Exercise 3.10. Define (informally) functions $f_n$ from $\mathbb{N}$ to $\mathbb{N}$ with the following properties:

1. $f_1$ is bijective and satisfies $f_1(x) \neq x$ but $f_1(f_1(x)) = x$ for all $x \in \mathbb{N}$;
2. $f_2$ is two-to-one: for every $y$ there are exactly two elements $x, x' \in \mathbb{N}$ with $f(x) = f(x') = y$;
3. $f_3$ is dominating all polynomials, that is, for every polynomial $p$ there is an $x$ such that for all $y > x$, $f_3(y) > p(y)$;
4. $f_4$ satisfies $f_4(x + 1) = f_4(x) + 2x + 1$ for all $x \in \mathbb{N}$;
5. $f_5(x) = \begin{cases} 0 & \text{if } x = 0; \\ f_5(x - 1) & \text{if } x > 0 \text{ and } x \text{ is not a square number}; \\ f_5(x - 1) + 1 & \text{if } x > 0 \text{ and } x \text{ is a square number}. \end{cases}$

Determine the range of the function $f_4$.

Exercise 3.12. Let $A = \{0, 1, 2\}$ and $F = \{f : A \to A \mid f = f \circ f\} = \{f : A \to A \mid \forall x (f(x) = f(f(x)))\}$. Show that $F$ has exactly 10 members and determine these.

Exercise 4.6. Which of the following sets is transitive and which is inductive?

1. $A = \{\emptyset, \{\emptyset\}\}$,
2. $B = \{\emptyset, \{\{\emptyset\}\}\}$,
3. $C = \{x \mid \forall y \in x \forall z \in y (z = \emptyset)\}$,
4. $D$ is the closure of $\{\emptyset, \mathbb{N} \times \mathbb{N}\}$ under the successor operation $x \mapsto S(x)$,
5. $E$ is the set of even numbers,
6. $F$ is the set of all natural numbers which can be written down with at most 256 decimal digits,
7. $G$ is the set of all finite subsets of $\mathbb{N}$,

8. $H = \mathcal{P}(G)$.

**Exercise 4.7.** Show that the following statements are equivalent for any inductive set $X$.

1. $X = \mathbb{N}$;
2. $X$ has no proper inductive subset;
3. $X$ is a subset of every inductive set;
4. $\forall x \in X \ (x = 0 \lor \exists y \in X \ (x = S(y)))$;
5. $X = N(Y)$ for every inductive set $Y$ where $N(Y)$ is the subset of those $y \in Y$ which are in every inductive subset of $Y$;
6. $X = N(Y)$ for some inductive set $Y$ where $N(Y)$ is defined as in the previous item.

**Exercise 4.9.** Assume that a property $p$ satisfies

$$p(1) \text{ and } \forall x \ (p(x) \Rightarrow p(S(S(x)))).$$ 

Consider the following subsets of natural numbers:

1. the set of all numbers;
2. the set of even numbers;
3. the set of odd numbers;
4. the set of square numbers;
5. the set of all powers of 35, that is, $\{1, 35, 1225, 42875, 1500625, \ldots\}$.

For which of these sets it is guaranteed that all elements and for which it is guaranteed that some elements satisfy $p$? Consider the property

$$q(x) \iff \forall A \subseteq \mathbb{N} \ ((1 \in A \land \forall y \in \mathbb{N} \ (y \in A \Rightarrow S(S(y)) \in A)) \Rightarrow x \in A).$$

Which of the above sets is equal to $\{x \in \mathbb{N} \mid q(x)\}$. That is, which set is the intersection of all sets $A$ for which

$$1 \in A \land \forall y \in \mathbb{N} \ (y \in A \Rightarrow S(S(y)) \in A)$$

is true?