MA 3205 – Set Theory – Homework for Week 3

Frank Stephan, fstephan@comp.nus.edu.sg, 6516-2759, Room S14#04-13.

Homework. The homework follows the lecture notes. Below a list of the homeworks for the tutorials from 29.08.2006 onwards. It is not mandatory to hand in homework; but it is recommended to solve the questions by yourself before going to the tutorials. Homework can be corrected on request.

Exercise 1.5. The property of being well-founded is an abstract property which applies also to some but not all graphs which are different from the universe of all sets. Here some examples of graphs. Which of them are well-founded? The answers should be proven.

1. the set $\{0, 1, \{0\}, \{1\}, \{0, 1, \{0\}\}, \{\{1\}\}, \{\{\{1\}\}\}, 512\}$ with (a, b) being an edge iff $a \in b$;
2. the set $\{0, 1, 2, 3\}$ with the edges $(0, 1), (1, 0), (2, 3)$;
3. the set $\mathbb{N}$ of the natural numbers with every edge being of the form $(n, n + 1)$;
4. the set $\mathbb{Z}$ of the integers with the edges being the pairs $(n, n + 1)$ for all $n \in \mathbb{Z}$;
5. the set $\mathbb{Q}$ of rational numbers with the edges being the pairs $(q, 2q)$ for all $q \in \mathbb{Q}$;
6. the set $\mathbb{Q}$ of rational numbers with the edges being the pairs $(q, q + 1)$ for all $q \geq 0$ and $(q, q - 1)$ for all $q \leq 0$.

Exercise 1.12. Which of the following sets of natural numbers are equal? Well-known mathematical theorems can be applied without proving them.

1. $A = \{1, 2\}$;
2. $B = \{1, 2, 3\}$;
3. C is the set of all prime numbers;
4. $D = \{d \mid \exists a, b, c > 0\, (a^d + b^d = c^d)\}$;
5. $E = \{e \mid e > 0 \wedge \forall c \in C\, (e \leq c)\}$;
6. $F = \{f \mid \forall c \in C\, (f \geq c)\}$;
7. $G = \{g \mid g \geq 2 \wedge \forall a, b > 1\, (4g \neq (a + b)^2 - (a - b)^2)\}$;

8. $H = \{h \mid h > 0 \land h^2 = h^h\}$;
9. $I = \{i \mid i + i = i \cdot i\}$;
10. $J = \{j \mid (j+1)^2 = j^2 + 2j + 1\}$;
11. $K = \{k \mid 4k > k^2\}$;
12. $L = \{l \mid \exists c \in C\,(l < c)\}$;
13. $M = \{m \mid \exists c \in C\,(m = c^2)\}$;
14. $N = \{n \mid \exists c, d \in C\,(n = cd)\}$;
15. $O = \{o \mid o \text{ has exactly three prime factors}\}$;
16. $P = \{p \mid p, p+2 \in C\}$.

Exercise 2.5. Determine the power-set of $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$. Is there any set X such that $\mathcal{P}(X)$ has exactly 9 elements?

Exercise 2.9. Given the set $\mathbb{N}$ of natural numbers, establish properties to define the following subsets of $\mathbb{N}$ using the Axiom of Comprehension:

1. the set of all numbers with exactly three divisors,
2. the set $\{0, 2, 4, 6, \ldots\}$ of all even numbers,
3. the set of all square numbers,
4. the set of all numbers whose binary representation contains exactly four times a 1.

For example, the set of prime numbers can be defined as the set $\{x \in \mathbb{N} \mid \exists$ unique $y, z \in \mathbb{N}\,(y \cdot z = x \land z < y \leq x)\}$, that is the set of all natural numbers with exactly two divisors.

Exercise 2.10. Show that every property p satisfies the following statements.

1. There are sets x, y such that $x \in y$ and either $p(x) \land p(y)$ or $\neg p(x) \land \neg p(y)$.
2. There is a set x with $x = \{y \in x \mid p(y)\}$.
3. There is a one-to-one function f such that $p(x)$ iff $p(y)$ for all $y \in f(x)$.

Exercise 2.18. Prove that the symmetric difference is associative, that is, for all sets A, B, C, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$. For this reason, one can just write $A \triangle B \triangle C$. Furthermore, prove that $A - B = A \cap (A \triangle B)$.