MA 3219 – Computability Theory

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Assignment for 26.01.2005. Can be corrected on request, it is not obligatory to hand the homework in.

1. Terminology. A set is decidable iff
(a) Every element is the double of the next smaller one.
(b) It is either finite or cofinite.
(c) Its characteristic function is computable by an URM.
(d) It is a superset of all prime numbers.

The domain of a function from $\mathbb{N}$ to $\mathbb{N}$ is
(a) The set of all numbers where it is defined.
(b) The set of all numbers which are mapped to themselves.
(c) The set of all numbers where a program computing the function does not terminate.
(d) The set of all numbers which are mapped to 0.
(e) The set of all numbers which are mapped to 1.

An URM not halt on some input iff
(a) It eventually goes to a line number which does not exist.
(b) The computation goes infinitely often through a loop.
(c) It transfers this input into the eighth register.

What does the acronym "URM" stand for?
(a) Universal Register Machine.
(b) Unlimited Register Machine.
(c) Universal Registrating Machine.
Give a short explination for the name.

A function $f : \mathbb{N} \to \mathbb{N}$ is URM-computable iff
(a) If it is undefined somewhere.
(b) If there is an URM which halts exactly on the those $x \in \mathbb{N}$ where $f(x)$ is defined such that the output of the URM is x.
(c) If there is an URM which halts exactly on the those $x \in \mathbb{N}$ where $f(x)$ is defined such that the output of the URM is $f(x)$.

A function is total iff
(a) The domain is decidable.
(b) The domain is the set $\mathbb{N}$.
(c) The domain is the set $\emptyset$.

2. Predicates. Which two of the following functions are predicates?
(a) $f_1(x) = x + 200$;
(b) $f_2(x, y) = 1$ if $M(x)$ halts with output y and $f_2(x, y) = 0$ otherwise where M is a given URM.
(c) $f_3(x) = 1$ if there is a y with $x = y \cdot y + 1$ and $f_3(x) = 0$ otherwise.
(d) $f_4(x) = y$ for the unique y such that there is a z with $0 \leq y \leq 2$ and $x = y + 3 \cdot z$.
(5) $f_5(x) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (x + 5)$.

3. Decidability. Prove that there is a URM which computes a total predicate $P(x)$ where $P(x) = 1$ iff x is a multiple of 2 or a multiple of 3. Note that $P(x) = 0$ iff $x = 6y + 1$ or $x = 6y + 5$ for some y. Write down the URM either as a program or as a flowchart.

4. Computable Function. Prove that the following function is a partial and computable function:

$$f(x) = \begin{cases} y & \text{if } x = 3y + 1; \\ 2y & \text{if } x = 3y + 2; \\ \uparrow & \text{otherwise, that is, there is no such } y. \end{cases}$$

Write down the URM witnessing this either as a program or as a flowchart.

5. Combining Functions. Let $g : \mathbb{N} \to \mathbb{N}$ be a partial computable function and f be a total computable function. Which two of the following functions are certainly computable (about the other two functions one cannot say anything without more knowledge on f and g).

$$\begin{aligned}
h_1(x) &= \begin{cases} f(g(x)) & \text{if } g(x) \text{ is defined;} \\ \uparrow & \text{if } g(x) \text{ is undefined;} \end{cases} \\
h_2(x) &= \begin{cases} 0 & \text{if } f(x) = g(x) \text{ and } g(x) \text{ is defined;} \\ 1 & \text{if } f(x) \neq g(x) \text{ and } g(x) \text{ is defined;} \\ 2 & \text{if } g(x) \text{ is undefined;} \end{cases} \\
h_3(x) &= \begin{cases} 0 & \text{if } g(x) \text{ is defined;} \\ \uparrow & \text{if } g(x) \text{ is undefined;} \end{cases} \\
h_4(x) &= \begin{cases} g(x) & \text{if } g(x) \text{ is defined;} \\ f(x) & \text{if } g(x) \text{ is undefined.} \end{cases}
\end{aligned}$$

Recall that the option $\uparrow$ in the case distinction for h_1 and h_3 means that $h_1(x)$ and $h_3(x)$ are undefined in the corresponding cases.