MA 3205 – Set Theory – Homework for Week 7

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Homework. The homework follows the lecture notes. Below the list of the homeworks for the tutorials from 03.10.2006 onwards. It is not mandatory to hand in homework; but it is recommended to solve the questions by yourself before going to the tutorials. Homework can be corrected on request.

Exercise 9.3. A graph (G, E) is called *bipartite* if there are two subsets X, Y of G such that $X \cap Y = \emptyset$ and every pair $(x, y) \in E$ is actually in $X \times Y \cup Y \times X$. An English-Russian dictionary is a bipartite graph because one can take X to be the words written in the Latin alphabet and Y to be the words written in the Cyrillic alphabet. An English-Spanish dictionary is not bipartite, for example some place names like "Los Angeles" appear in the same spelling in both languages. By the way, the mentioned name is of Spanish origin and has the English translation "the angels". Which of the following graphs are bipartite? The set of vertices is $\mathbb{N}$ and the set E_n of edges is specified below, note that $E_n \subseteq \mathbb{N} \times \mathbb{N}$.

1. $(x, y) \in E_1 \Leftrightarrow x = y$,
2. $(x, y) \in E_2 \Leftrightarrow x < y$,
3. $(x, y) \in E_3 \Leftrightarrow 12 < x + y < 18$,
4. $(x, y) \in E_4 \Leftrightarrow x > 4 \wedge (y = x^2 \vee y = x^4)$,
5. $(x, y) \in E_5 \Leftrightarrow \exists z > 0\, (x = 2^z \wedge y = 3^z)$,
6. $(x, y) \in E_6 \Leftrightarrow \exists z > 0\, (x \in \{2^z, 3^z\} \wedge y \in \{5^z, 7^z\})$,
7. $(x, y) \in E_7 \Leftrightarrow y = S(x) \wedge x$ is even.

Exercise 9.6. Let $A = \mathbb{N} - \{0, 1\} = \{2, 3, 4, \ldots\}$ and let $<_{div}$ be given by $x <_{div} y \Leftrightarrow \exists z \in A\, (x \cdot z = y)$. That is, $x <_{div} y$ iff x is a proper divisor of y, so $2 <_{div} 8$ but $2 \not<_{div} 2$ and $2 \not<_{div} 5$. Prove that $(A, <_{div})$ is a partially ordered set.

Exercise 9.9. Prove that the following relations are partial orderings on $\mathbb{N}^{\mathbb{N}}$:

- $f \sqsubset_1 g \Leftrightarrow \exists n\, \forall m > n\, (f(m) < g(m))$;
- $f \sqsubset_2 g \Leftrightarrow \forall n\, (f(n) \leq g(n)) \wedge \exists m\, (f(m) < g(m))$;
- $f \sqsubset_3 g \Leftrightarrow \forall n\, (f(n) \leq g(n)) \wedge \exists n\, (f(n) < g(n)) \wedge \exists n\, \forall m > n\, (f(m) = g(m))$;

- $f \sqsubset_4 g \Leftrightarrow f(0) < g(0)$.

Determine for every ordering a pair of incomparable elements f, g such that neither $f \sqsubset_m g$ nor $g \sqsubset_m f$ nor $f = g$. For which of these orderings is it possible to choose the f of this pair (f, g) of examples such that $f(n) = 0$ for all n?

Exercise 9.13. Let $A = \mathbb{N} - \{0, 1\}$ and $<_{div}$ given by $x <_{div} y \Leftrightarrow \exists z \in A\,(x \cdot z = y)$ as in Exercise 9.6. Define a relation E on $A \times A$ by putting (x, y) into E iff there is a prime number z with $x \cdot z = y$. So $(2, 4) \in E$, $(2, 6) \in E$, $(2, 10) \in E$ but $(2, 7) \notin E$, $(2, 8) \notin E$ and $(2, 20) \notin E$. Show that $(A, <_E)$ and $(A, <_{div})$ are identical partially ordered sets.