# Homework for 30.09.2004

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**Homework.** The homework follows the lecture notes. What cannot be done as scheduled, will be done the week afterwards. This homework contains the new tasks for 30.09.2004.

Lecture is Mon 16.00h - 17.30h and Thu 16.00h - 16.45h. Tutorial is Thu 16.45h - 17.30h. Midterm Exam Mon 27.09.2004 16.00h. The room is S13#05-03.

http://www.comp.nus.edu.sg/~fstephan/homework.ps
http://www.comp.nus.edu.sg/~fstephan/homework.pdf

**Exercise 8.9.** Let $D = \{f : \mathbb{N} \to \mathbb{N} \mid \forall n\,(f(S(n)) \leq f(n))\}$ be the set of all decreasing functions. Show that $D$ is countable.

**Exercise 8.12.** Let the elements of $A$ be ordered such that the symbol $\emptyset$ comes first and the comma comes last. Let $<_{ll}$ be the length-lexicographic ordering on the set $A^*$ of all strings over $A$. Now let $f : V_\omega \to A^*$ map every set $x$ in $V_\omega$ to first expression describing $x$. Then $\emptyset <_{ll} \{\}$, thus the symbol "$\{\}$" is never used to describe the empty set; this convention is also applied in this text. Now prove the following facts:

1. the length of $f(x)$ is odd for every $x \in V_\omega$;
2. if $f(x) = \{y\}$ then $f(S(x)) = \{y, \{y\}\}$;
3. $f(2) = \{\emptyset, \{\emptyset\}\}$;

Furthermore, find a formula giving the length of $f(n)$ for every $n$ and determine which of the following numbers is the length of $f(10)$: 42, 100, 1000, 1001, 1022, 1023, 1024, 2047, 4096, $256^2$, $10^{10} - 1$, $2^{256}$, $1023^{1023}$.

If the length of $f(x)$ is $n$ and $f(y)$ is $m$, what is the length of $f((x, y))$ for the ordered pair $(x, y)$?

**Exercise 8.13.** Let $\mathbb{A}$ be the set of algebraic real numbers, that is, the set of all $r \in \mathbb{R}$ for which there are $n \in \mathbb{N}$ and $z_0, z_1, \ldots, z_n \in \mathbb{Z}$ such that $z_n \neq 0$ and $z_0 + z_1 r + z_2 r^2 + \ldots + z_n r^n = 0$. Note that such a polynomial of degree $n$ can have up to $n$ places $r$ which are mapped to 0. Show that $\mathbb{A}$ is countable by giving a one-to-one mapping from $\mathbb{A}$ into $\mathbb{N}$.

**Exercise 9.3.** A graph $(G, E)$ is called *bipartite* if there are two subsets $X, Y$ of $G$ such that $X \cap Y = \emptyset$ and every pair $(x, y) \in E$ is actually in $X \times Y \cup Y \times X$. An English-Russian dictionary is a bipartite graph because one can take $X$ to be the words written in the Latin alphabet and $Y$ to be the words written in the Cyrillic alphabet. An English-Spanish dictionary is not bipartite, for example some place names like "Los Angeles" appear in the same spelling in both languages. By the way, the mentioned name is of Spanish origin and has the English translation "the angels". Which of the following graphs are bipartite? The set of vertices is $\mathbb{N}$ and the set $E_n$ of edges is specified below, note that $E_n \subseteq \mathbb{N} \times \mathbb{N}$.

1. $(x, y) \in E_1 \Leftrightarrow x = y$,
2. $(x, y) \in E_2 \Leftrightarrow x < y$,
3. $(x, y) \in E_3 \Leftrightarrow 12 < x + y < 18$,
4. $(x, y) \in E_4 \Leftrightarrow x > 4 \wedge (y = x^2 \vee y = x^4)$,
5. $(x, y) \in E_5 \Leftrightarrow \exists z > 0\, (x = 2^z \wedge y = 3^z)$,
6. $(x, y) \in E_6 \Leftrightarrow \exists z > 0\, (x \in \{2^z, 3^z\} \wedge y \in \{5^z, 7^z\})$,
7. $(x, y) \in E_7 \Leftrightarrow y = S(x) \wedge x$ is even,

**Exercise 9.6.** Let $A = \mathbb{N} - \{0, 1\} = \{2, 3, 4, \ldots\}$ and let $<_{div}$ be given by $x <_{div} y \Leftrightarrow \exists z \in A\, (x \cdot z = y)$. That is, $x <_{div} y$ iff $x$ is a proper divisor of $y$, so $2 <_{div} 8$ but $2 \not<_{div} 2$ and $2 \not<_{div} 5$. Prove that $(A, <_{div})$ is a partially ordered set.

**Exercise 9.9.** Prove that the following relations are partial orderings on $\mathbb{N}^{\mathbb{N}}$:

- $f \sqsubset_1 g \Leftrightarrow \exists n\, \forall m > n\, (f(m) < g(m))$;
- $f \sqsubset_2 g \Leftrightarrow \forall n\, (f(n) \leq g(n)) \wedge \exists m\, (f(m) < g(m))$;
- $f \sqsubset_3 g \Leftrightarrow \forall n\, (f(n) \leq g(n)) \wedge \exists n\, (f(n) < g(n)) \wedge \exists n\, \forall m > n\, (f(m) = g(m))$;
- $f \sqsubset_4 g \Leftrightarrow f(0) < g(0)$.

Determine for every ordering a pair of incomparable elements $f, g$ such that neither $f \sqsubset_m g$ nor $g \sqsubset_m f$ nor $f = g$. For which of these orderings is it possible to choose the $f$ of this pair $(f, g)$ of examples such that $f(n) = 0$ for all $n$?

**Exercise 9.13.** Let $A = \mathbb{N} - \{0, 1\}$ and $<_{div}$ given by $x <_{div} y \Leftrightarrow \exists z \in A\, (x \cdot z = y)$ as in Exercise 9.6. Define a relation $E$ on $A \times A$ by putting $(x, y)$ into $E$ iff there is a prime number $z$ with $x \cdot z = y$. So $(2, 4) \in E$, $(2, 6) \in E$, $(2, 10) \in E$ but $(2, 7) \notin E$, $(2, 8) \notin E$ and $(2, 20) \notin E$. Show that $(A, <_E)$ and $(A, <_{div})$ are identical partially ordered sets.