Homework for 02.09.2004

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Homework. The homework follows the lecture notes. What cannot be done as scheduled, will be done the week afterwards. This homework contains the left-overs from 26.08.2004.

Lecture is Mon 16.00h - 17.30h and Thu 16.00h - 16.45h. Tutorial is Thu 16.45h - 17.30h. The room is S13#05-03.

http://www.comp.nus.edu.sg/~fstephan/homework.ps

Exercise 1.12. Which of the following sets of natural numbers are equal? Well-known mathematical theorems can be applied without proving them.

1. \(A = \{1, 2\}\);
2. \(B = \{1, 2, 3\}\);
3. \(C\) is the set of all prime numbers;
4. \(D = \{d \mid \exists a, b, c > 0 (a^d + b^d = c^d)\}\);
5. \(E = \{e \mid e > 0 \land \forall c \in C\ (e \leq c)\}\);
6. \(F = \{f \mid \forall c \in C\ (f \geq c)\}\);
7. \(G = \{g \mid g \geq 2 \land \forall a, b > 1 (4g \neq (a + b)^2 - (a - b)^2)\}\);
8. \(H = \{h \mid h > 0 \land h^2 = h^b\}\);
9. \(I = \{i \mid i + i = i \cdot i\}\);
10. \(J = \{j \mid (j + 1)^2 = j^2 + 2j + 1\}\);
11. \(K = \{k \mid 4k \geq k^2\}\);
12. \(L = \{l \mid \exists c \in C\ (l < c)\}\);
13. $M = \{m \mid \exists c \in C (m = c^2)\}$;
14. $N = \{n \mid \exists c, d \in C (n = cd)\}$;
15. $O = \{o \mid o \text{ has exactly three prime factors}\}$;
16. $P = \{p \mid p, p + 2 \in C\}$.

**Exercise 2.9.** Establish properties to define the following sets as subsets of the natural numbers using the Axiom of Comprehension:

1. the set of all numbers with exactly three divisors,
2. the set $\{0, 2, 4, 6, \ldots\}$ of all even numbers,
3. the set of all square numbers,
4. the set of all numbers whose binary representation contains exactly four times a 1.

For example, the set of prime numbers can be defined as the set $\{x \in \mathbb{N} \mid \exists \text{ unique } y, z \in \mathbb{N} (y \cdot z = x \land z < y \leq x)\}$, that is the set of all natural numbers with exactly two divisors.

**Exercise 2.10.** Show that every property $p$ satisfies the following statements.

1. There are sets $x, y$ such that $x \in y$ and either $p(x) \land p(y)$ or $\neg p(x) \land \neg p(y)$.
2. There is a set $x$ with $x = \{y \in x \mid p(y)\}$.
3. There is a one-to-one function $f$ such that $p(x)$ iff $p(y)$ for all $y \in f(x)$.

**Exercise 2.18.** Prove that the symmetric difference is associative, that is, for all sets $A, B, C$, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$. For this reason, one can just write $A \triangle B \triangle C$. Furthermore, prove that $A - B = A \cap (A \triangle B)$.

**Exercise 3.9.** Define (informally) functions $f_n$ from $\mathbb{N}$ to $\mathbb{N}$ with the following properties:

1. $f_1$ is bijective and satisfies $f_1(x) \neq x$ but $f_1(f_1(x)) = x$ for all $x \in \mathbb{N}$;
2. $f_2$ is two-to-one: for every $y$ there are exactly two elements $x, x' \in \mathbb{N}$ with $f(x) = f(x') = y$;
3. $f_3$ is dominating all polynomials, that is, for every polynomial $p$ there is an $x$ such that for all $y > x$, $f_3(y) > p(y)$;
4. $f_4$ satisfies $f_4(x + 1) = f_4(x) + 2x + 1$ for all $x \in \mathbb{N}$;

5. $f_5(x) = \begin{cases} 
0 & \text{if } x = 0; \\
 f_5(x - 1) & \text{if } x > 0 \text{ and } x \text{ is not a square number;} \\
f_5(x - 1) + 1 & \text{if } x > 0 \text{ and } x \text{ is a square number.}
\end{cases}$

Determine the range of the function $f_4$.

**Exercise 3.11.** Let $A = \{0, 1, 2\}$ and $F = \{f : A \to A \mid f = f \circ f\} = \{f : A \to A \mid \forall x (f(f(x)) = f(f(x)))\}$. Show that $F$ has exactly 10 members and determine these.