MA 3205 – Set Theory – Homework for Week 8

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Homework. The homework follows the lecture notes. Below the list of the homeworks for the tutorials from 10.10.2006 onwards. It is not mandatory to hand in homework; but it is recommended to solve the questions by yourself before going to the tutorials. Homework can be corrected on request.

Exercise 10.6. Let $(A, <)$ be a linearly ordered set and $B = A^\mathbb{N}$. Define

$$f <_{\text{lex}} g \iff \exists k \in \mathbb{N} \ (f|k = g|k \wedge f(k) < g(k)).$$

Here $f|k$ is the restriction of $f$ to $k$: $f|k = \{ (x, f(x) \mid x \in k \}$.

Furthermore, let $C = A^\ast$. The lexicographic ordering on $A^\ast$ is defined such that either the smaller word is shorter than the longer one or that the first word has a member of $A$ strictly before the second one at the first position where they differ. That is, if $m$ is the domain of $f$ and $n$ the domain of $g$, then

$$f <_{\text{lex}} g \iff \exists k \in S(m) \cap n \ ((f|k = g|k) \wedge (k = m \lor (k < m \wedge f(k) < g(k)))).$$

Show that $(B, <_{\text{lex}})$ and $(C, <_{\text{lex}})$ are linearly ordered sets. Assuming that $A = \{0,1,2,\ldots,9\}$ with the usual ordering, put the following elements of $C$ into lexicographic order: 120, 88, 512, 500, 5, 121, 900, 0, 76543210, 15, 7, 007, 00.

Exercise 10.10. Determine which of the following subsets of the real numbers $\mathbb{R}$ have a lower and upper bound. If so, determine the infimum and supremum and check whether these are even the least and greatest element of these sets.

1. $A = \{ a \in \mathbb{R} \mid \exists b \in \mathbb{R} \ (a^2 + b^2 = 1) \}$;
2. $B = \{ b \in \mathbb{R} \mid b^3 - 4 \cdot b < 0 \}$;
3. $C = \{ c \in \mathbb{R} \mid \sin(c) > 0 \}$;
4. $D = \{ d \in \mathbb{R} \mid d^2 < \pi^3 \}$;
5. $E = \{ e \in \mathbb{R} \mid \sin(\frac{\pi}{2} \cdot e) = \frac{e}{101} \}$.

Exercise 10.22. Show that in a complete ordered set $(A, <)$ every nonempty subset which is bounded from below has an infimum in $A$. 