\mathbf{x} \not\sqsubset \mathbf{x}]$.

Well-known automatic orderings are $<_{\mathbf{lex}}$, $<_{\mathbf{ll}}$, $<_{\mathbf{sh}}$ and $\prec$.

An ordering is called linear iff

3. $\forall \mathbf{x}, \mathbf{y} \in \mathbf{A}\,[\mathbf{x} \sqsubset \mathbf{y} \vee \mathbf{x} = \mathbf{y} \vee \mathbf{y} \sqsubset \mathbf{x}]$.

The orderings $<_{\mathbf{lex}}$ and $<_{\mathbf{ll}}$ are linear, the orderings $<_{\mathbf{sh}}$ and $\prec$ are not linear.

Repetition: Ordinals

Cantor designed a way to represent small ordinals as sums of descending chains of ω-powers: $\omega^{\mathbf{4}} + \omega^{\mathbf{2}} + \omega^{\mathbf{2}} + \omega$. Here $\omega^{\mathbf{k+1}}$ is the first ordinal which cannot be written as a finite sum of ordinals up to $\omega^{\mathbf{k}}$; ω is the first ordinal which cannot be written as $\mathbf{1} + \mathbf{1} + \ldots + \mathbf{1}$.

Write $\omega^{\mathbf{3}} \cdot \mathbf{2} + \omega \cdot \mathbf{3} + \mathbf{4}$ instead of $\omega^{\mathbf{3}} + \omega^{\mathbf{3}} + \omega + \omega + \omega + \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{1}$.

Example to add ordinals:
$(\omega^{\mathbf{8}} \cdot \mathbf{5} + \omega^{\mathbf{7}} \cdot \mathbf{2} + \omega^{\mathbf{4}}) + (\omega^{\mathbf{7}} + \omega^{\mathbf{6}} + \omega + \mathbf{1}) = \omega^{\mathbf{8}} \cdot \mathbf{5} + \omega^{\mathbf{7}} \cdot \mathbf{3} + \omega^{\mathbf{6}} + \omega + \mathbf{1}$.

Theorem of Delhomme: The ordinals below $\omega^{\mathbf{k}}$ with $\mathbf{k} \in \mathbb{N}$ have an automatic representation plus addition and comparison algorithm. This is impossible for larger sets of ordinals.

Rational Relations

Automatic Relation: Finite automaton reads all inputs involved at the same speed with $\#$ supplied for exhausted inputs.

Rational Relation: Nondeterministic finite automaton reads all inputs individually and can read them at different speed.

The first type of automata is called synchronous, the second type is called asynchronous.

There are many relations which are rational but not automatic.

Formal definition

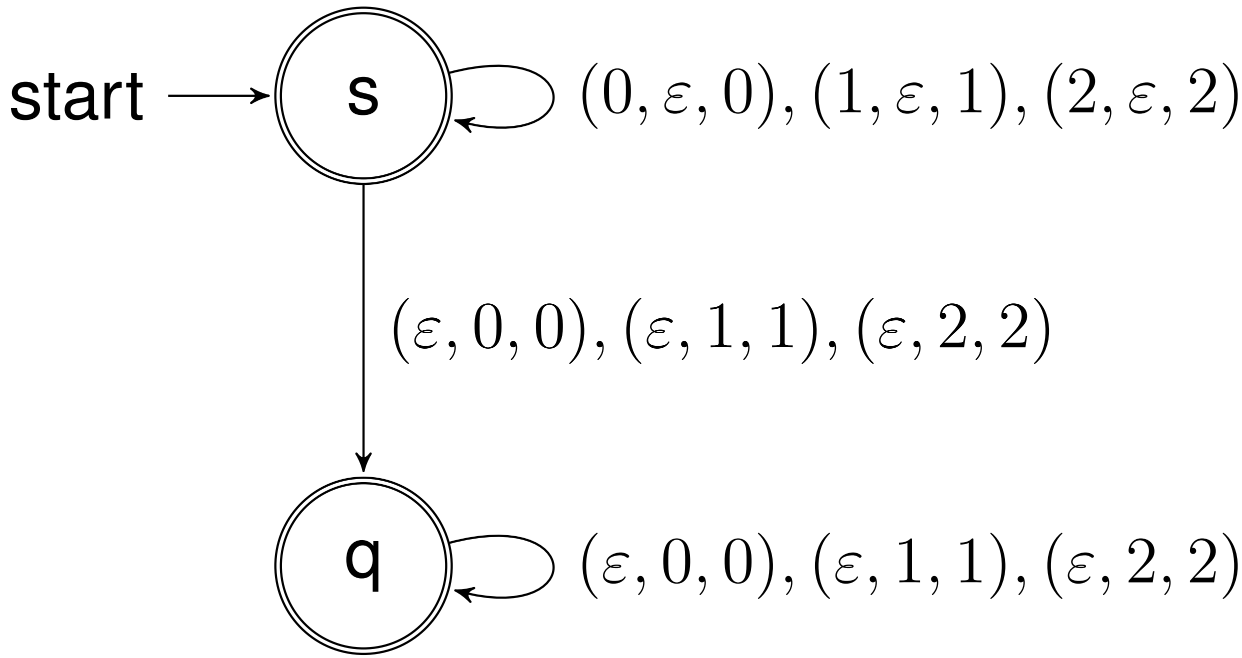
A rational relation $\mathbf{R} \subseteq (\Sigma^*)^{\mathbf{n}}$ is given by an non-deterministic finite state machine which can process $\mathbf{n}$ inputs in parallel and does not need to read them in the same speed. Transitions from one state $\mathbf{p}$ to a state $\mathbf{q}$ are labelled with an $\mathbf{n}$-tuple $(\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n})$ of words $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n} \in \Sigma^*$ and the automaton can go along this transition iff for each input $\mathbf{k}$ the next $|\mathbf{w_k}|$ symbols in the input are exactly those in the string $\mathbf{w_k}$ (this condition is void if $\mathbf{w_k} = \varepsilon$) and in the case that the automaton goes on this transition, $|\mathbf{w_k}|$ symbols are read from the $\mathbf{k}$-th input word.

A word $(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ is in $\mathbf{R}$ iff there is a run of the machine with transitions labelled by $(\mathbf{w_{1,1}}, \mathbf{w_{1,2}}, \ldots, \mathbf{w_{1,n}})$, $(\mathbf{w_{2,1}}, \mathbf{w_{2,2}}, \ldots, \mathbf{w_{2,n}})$, ..., $(\mathbf{w_{m,1}}, \mathbf{w_{m,2}}, \ldots, \mathbf{w_{m,n}})$ ending up in an accepting state such that $\mathbf{x_1} = \mathbf{w_{1,1}} \mathbf{w_{2,1}} \ldots \mathbf{w_{m,1}}$, $\mathbf{x_2} = \mathbf{w_{2,1}} \mathbf{w_{2,2}} \ldots \mathbf{w_{m,2}}$, ..., $\mathbf{x_n} = \mathbf{w_{1,n}} \mathbf{w_{2,n}} \ldots \mathbf{w_{m,n}}$.

Example 10.2: String Concatenation

Concatenation: $\mathbf{0100 \cdot 1122 = 01001122}$; not an automatic relation.

The following automaton witnesses that it is a rational relation over alphabet $\mathbf{\Sigma = \{0, 1, 2\}}$.

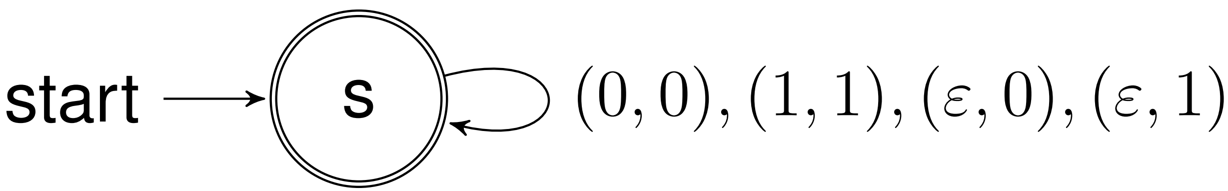


Sample Run: $\mathbf{(s01, s210, s01210) \Rightarrow (0s1, s210, 0s1210) \Rightarrow (01s, s210, 01s210) \Rightarrow (01q, 2q10, 012q10) \Rightarrow (01q, 21q0, 0121q0) \Rightarrow (01q, 210q, 01210q)}$.

Example 10.3: Subsequence

A string $\mathbf{x}$ is a subsequence of $\mathbf{y}$ iff it can be obtained by from $\mathbf{y}$ by deleting symbols at some positions. For example $\mathbf{12112}$ is a subsequence of $\mathbf{010200100102}$ and of $\mathbf{1211212}$ but not of $\mathbf{321123}$.

The following one-state automaton recognises this relation for the binary alphabet $\{\mathbf{0}, \mathbf{1}\}$.

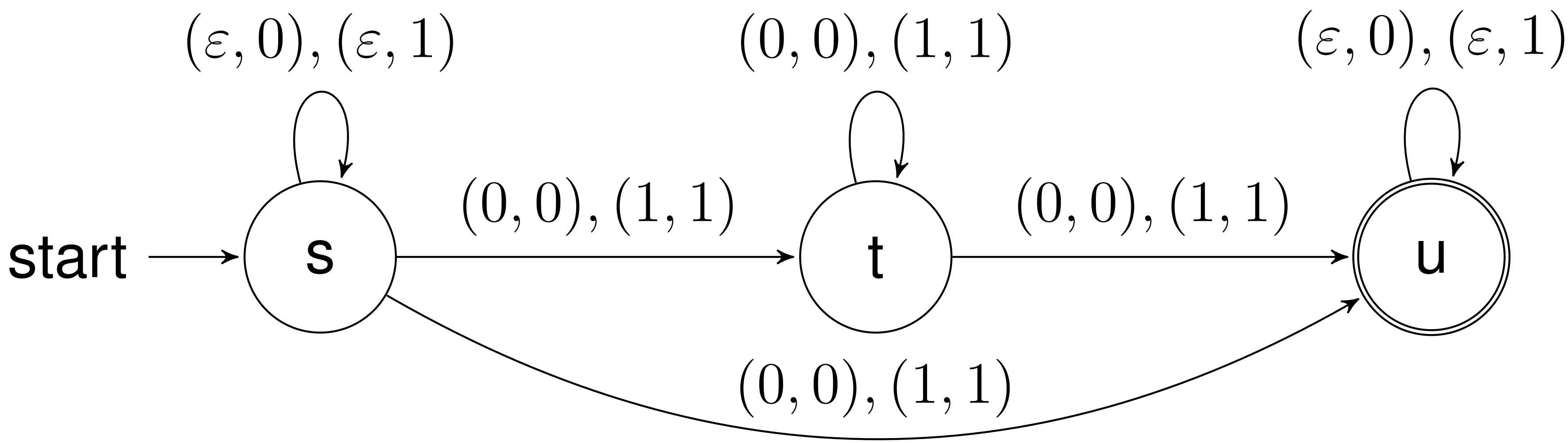


In general, there are one initial accepting state $\mathbf{s}$ with self-loops from $\mathbf{s}$ to $\mathbf{s}$ labelled with $(\varepsilon, \mathbf{a})$ and $(\mathbf{a}, \mathbf{a})$ for all $\mathbf{a} \in \Sigma$.

If $\mathbf{x} = \mathbf{0101}$ and $\mathbf{y} = \mathbf{00110011}$ then the automaton can accept this subsequence relation $(\mathbf{x}, \mathbf{y})$ using transitions labelled $(\mathbf{0}, \mathbf{0}), (\varepsilon, \mathbf{0}), (\mathbf{1}, \mathbf{1}), (\varepsilon, \mathbf{1}), (\mathbf{0}, \mathbf{0}), (\varepsilon, \mathbf{0}), (\mathbf{1}, \mathbf{1}), (\varepsilon, \mathbf{1})$.

Example 10.4: Substring

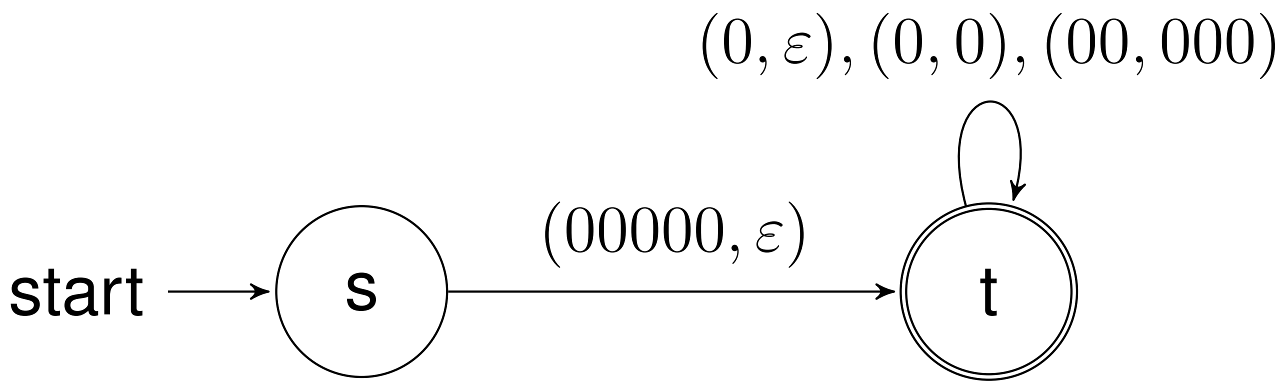
The following automaton recognises the relation of all $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x}$ is a nonempty substring of $\mathbf{y}$, that is, $\mathbf{x} \neq \varepsilon$ and $\mathbf{y} = \mathbf{vxw}$ for some $\mathbf{v}, \mathbf{w} \in \{\mathbf{0}, \mathbf{1}\}^*$.



In $\mathbf{s}$: Parsing $(\varepsilon, \mathbf{v})$;
From $\mathbf{s}$ to $\mathbf{u}$: Parsing $(\mathbf{x}, \mathbf{x})$;
In $\mathbf{u}$: Parsing $(\varepsilon, \mathbf{w})$.

Exercise 10.5: Rational Relations

Rational relations got their name, as one can use them in order to express relations between the various inputs words which are rational. For example, over alphabet $\{0\}$, the relation of all (x, y) with $|x| \geq \frac{2}{3}|y| + 5$ is recognised as follows:



Make automata which recognise the following relations:

(a) $\{(x, y) \in (0^*, 0^*) : 5 \cdot |x| = 8 \cdot |y|\}$;

(b) $\{(x, y, z) \in (0^*, 0^*, 0^*) : 2 \cdot |x| = |y| + |z|\}$;

(c) $\{(x, y, z) \in (0^*, 0^*, 0^*) : 3 \cdot |x| = |y| + |z| \vee |y| = |z|\}$.

Transducers

Transducers are automata recognising rational relations where the first parameter $\mathbf{x}$ is called input and the second parameter $\mathbf{y}$ is called output.

In particular one is interested in transducers computing functions: Two possible runs accepting $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}, \mathbf{z})$ must satisfy $\mathbf{y} = \mathbf{z}$. Given $\mathbf{x}$, there is a run accepting some pair $(\mathbf{x}, \mathbf{y})$ iff $\mathbf{x}$ is in the domain of the function.

Two main concepts:

Mealy machines:
Input/Output pairs on transition-labels.

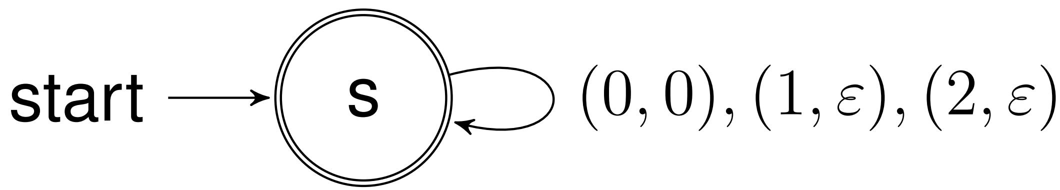
Moore machines:
Input on transition-labels; output in states.

Mealy machine

A Mealy machine computing a rational function $\mathbf{f}$ is a non-deterministic finite automaton such that each transition is attributed with a pair $(\mathbf{v}, \mathbf{w})$ of strings and whenever the machine follows a transition $(\mathbf{p}, (\mathbf{v}, \mathbf{w}), \mathbf{q})$ from state $\mathbf{p}$ to state $\mathbf{q}$ then one says that the Mealy machine processes the input part $\mathbf{v}$ and produces the output part $\mathbf{w}$.

Every automatic function is also a rational function and computed by a transducer, but not vice versa.

Mealy machine can compute π with $\pi(\mathbf{0}) = \mathbf{0}$, $\pi(\mathbf{1}) = \varepsilon$, $\pi(\mathbf{2}) = \varepsilon$, $\pi(\mathbf{v} \cdot \mathbf{w}) = \pi(\mathbf{v}) \cdot \pi(\mathbf{w})$:

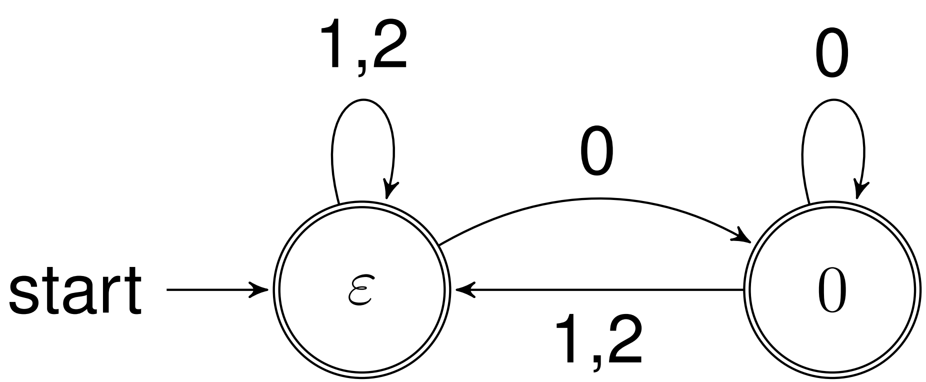


Moore machine

Moore machine

Non-deterministic finite automaton such that:
Possibly several starting states and final states;
Transitions (q, a, p) with input symbol $a \in \Sigma$;
States q labelled with output string $w_q \in \Sigma^*$;
Word $a_1 \ldots a_n$ translated into $w_{q_0} w_{q_1} \ldots w_{q_n}$ iff q_0 is a starting state and q_n is a final state and (q_m, a_{m+1}, q_{m+1}) is a valid transition for all $m < n$.

Moore machine erasing all $1, 2$ and preserving 0 computing function π with $\pi(012012) = 00$.



Example: Function f

Let $\mathbf{f(a_1a_2\ldots a_n) = 012a_1a_1a_2a_2\ldots a_na_n012}$, so $\mathbf{f(01) = 0120011012}$. Moore machine for $\mathbf{f}$:

state	starting	acc/rej	output	on 0	on 1	on 2
s	yes	rej	012	p, p'	q, q'	r, r'
p	no	rej	00	p, p'	q, q'	r, r'
q	no	rej	11	p, p'	q, q'	r, r'
r	no	rej	22	p, p'	q, q'	r, r'
s'	yes	acc	012012	–	–	–
p'	no	acc	00012	–	–	–
q'	no	acc	11012	–	–	–
r'	no	acc	22012	–	–	–

Quiz: Write Mealy machine for $\mathbf{f}$.

Example: Function g

Let $g(a_1 a_2 \ldots a_n) = (\max(\{a_1, a_2, \ldots, a_n\}))^n$. So $g(\varepsilon) = \varepsilon$, $g(000) = 000$, $g(0110) = 1111$ and $g(00212) = 22222$.

Nodes $\{s, q_0, r_0, q_1, r_1, q_2, r_2\}$;
Starting nodes: s;
Accepting nodes: s, r_0, r_1, r_2.
Output in nodes: $w_s = \varepsilon$; $w_{q_0} = w_{r_0} = 0$; $w_{q_1} = w_{r_1} = 1$; $w_{q_2} = w_{r_2} = 2$.
Transitions: $(s, 0, r_0)$, $(s, 1, r_1)$, $(s, 2, r_2)$, $(s, 0, q_1)$, $(s, 0, q_2)$, $(s, 1, q_2)$, $(r_0, 0, r_0)$, $(q_1, 0, q_1)$, $(q_1, 1, r_1)$, $(r_1, 0, r_1)$, $(r_1, 1, r_1)$, $(q_2, 0, q_2)$, $(q_2, 1, q_2)$, $(q_2, 2, r_2)$, $(r_2, 0, r_2)$, $(r_2, 1, r_2)$, $(r_2, 2, r_2)$.

Quiz: Write a Mealy machine for g.

Quiz

Determine the minimum number $\mathbf{m}$ such that every rational function can be computed by a non-deterministic Moore machine with at most $\mathbf{m}$ starting states.

Note that $\mathbf{m}$ cannot be $\mathbf{1}$ as there is a function which maps ε to $\mathbf{0}$ and every non-empty word to $\mathbf{1}$. Give the Moore machine for this function.

Can the same be done to rule out $\mathbf{m} = \mathbf{2}$?

Exercise 10.10

A Moore machine / Mealy machine is deterministic, if it has exactly one starting state and each transition reads exactly one input symbol and for each state and each input symbol there is at most one transition which applies.

Make a deterministic Moore machine and a deterministic Mealy machine which do the following with binary inputs: As long as the symbol $\mathbf{1}$ appears on the input, the symbol is replaced by $\mathbf{0}$; if at some time the symbol $\mathbf{0}$ appears, it is replaced by $\mathbf{1}$ and from then onwards all symbols are copied from the input to the output without any further change.

Examples for input $\mapsto$ output are $\mathbf{0001} \mapsto \mathbf{1001}$, $\mathbf{110} \mapsto \mathbf{001}$, $\mathbf{1111} \mapsto \mathbf{0000}$, $\mathbf{0} \mapsto \mathbf{1}$ and $\mathbf{110011} \mapsto \mathbf{001011}$.

Exercise 10.11

Let the alphabet be $\{0, 1, 2\}$ and let $R = \{(x, y, z, u) : u$ has has $|x|$ many 0s, $|y|$ many 1s and $|z|$ many 2s$\}$.

Is R a rational relation? Prove your result.

Theorem of Nivat

Theorem [Nivat 1968]
Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ be disjoint alphabets. Let π_k preserve the symbols from Σ_k and erase all other symbols.
Now a relation $R \subseteq \Sigma_1^* \times \Sigma_2^* \times \ldots \times \Sigma_m^*$ is rational iff there is a regular set P over a sufficiently large alphabet such that $(w_1, w_2, \ldots, w_n) \in R \Leftrightarrow \exists v \in P\, [\pi_1(v) = w_1 \wedge \pi_2(v) = w_2 \wedge \ldots \wedge \pi_m(v) = w_m]$.

Rational Structures

A structure $(\mathbf{A}, \mathbf{R_1}, \mathbf{R_2}, \ldots, \mathbf{R_k}, \mathbf{f_1}, \mathbf{f_2}, \ldots, \mathbf{f_h})$ is rational iff all the $\mathbf{A}$ is regular and $\mathbf{R_1}, \mathbf{R_2}, \ldots, \mathbf{R_k}$ are rational relations and $\mathbf{f_1}, \mathbf{f_2}, \ldots, \mathbf{f_h}$ are rational functions.

Furthermore, structures isomorphic to a rational structure might also be called rational.

Every automatic structure is by definition also a rational structure, but not vice versa.

The monoid $(\{\mathbf{0}, \mathbf{1}\}^*, \cdot, \varepsilon)$ with $\cdot$ being the concatenation is a rational structure but not an automatic one.

Theorem of Khoussainov and Nerode

The Theorem of Khoussainov and Nerode does not hold for rational structures.

- There are relations and functions which are first-order definable from rational relations without being a rational relation;
- There is no algorithm to decide whether a given first-order formula in a rational structure is true.

However, certain structures which are not automatic, are still rational.

Exercise 10.13: Random Graph

There is a rational representation of the random graph. Instead of coding (V,E) directly, one first codes a directed graph (V,F) with the following properties:

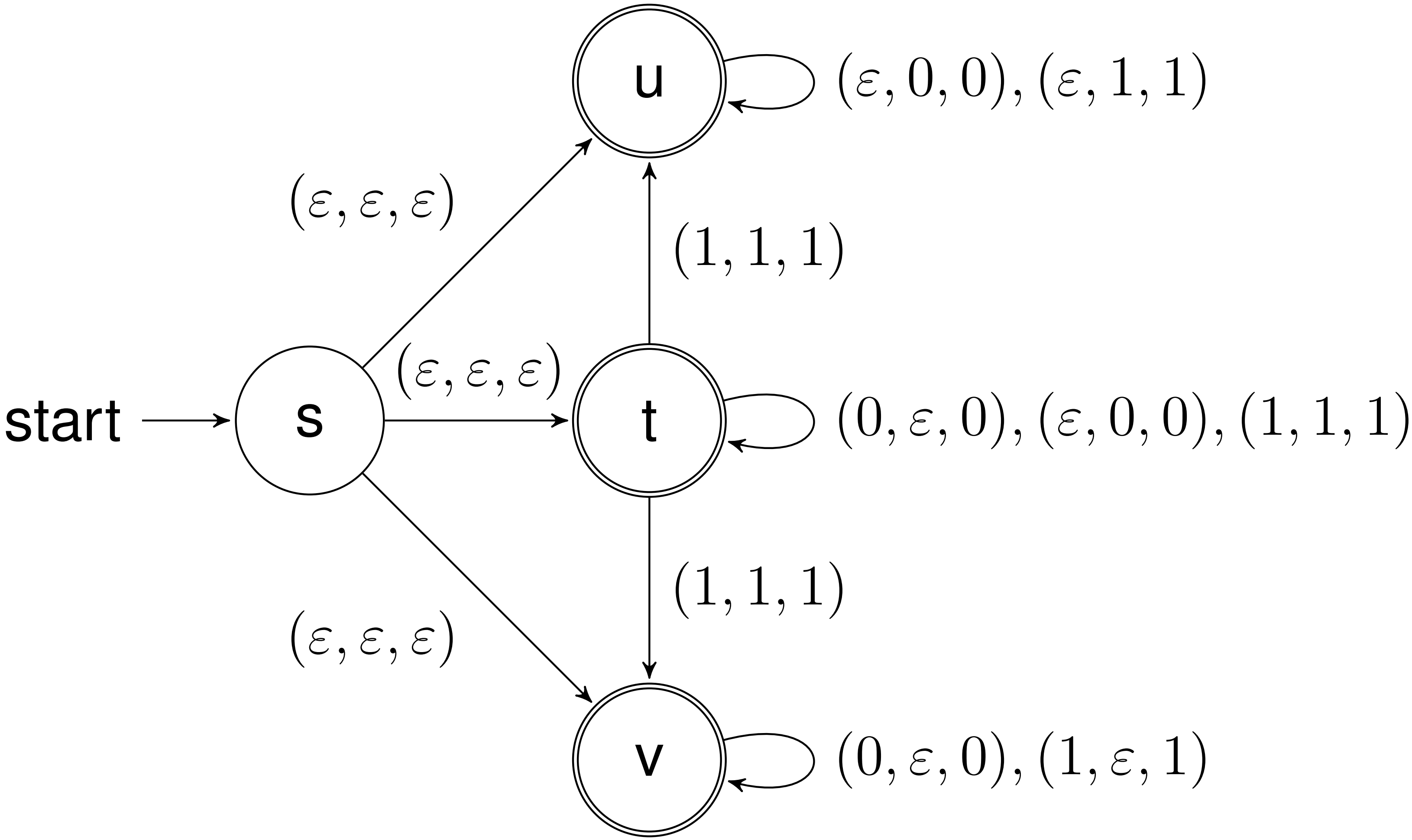
- For each $x, y \in V$, if $(x,y) \in F$ then $|x| < |y|/2$;
- For each finite $W \subseteq V$ there is a y with $\forall x\,[(x,y) \in F \Leftrightarrow x \in W]$.

This is done by letting $V = \{00, 01, 10, 11\}^+$ and defining that $(x,y) \in F$ iff there are n, m, k such that $y = a_0 b_0 a_1 b_1 \ldots a_n b_n$ and $a_m = a_k = 0$ and $a_h = 1$ for all h with $m < h < k$ and $x = b_m b_{m+1} \ldots b_{k-1}$. Give a transducer recognising F and show that this F satisfies the two properties above.

Now let $(x,y) \in E \Leftrightarrow (x,y) \in F \vee (y,x) \in F$. Show that (V,E) is the random graph.

Multiplication of Natural Numbers

The multiplicative monoid $(\mathbb{N} - \{0\}, *, 1)$ has a rational representation. Here one would represent $2^{n_1} \cdot 3^{n_2} \cdot \ldots \cdot p_k{}^{n_k}$ with $n_k > 0$ by $0^{n_1}10^{n_2}1 \ldots 0^{n_k}1$ and 1 by ε.



Exercises 10.16-10.18

Let $\mathbf{R}$ be a binary rational relation and let $\mathbf{L}$ be any language. Now define $\mathbf{R}(\mathbf{L}) = \{\mathbf{v} : \exists \mathbf{w} \in \mathbf{L}\,[\mathbf{R}(\mathbf{v},\mathbf{w})]$. Similarly for a ternary rational relation $\mathbf{S}$, let $\mathbf{S}(\mathbf{L},\mathbf{H}) = \{\mathbf{u} : \exists \mathbf{v} \in \mathbf{L}\,\exists \mathbf{w} \in \mathbf{H}\,[\mathbf{S}(\mathbf{u},\mathbf{v},\mathbf{w})]\}$.

Exercise 10.16: Show that if $\mathbf{L}$ and $\mathbf{H}$ are regular, so are $\mathbf{R}(\mathbf{L})$ and $\mathbf{S}(\mathbf{L},\mathbf{H})$.

Exercise 10.17: Show that if $\mathbf{L}$ is context-free, so is $\mathbf{R}(\mathbf{L})$.

Exercise 10.18: If $\mathbf{L},\mathbf{H}$ are context-free, is then also $\mathbf{S}(\mathbf{L},\mathbf{H})$ context-free? Prove the answer.

Exercise 10.19-10.21

Exercise 10.19: Construct a transducer $\mathbf{S}$ which recognises a triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ iff $\mathbf{v}$ and $\mathbf{w}$ have a common subsequence of length at least $|\mathbf{u}|$.

Exercise 10.20: Is there a transducer $\mathbf{R}$ which recognises a pair $(\mathbf{v}, \mathbf{w})$ iff $\mathbf{v}$ is the mirror image of $\mathbf{w}$?

Exercise 10.21: Is there a transducer $\mathbf{T}$ which recognises a pair $(\mathbf{v}, \mathbf{w})$ iff $\mathbf{v}$ occurs in $\mathbf{w}$ two times as a subword?